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COMPLETE MINIMAL SURFACES EMBEDDED IN $\mathbf{R}^3$ WITH TOTAL CURVATURE 12π.

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From the point of view of classical differential geometry, imbedded complete minimal surfaces are interesting objects. A fundamental question about this subject is the classification of such surfaces.

We denote by $\mathcal{F}$ the set of complete orientable minimal surfaces in $\mathbf{R}^3, X : M \to \mathbf{R}^3$, with finite total curvature, $C(M)$. Here M is a non-compact Riemann surface and $C(M) = \int_M | K | dM$ where K is the Gaussian curvature of M. Also, $\mathcal{F}_0$ is the subset of $\mathcal{F}$ such that $X : M \to \mathbf{R}^3$ is in $\mathcal{F}_0$ if and only if X is an embedding.

Some basic properties of $\mathcal{F}$ were studied by R. Ossermann [9] and [10], who showed that if $X : M \to \mathbf{R}^3$ is in $\mathcal{F}$ then M is conformally equivalent to a compact Riemann surface of genus $\gamma, \bar{M}_\gamma$ punctured at N points and $C(M) = 4\pi m, m \in Z, m \geq 0$. In this situation, we say that M has genus γ and N ends. Also, Ossermann caracterized the catenoid and the Enneper surface as the unique surfaces in $\mathcal{F}$ with $C(M) = 4\pi$. So, the catenoid is the unique example in $\mathcal{F}_0$ with $C(M) = 4\pi$ (that is $m = 1$).

On the other hand, Jorge and Meeks [7] have proved that if $X : M \to \mathbf{R}^3$ is in $\mathcal{F}_0$ then $C(M) = 4\pi m, m = \gamma + N - 1$, where M is a conformally equivalent to $\bar{M}_\gamma - \{q_1, \ldots, q_N\}$. Recently some authors have shown the non-existence of examples in $\mathcal{F}_0$, for some values of γ and N. Lopez and Ros [8] have proven the non-existence of examples for $\gamma = 0$ and $N \geq 3$ and R. Schoen [11] has proven the non-existence of examples with $\gamma > 0$ and $N = 2$. So, from these results, for a complete description of $\mathcal{F}_0$ we can concentrate our attention to the surfaces, $X : M = \bar{M}_\gamma - \{q_1, \ldots, q_N\}$ of $\mathcal{F}_0$, where $\gamma \geq 1$ and $N \geq 3$.

The author [1] has constructed an example of a complete minimal immersion in $\mathbf{R}^3$ of genus one and three embedded ends. We will call this surface "Surface S_1". Later, D. Hoffman and W. Meeks [4] have proven that S_1 is embedded (that is, $S_1 \in \mathcal{F}_0$) and, inspired by this surface, they were able to construct, see [6], for each genus $\gamma \geq 2$ an example in $\mathcal{F}_0$ with three ends. The existence of a one parameter family $S_y, y > 1$, in $\mathcal{F}_0$ of genus one and three ends obtained from S_1, by a differentiable perturbation, was announced in [5]. Each S_y is conformally equivalent to $C/L(iy) - \{\pi(0), \pi(1/2), \pi(iy/2)\}$, where $L(iy) = \{m + niy; m, n \in Z\}$ and $\pi : C \to C/L(iy)$ is the canonical projection. We remark that S_1 has a planar end and two ends of catenoid types and $S_y, y > 1$, has three ends of catenoid types. Also, the author [2] has proven that S_1 is the only surface in $\mathcal{F}_0$, of genus $\gamma = 1$, three ends and with one planar end.

The main result that we can prove is that $S_y, y \geq 1$, are all the examples in $\mathcal{F}_0$ with the topology of a torus ($\gamma = 1$) and three ends ($N = 3$). This results follows from Theorems 1 and 2 below. Theorem 1 is a theorem of uniqueness, but we need the result of existence of Hoffman and Meeks [5] to conclude the classification. This result of existence can be obtained, through the maximum principle, as an easy consequence of the results developed in the proof of Theorem 1. So, to make our work self-contained we present in Theorem 2 a proof of the existence of the surfaces $S_y, y \geq 1$.

We say that two minimal surfaces M_1 and M_2 are the same if there exists a rigid motion and a homothety in $\mathbf{R}^3$ that carries M_1 onto M_2.

Theorem 1 (Uniqueness): *Let $X : M \to \mathbf{R}^3$ be a complete minimal surface embedded in $\mathbf{R}^3$ of genus one and three ends. Then,*

a) M is conformally equivalent to $M_y = C/L(iy) - \{\pi(0), \pi(1/2), \pi(iy/2)\}$, where $y \geq 1$, $L(iy) = \{m + niy \in C; m, n \in Z\}$ and $\pi : C \to C/L(iy)$ is the canonical projection.

b) For each $y \geq 1$ there exists at most one such surface X.

