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Covering a body using unequal spheres and the problem of finding covering holes

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Abstract

This article deals with partial coverings of convex bodies using unequal spheres S_i, $i \in N$, where N is an index set. For the matter of this work, it is assumed that the covering spheres structure had already been obtained and the objective is just to certify that there are no "holes" in it. Let G be the undirected graph $G(V, E)$ where V is the set of centers of the spheres and E is the set of the edges, such that edge $e_{ij} \in E$ if spheres S_i and S_j overlap each other. A method involving the geometrical properties of the cliques K_3 and K_4, as subgraphs of G, will be presented, which permits to identify the presence of "holes" in the covering structure.

1 Introduction

This article deals with *partial coverings*, which may not integrally cover a target, in contrast with *full coverings*, which totally cover a target. Articles and theses dealing with full and partial coverings are abundant in the literature [4, 6, 7, 8, 9]. Some of those works also deal with "holes" or "cavities" present in partial covering structures [1, 2, 3, 5].

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In this work we consider partial covering structures B formed of solid spheres, usually with different radii, used to cover compact and convex subsets T of $\mathbf{R}^3$. Holes in those covering structures B are void spaces inside the solid formed by the union of the covering spheres.

Partial coverings are important in pratical applications like the Gamma Knife radiosurgery treatment, where a brain tumor is modeled as the subset T and the shots of radiation are modeled as spheres. The covering structures B normally employed present some remarking characteristics:

- B is connected (it is composed of agglutinated spheres);
- There are no spheres in the interior of other spheres;
- The bigger spheres form the inner part of B.

One interesting question can then be posed: given a covering structure B, are there holes in it? A standard way to deal with this kind of situation is by applying homology.

Since the covering problem at hand is a concrete and well defined problem in the three dimensional space, this work exploits the geometric properties of cliques K_3 and K_4 in a graph G, derived from the covering structure B, to create an algorithm which generates a subgraph $H \subseteq G$ that retains all the information necessary to identify the presence of holes.

2 Methodology

Given a partial covering structure $B = \bigcup_{i \in N} S_i$, let $G(V, E)$ be an undirected simple graph defined by:

- $V = \{S_i \mid i \in N\}$;
- $E = \{\{S_i, S_j\} \mid S_i \cap S_j \neq \emptyset,\ i \neq j\}$.

G is an abstract simplification of the geometric properties of the covering structure B. As a consequence, it is observable that G has some special properties:

- G is always connected;
- If G has a cycle, it is the sum of K_3 cliques;
- Whenever present in B, $3D$ void spaces are always inside K_4 cliques.

We will focus on (assume that our graphs are) simple graphs G having the above properties, which are the basis for a straightforward (and intuitive) definition of covering holes:

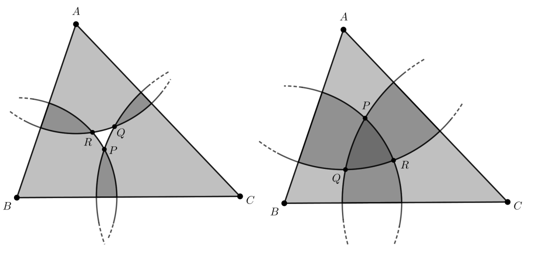
- If G is a tree then B has no holes;
- There is a $2D$ hole in a K_3 clique if the union of the spheres at each of its vertices don't cover the triangle T formed by its vertices. This K_3 is an *uncovered* K_3 (UK_3). Otherwise, this K_3 is a *covered* K_3 (CK_3);
- There is a $3D$ hole in a K_4 clique if the union of the spheres at each of its vertices don't cover the tetrahedron H formed by its vertices. This K_4 is an *uncovered* K_4 (UK_4). Otherwise, this K_4 is a *covered* K_4 (CK_4).

The covering holes definition now permits to sketch a method to find holes in B:

- Build G based on B;
- Find all K_3 and K_4 of G. If not present (G is a tree), then B has no holes;
- If all K_3 are CK_3 and all K_4 are CK_4, then B has no holes;
- Otherwise, B *may have* holes (in UK_3's or UK_4's).

Subgraphs of G may have some special geometrical configurations: "linear K_3: LK_3", "flat K_4: FK_4" and "overlapped K_4's". The test instances didn't produce LK_3's but it was possible to identify FK_4's in some instances. The overlapped K_4's are a real challenge because they represent a covering redundancy that must be properly taken into account.

