Matemática Contemporânea, Vol. 1, 17–23
http://doi.org/10.21711/231766361991/rmc13

KALUZA-KLEIN METRICS WITH POSITIVE SECTIONAL CURVATURE

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Let $G - - - P \xrightarrow{\pi} M$ be a smooth principal bundle with connection form ω. A natural class of metrics in P is defined by the following procedure. Fix a Riemannian metric $\langle\ ,\ \rangle_M$ in M and a bi-invariant metric $\langle\ ,\ \rangle_G$ in G and define

$$g_t(X,\, Y) := \langle \pi_* X,\, \pi_* Y\rangle_M + t\langle \omega(X),\, \omega(Y)\rangle_G$$

where $X,\ Y \in T_pP$, π_* is the derivative of π and t a positive real number. These metrics are often called Kaluza-Klein metrics.
We have:
a) G acts by isometries in $(P,\ g_t)$.
b) π is a Riemannian submersion.
c) The fibers of P are totally geodesic.
These facts follow trivially from O'Neill's equations [0]. We must observe that the parameter t allows the sectional curvature of the fibers to change uniformly.

We want to focus our study on the positivity of the sectional curvature for $(P,\ g_t)$. Since the fibers are totally geodesic and isometric to $(G,\ \langle\ ,\ \rangle_G)$, by a theorem of Wallach $[W_1]$ we must have $G = S^1$, $SO(3)$ or S^3.

Studying 2-planes in T_pP that define positive sectional curvature, Weinstein $[W_3]$ introduced the concept of fatness in fiber bundles.
Definition: Let $G - - - P \to M$ be a smooth principal bundle e ω a principal connection form with curvature form Ω. We call ω fat if for every non null linear functional $m : \mathfrak{g} \to \mathbb{R}$, $m \circ \Omega$ is a nondegenerate real 2-form in P, where $\mathfrak{g}$ is the Lie algebra of G.

There are topological obstructions to the existence of a fat connection. In the case $G = S^1$ we have that M admits a sympletic structure. If $G = S^3$ then there is a cohomology class $v \in H^4(M, \mathbb{R})$ with $v^k \neq 0$ for $k = 1, 2, \ldots, \frac{\dim M}{4}$.

Consider in P the metric g_t. The fact that ω is fat translates in the following geometric property: ω is fat if and only if the vertizontal curvatures (that is, the sectional curvatures defined by planes spanned by one horizontal and one vertical vector) are positive.

A natural class of bundles for the search of fat connections is:

$$S^3 - - - P \rightarrow HP^n$$

where HP^n are the quaternionic projective spaces. We will look for the simplest case $S^3 - - - P \rightarrow HP^1 \simeq S^4$. These bundles are parametrized by the integers and Derdziński-Rigas [D-R] proved that only one, just the Hopf bundle, can admit fat connections. The connections, in the Hopf bundle, that are interesting from the point of view of physics are the instanton connections, that is, the critical points of the Yang-Mills functional. The natural question is: *"Are the instanton connections fat?"*

The Yang-Mills theory

Let $G - - - P \rightarrow M$ ($P(M, G)$) be a smooth principal bundle. Recall that an inner automorphism of P is a G-equivariant diffeomorphism of P, which projects down to the identity on M. The group of all inner automorphisms is called the gauge group of $P(M, G)$ and will be denoted G_P. Let C_P be a space of all principal connections on $P(M, G)$. G_P acts on C_P by pull-back

$$\begin{aligned} G_P \times C_P &\rightarrow C_P \\ f, \omega &\rightarrow f^*\omega \end{aligned}$$

The curvature form Ω of a connection ω may be considered a 2-form in M with values in the adjoint bundle $AdP = P \times_{Ad} \mathfrak{g}$. Then we can define the

Yang-Mills functional by

$$\begin{aligned} \mathcal{YM}: \ C_P &\to \mathbb{R} \\ \omega &\to \frac{1}{2}\int_M \|\Omega\|^2. \end{aligned}$$

We have:

a) $\mathcal{YM}$ is an invariant of the action of the gauge group.

b) The critical points of $\mathcal{YM}$ are called instanton connections.

c) If M is a four dimensional Riemannian manifold the connections with curvature Ω, such that Ω is self-dual ($*\Omega = \Omega$ where $*$ is the Hodge star operator) are instanton connections which are absolute minima for $\mathcal{YM}$ (see [L]).

We will give a description of the set of self-dual connections modulo gauge equivalence (moduli space $\mathcal{M}$) for the Hopf bundle. For this, we will make use of associated vector bundles. Let H denote the quaternion numbers, $S^3 = \{a \in H, \ \|a\| = 1\}$ and $S^7 = \{(a, \ b) \in H \times H, \ \|a\|^2 + \|b\|^2 = 1\}$. Let S^3 act on H by $a \times \xi = \overline{a}\xi$ where $a \in S^3 \subseteq H$. Then we have the associated vector bundle

$$H - - - - - - S^7 \times_{S^3} H \to S^4.$$

The tautological bundle over S^4 is defined by

$$N_{S^4} = \{([v_1, \ v_2], \ w) \in HP^1 \times H^2, \ w \in [v_1, v_2]\}$$

It is easy to see that $N_{S^4} \simeq S^7 \times_{S^3} H$. $\langle (v_1, \ v_2), \ (w_1, \ w_2) \rangle = v_1\overline{w}_1 + v_2\overline{w}_2$ is a quaternionic inner product in H^2 and Real $\langle \ , \ \rangle$ is the canonical inner product in $\mathbb{R}^8 \simeq H^2$. The 1-form with values in ImH (pure imaginary quaternion numbers) $\omega = Im(v_1 d\overline{v}_1 + v_2 d\overline{v}_2)$ defines a principal connection in the Hopf bundle. The vertical tangent space defined by ω is

$$VTS^7_{(v_1, \ v_2)} = \{(av_1, \ av_2), \ a \in ImH\}.$$

The formula for the curvature is complicated, but at the point (0, 1) we have

$$\Omega_{(0, 1)} = dv_1 \wedge d\overline{v}_1.$$

A reference for quaternion diferentials is [A].