Fufthermore if $y = 1, X : M \to \mathbf{R}^3$ is the surface S_1 and if $y > 1, X : M \to \mathbf{R}^3$ has three ends of catenoid types.

Theorem 2 (Existence): *There exists a smooth one-parameter family,* $S_y, S_y : M_y \to \mathbf{R}^3, y \geq 1$, *of complete embedded minimal surfaces in* $\mathbf{R}^3$ *of genus one, three ends and finite total curvature such that:*

a) M_y *is as in (a) of Theorem 1,*

b) S_1 *has a planar end and two ends of catenoid types. That is,* S_1 *is the surface that appears in* [1],[2] *and* [4]*and*

c) $S_y, y > 1$, *has three ends of catenoid types.*

Corollary 1 $S_y, y \geq 1$, *are all complete minimal surfaces embedded in* $\mathbf{R}^3$ *of genus one, three ends and finite total curvature.*

Corollary 2 *The surfaces* $S_y, y \geq 1$, *are all complete minimal surfaces embedded in* $\mathbf{R}^3$ *with total curvature* 12π.

The complete proofs of Theorems 1 and 2 will appear in [3]. Here we will make some considerations about these proofs.

Complete minimal surfaces in $\mathbf{R}^3$.

In [9], we find the following representation of complete minimal surfaces, called Enneper-Ossermann-Weierstrass representation.

Theorem A: *Let* $X : M \to \mathbf{R}^3$ *be a complete minimal immersion of finite total curvature. Then,*

a) M *is conformally equivalent to a compact Riemann surface of genus* $\gamma, \bar{M}_\gamma$, *punctured at* N *points,*

b) there exist a meromorphic function g *and a meromorphic differential* η *in* $\bar{M}_\gamma$ *such that* η *is holomorphic in* M *and* $q \in M$ *is a pole of order* m *of* g *if and only if* g *is a zero of order* $2m$ *of* η. *Furthermore* $g : M \to C \cup \{\infty\}$ *is the Gauss normal map of the immersion,*

c) if α *is a closed path in* M, *then*

$$Re \int_\alpha g\eta = 0, \quad \overline{\int_\alpha \eta} = \int_\alpha g^2\eta \quad \textit{and}$$

d) every divergent path in M has infinite length.

Conversely, let M be as in a) and let g and η be a meromorphic function and a meromorphic differential in $\bar{M}_\gamma$, respectively. If (g,γ) satisfies b) and c) then $X : M \to \mathbf{R}^3$,

$$X(q) = Re\int_{q_0}^{q}((1-g^2\eta), i(1+g^2\eta), 2g\eta), q_0 \in M$$

is a minimal immersion with finite total curvature. Furthermore if X satisfies d) X is complete.

The pair (g,η) is called representation of Weierstrass (or representation of Enneper-Ossermann-Weierstrass) of the immersion X.

Let $X : M = \bar{M}_\gamma - \{q_1, \ldots, q_N\} \to \mathbf{R}^3$ be a complete minimal immersion with finite total curvature. If $D_j \subset \bar{M}_\gamma, j = 1, \ldots, N$ is a small topological disk with $q_j \in D_j$, then $F_j = X(M \cap D_j)$ is an end of the immersion X. Let $z : D_j \to C$ be coordinates in D_j such that $z(q_j) = 0$. Suppose that (g,η) is the representation of Weierstrass of X. Since g is the Gauss map of X we can suppose, after a rotation of X in $\mathbf{R}^3$, that $g(q_j) = 0$. In this situation, see [7], F_j is an embedded end if and only if q_j is a pole of order two of η. Then, around q_j, we have the local expressions

$$g(z) = a_n z^n + o(z)^{n+1}, a_n \neq 0, m \geq 1 \text{ and } \eta(z) = \frac{b}{z^2} + o(z)^{-1}, b \neq 0. \quad (1)$$

We say that the embedded end is of *catenoid type* if $n = 1$ and is a *planar end* of order $n-1$ if $n > 1$. In this last case, the coordinate $X^3(g) = 2Re\int^g g\eta$, $q \in M \cap D_j$, is bounded and the immersion approaches a plane parallel to the plane $X^3 = 0$. That is, from (1) we conclude that the differential $g\eta$ has a pole of order one at q_j (respectively $g\eta$ is holomorphic at q_j) if F_j is a catenoid end (respectively a planar end). Also, from (c) of Theorem A above, we obtain the following information about the residue of the meromorphic differential $g\eta$ at q_j.