The main objective now is to present a method to classify K_3's and K_4's. Figure 1 shows the geometrical idea to classify K_3's as UK_3's or CK_3's.



(a) A UK_3triangle (b) A CK_3 triangle

Figure 1: K_3 triangle coverings

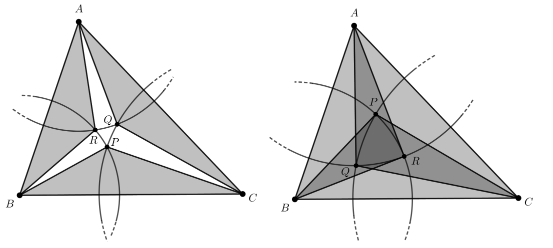
The intersection points P, Q and R always exist, by definition of G. This geometrical configuration leads to the *Areas test*: Let S be the area of a K_3 triangle ABC. Let S_P, S_Q and S_R be the areas of the triangles PBC, QAC and RAB. Then:

- If $S_P + S_Q + S_R < S$ then ABC is a UK_3 triangle;
- If $S_P + S_Q + S_R \geq S$ then ABC is a CK_3 triangle.

Figure 2 presents geometrically the inequalities used in the Areas test for an UK_3 triangle and a CK_3 triangle.

A similar geometric argument is employed to classify K_4's as UK_4's or CK_4's, but now using volumes and the *Volumes test.* Barycentric coordinates are used to simplify the calculation of the areas and volumes tests.

It is now possible to make a final statement regarding holes in a covering structure B based on the subgraph H defined below:



(a) A UK_3 triangle (b) A CK_3 triangle

Figure 2: K_3 triangle coverings

> "The covering structure B has no holes if there is a spanning subgraph $H \subset G$ composed only of CK_3's and non-overlapped CK_4's"

An algorithm to eliminate redundancies and select the right CK_3's and CK_4's to create H is currently under development and implementation. The pseudocode of this algorithm is presented below:

3 Results

Arbitrary data from previous partial covering works will now be used. Table 1 presents the solids T used for the covering instances. Table 2 presents the characteristics of graphs G derived from these covering instances.

As an example of the application of the algorithm, let's consider the original graph G for oblate ellipsoid covering 1: $3\,UK_3$, $30\,CK_3$, $5\,FK_4$, $5\,UK_4$ and $11\,CK_4$. After deleting edges $E_{1,4}$, $E_{3,4}$ and $E_{3,5}$ we obtain a spanning subgraph H of G for oblate ellipsoid covering 1: $22\,CK_3$ and $7\,CK_4$. The existence of this subgraph implies that the covering has no holes, according to the presented definition.

Algorithm 1 Find holes in a partial covering structure B

Input: A covering structure B
Output: A graph H and the number of holes N_H in B

```
N_H = 0
Build graph G based on geometric information of B
H = G
Find all K_3's and K_4's, and classify them
if (There are no cliques) or (All K_3 are CK_3 and All K_4 are CK_4) then
  return H and N_H {leave algorithm}
end if
loop
  Delete edges of F_4's and overlapped K_4's to eliminate UK_3's and UK_4's
  Update H
end loop
return H and N_H = #UK_3 + #UK_4 {a simplified number of holes}
```

Solid T	Length	Width	Height
Parallelepiped	15	9	9
Cube	12	12	12
Sphere	12	12	12
Prolate ellipsoid	15	9	9
Oblate ellipsoid	9	15	15

Table 1: Characteristics of the selected solids

4 Conclusion

The results so far are encouraging. The basic ideas proposed in this text proved to be useful in the search for covering holes. Unfortunately, the algorithm to find H starting with G demands some improvements, mainly in the area of identification of the redundant CK_4's in an overlapped CK_4's configuration.

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Solid T	G	$\#K_3$			$\#K_4$		
	V, E	LK_3	UK_3	CK_3	FK_4	UK_4	CK_4
Parallelepiped 1	9,18	0	0	13	0	0	3
Parallelepiped 2	9,17	0	0	10	0	0	1
Cube	25,96	0	10	134	0	23	73
Sphere 1	11,30	0	0	33	0	0	15
Sphere 2	11,28	0	0	27	0	0	9
Prolate ellipsoid 1	9,26	0	3	34	0	3	34
Prolate ellipsoid 2	9,24	0	0	27	1	0	12
Oblate ellipsoid 1	10,27	0	3	30	5	5	11
Oblate ellipsoid 2	10,29	0	3	35	5	5	15

Table 2: Cliques composition of G's derived from some covering structures

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