The horizontal tangent space in (0, 1) is generated by $\dfrac{\partial}{\partial v_1}$ and therefore $\Omega_{(0,1)}$ is the pull-back by π of a self-dual 2-form in $[0, 1] \in HP^1 \simeq S^4$. The sympletic group $Sp(2)$, acts on the right on S^7 by inner bundle automorphisms. This action is transitive, by isometries and preserves ω. From this we can ensure that Ω is self-dual at all points of S^7. We now look for local expressions for ω and Ω. A local parametrization for $HP^1 \simeq S^4$ is

$$\begin{aligned} \mathbb{R}^4 \simeq H &\rightarrow S^4 \simeq HP^1 \\ x &\rightarrow [x, 1]. \end{aligned}$$

With this parametrization we have the local section for the Hopf bundle

$$\begin{aligned} r : H &\rightarrow S^7 \\ x &\rightarrow \frac{(x, 1)}{\sqrt{1+|x|^2}} \end{aligned}$$

and after a simple calculation

$$\begin{aligned} A = r^*\omega &= Im\left(\frac{x d\bar{x}}{1+|x|^2}\right) \\ F = r^*\Omega &= \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2}. \end{aligned}$$

This is the basic instanton [A]. Since the self-duality equations are conformally invariant, the pull-back of the basic instanton by conformal transformations S^4 will give us new self-dual connections.

The group of conformal transformations of S^4, that preserves the orientation, is $SL(2, H)/\pm 1$ acting by fractionary linear transformations in the quaternionic variable. The group $SL(2, H)$ acts on the tautological bundle by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} ([v_1, v_2], (w_1, w_2)) = ([v_1 a + v_2 c, v_1 b + v_2 d], (w_1 a + w_2 c, w_1 b + w_2 d))$$

This action projects to the action of the conformal group $SL(2, H)/\pm 1$ on S^4. If $\nabla = d + A$ is the connection in N_{S^4}, defined by the basic instanton ω, $f^*\nabla$ is self-dual for $f \in SL(2, H)$ acting in N_{S^4}

and $f^*\nabla = \nabla$ if and only if $f \in Sp(2) \subseteq SL(2, H)$. In [A - H - S],it is shown that we have calculated all the equivalent classes of self-dual connections, that is, $\mathcal{M} = SL(2, H)/Sp(2)$. Then one parametrization for $\mathcal{M}$ is given by the transformations $T_{\lambda, b}(x) = \lambda(x+b)$, $\lambda \in \mathbb{R}$, $0 < \lambda \leq 1$ and $b \in H$ ([F-U]). These transformations acting in the connection $\nabla = d + A$ give

$$A^\lambda = T^*_{\lambda, b}A = Im(\frac{(x-b)dx}{\lambda^2 + |x-b|^2})$$
$$F^\lambda = T^*_{\lambda, b}F = \lambda^2 \frac{dx \wedge d\overline{x}}{(\lambda^2 + |x-b|^2)^2} \quad ,$$

where A^λ is called the instanton with center b and scale λ.

We now have a good description of the moduli space $\mathcal{M}$ as a five dimensional ball B^5 with the basic instanton in the center and the lines between the center and $b \in \partial B^5 = S^4$ parametrized by the connections A^λ. The basic instanton ω defines a one parameter family of Kaluza-Klein metrics g_t, such that g_1 is nothing more than the canonical metric in S^7. Now the natural question is: *"How far we can go along the radial lines in the moduli space such that the metrics g_t^λ defined by A^λ still have positive sectional curvature?"* In order to answer this question we will need the theorem:

Theorem [C - D - R]: *Let $G - - - P \to M$ be a principal smooth bundle with connection ω over a compact manifold M, where $G = S^3$, $S0(3)$ or S^1. Fix a Riemannian metric $\langle\ ,\ \rangle_M$ in M and a bi-invariant metric $\langle\ ,\ \rangle_G$ in G. The following conditions are equivalent:*

a) There is a $t_0 > 0$, such that for $0 < t \leq t_0$, (P, g_t) has positive sectional curvature.

b) For any point x in M and any non-zero element u over x of AdP, we have

$$R_M(X, Y, X, Y) \sum_{k=1}^{n} (\langle u, \Omega(X, X_k)\rangle)^2 > (\langle u, (\nabla_X \Omega)(X, Y)\rangle)^2$$

for any orthonormal vectors X, Y in T_xM, where R_M is the curvature tensor of $(M, \langle\ ,\ \rangle_M)$, Ω is the curvature form of ω, ∇ is the covariant derivative *of AdP induced by ω and $\{X_1, \dots, X_n\}$ is an arbitrary orthonormal basis of T_xM.*

The proof of this theorem is based on calculations done by Jensen [J] of the components of the curvature tensor R_M and an asymptotic analysis of a one parameter family of inequalities.

Corollary [C - D - R]: *$(P,\ g_t)$ has positive sectional curvature for all t small enough if and only if the sectional curvatures of the planes spanned by $\{X + sV,\ Y\}$, $s \in \mathbb{R}$ and X, Y horizontal vectors and V a vertical vector, is positive.*

The theorem above applied to the instantons A^λ gives:

Theorem [C - D - R]: *For all λ in $(\frac{-1+\sqrt{5}}{2}, 1]$ the instanton A^λ, is such that, $(S^7,\ g_t^\lambda)$ has positive sectional curvature for all t small enough.*

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