$$Res_{q_j} g\eta = \alpha_j \in \mathbf{R}.$$

Suppose that $X : D \to \mathbf{R}^3, D = \{z \in C; 0 <| z |\leq \xi\}$ is a parametrization of the embedded end F_j in terms of the local coordinates $z : D_j \to C$, with

$z(q_j) = 0$. Then we can prove, see [11], page 801, that there exist a punctured neighborhood D^1 of $0 \in C, R_1 > 0$ and $z_1 \in C$ such that $F_j^1 = X(D^1) \subset \mathbf{R}^3$, is a graph over $\{z \in C; | z - z_1 | \geq R_1\}$. That is, $X(D^1)$ can be parametrized by

$$X^3(z) = -\alpha_j \log R + b + o(R^{-1}), \quad | z - z_1 | = R, \tag{2}$$

where $a, b \in \mathbf{R}$ and $o(R^{-1})$ is such that $o(R^{-1})R \to 0$ if $R \to \infty$. The number $-\alpha_j$ is called the logarithmic growth of the end F_j. We observe that if $\alpha_j \neq 0 (\alpha_j = 0)$ we have an end of catenoid type (a planar end).

Elliptic Functions and minimal immersion

Let λ_j be, $j = 1, 2$ complex numbers such that $Im(\lambda_2/\lambda_1) > 0$. The lattice generated by λ_1 and λ_2 is the set $L = L(\lambda_1, \lambda_2) = \{m\lambda_1 + n\lambda_2; m, n \in Z\}$. Two points $z_1, z_2 \in C$ are L-congruent if $z_1 - z_2 \in L$. Otherwise they are incongruent and we write $z_1 \not\equiv z_2$. The canonical projection $\pi : C \to C/L$ induces over C/L a complex structure. So, C/L is a compact Riemann surface of genus one.

Definition 1: An elliptic function of L is a meromorphic function $f : C \to C \cup \{\infty\}$ such that $f(z + \Omega) = f(z)$ for every $\Omega \in L$ and $z \in C$. Also $\eta = f(z)dz$ is a differential elliptic of L where z is the global coordinate of C and f is an elliptic function.

It is easy to see that the canonical projection $\pi : C \to C/L$ identifies the elliptic functions and the elliptic differentials of L with the meromorphic functions and meromorphic differentials of C/L, respectively.

The most important elliptic functions is the P-functions of Weierstrass defined by

$$P(z) = \frac{1}{z^2} + \sum_{\Omega \in L - \{0\}} \left[\frac{1}{(z - \Omega)^2} - \frac{1}{\Omega^2} \right].$$

$P(z)$ is a meromorphic function of order two in C/L with a double pole at the point $\pi(0)$. $P'(z)$ is an elliptic function of L with a triple pole at the points $z \in C$ such that $z = 0$ and z is a zero of P' if and only if $z \equiv \lambda_1/2$ or $z \equiv \lambda_2/2$

or $z \equiv \frac{\lambda_1+\lambda_2}{2}$. So, $P'(z)$ is a meromorphic function in C/L of order three with a triple pole at $\pi(0)$ and zeros at the points $\pi(\lambda_1/2), \pi(\lambda_2/2)$ and $\pi(\frac{\lambda_1+\lambda_2}{2})$.

Let $FM = \{\tau = x + iy \in C; x^2 + y^2 \geq 1, y > 0, | x | \leq \frac{1}{2}\}$ and let $L(\tau)$ be the lattice in C given by $L(\tau) = \{m + n\tau; m, n \in Z\}$, where $\tau \in FM$. Then $C/L(\tau)$ with the complex structure induced by $\pi : C \to C/L(\tau)$ are all Riemann surfaces of genus one. That is, see [12], we have the following theorem:

Theorem B: *Let L be a lattice in C, and let $\bar{M} = C/L$ equipped with the complex structure induced by L. Then there exists $\tau \in FM$ such that $C/L(\tau)$ is conformally equivalent to $\bar{M}$. Furthermore, if $C/L(\tau)$ and $C/L(\tau'), \tau, \tau' \in FM$ are conformally equivalents, then $\tau = \tau'$ or $\tau, \tau' \in \partial FM$ and $\tau' = -\bar{\tau}$.*

The Theorem B shows that to prove Theorems 1 and 2 it is sufficient to consider only the compact Riemann surfaces of genus one, $\bar{M} = C/L(\tau)$, where $\tau \in FM$.

The delicate part of the proof of Theorem 1 is the following technical lemmas and its corollaries.

Lemma 1: *Let $X : M = C/L(\tau) - \{q_1, q_2, q_3\} \to \mathbf{R}^3$ be a complete minimal surface embedded in $\mathbf{R}^3$, with finite total curvature, where $\tau = x + iy \in FM$. Then,*

a) $x = 0$. That is, $L(\tau) = L(iy)$ is a rectangular lattice and

b) M is conformally equivalent to $M_y = C/L(iy) - \{\pi(0), \pi(1/2), \pi(iy/2)\}$.

Lemma 2: *Let $X_y : M_y \to R^3$ be, $y \geq 1$ a complete embedded minimal surface with finite total curvature, where M_y is as in b) of the Lemma 1. Then after a rigid motion and a homothety of X_y in $\mathbf{R}^3$ the Weierstrass representation (g_y, η_y) of X_y is given by*

$$g = g_y = \frac{P'(z)}{2}\left[\frac{a(y)}{P(z) - e_2} + \frac{b_1(y)}{(e_1 - e_2)(P(z) - e_1)} + \frac{b_2(y)}{(e_3 - e_2)(P(z) - e_3)}\right], \quad (3)$$

$$\eta = \eta_y = (P(z) - e_2)dz,$$

where $e_j = P(w_j), w_1 = 1/2, w_2 = \frac{1+iy}{2}, w_3 = iy/2,$

$$a(y) + \frac{b_1(y)}{e_1 - e_2} + \frac{b_2(y)}{e_3 - e_2} = 0$$

and $b_1 = b_1(y), b_2 = b_2(y), b_j : [1, \infty) \to \mathbf{R}$, *is a pair of differentiable solutions of equations*

$$\frac{\pi b_1^2}{e_1 - e_2} + \frac{\pi b_2^2}{e_3 - e_2} = y(2\eta_1 + e_2) - \pi \tag{4}$$

and

$$y(2\eta_1 + e_2)^2 - 2\pi(2\eta_1 + e_2) = 2\pi b_1 b_2, \tag{5}$$

in the lattice $L(iy)$ *with the conditions*

$$b_2 \geq -b_1 > 0. \tag{6}$$

Here, $2\eta_1 = -\int_\alpha P(z)dz$, *where* $\alpha(t) = 1/3 + it, 0 \leq t \leq 1$.

Furthermore, if $b_1(y), b_2(y)$ *is a pair of solutions of equations (4) and (5) with the conditions (6), then (3) is the representation of Weierstrass of a complete minimal immersion of* M_y *in* $\mathbf{R}^3$ *with embedded ends.*

To complete the proof of Theorem 1 it is necessary to examine all the solutions of equations (4) and (5) with the condition (6). This is contained in the following lemma.

Lemma 3: *Let* $L(iy) = \{m + niy \in C; m, n \in Z\}$ *be a lattice, where* $y \geq 1$. *Then there exists one and only one pair* $b_1 = b_1(y), b_2 = b_2(y)$ *of solutions of the equations (4) and (5) with the condition (6). Furthermore,*

a) The fuctions $b_1, b_2 : [1, \infty)$ *are differentiable,*

b) $b_2 = -b_1 = \sqrt{\pi/2}$ *if* $y = 1$,

c) $1 > -b_1/b_2 > 1/2$ *if* $y > 1$.

Also, with these values of b_1 *and* b_2 *we have*

a[1]*) if* $y = 1$, *the Weierstrass*[1] *representation (3) is exactly the same of the surface* S_1 *described in the introduction. So,* $X_1 = S_1$ *of the Lemma 2 is embedded.*

b^1) if $y > 1$, (4) is the Weierstrass[1] representation of a complete minimal immersion, with three ends of catenoid types.

Lemmas 1, 2 and 3 are sufficient to conclude the proof of Theorem 1. Now, we observe that from the representation of Weierstrass (3) we obtain that the differential $g\eta$ has residues $-(b_1 + b_2), b_1$ and b_2 at the points $q_0 = \pi(0), q_1 = \pi(1/2)$ and $q_2 = \pi(iy/2)$, respectively. Let F_j be the ends of the immersion X_y associated to the points $q_j, j = 0,1,2$, respectively. Then, it follows from (2) that the logarithmic growths of the end F_j are respectively, $b_1 + b_2, -b_1$ and $-b_2$. These observations and Lemmas 2) and 3) show the following situation : For $y = 1, b_2 = -b_1 > 0, -(b_1 + b_2) = 0$ and the surface $X_1 = S_1$ is embedded. For $y > 1, b_2 > -b_1 > b_1 + b_2 > 0$. This shows that the end F_2 (logarithmic growth $-b_2 < 0$) is contained in the half-space $x^3 < 0$ and the ends F_0 and F_1 (respectively with logarithmic growths $b_1 + b_2 > 0$ and $-b_1 > 0$) are contained in the half-space $x^3 > 0$. Furthermore, as $-b_1 > b_1 + b_2$, the end F_0 is always below the end F_1. This geometric placement is the key to prove that X_y is embedded also for every $y > 1$. For the details see [3].

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