Matemática Contemporânea, Vol. 46, 94–102
http://doi.org/10.21711/231766362020/rmc469

Decomposition of graphs into trees with bounded maximum degree

Fábio Botler

Abstract

A decomposition of a graph G is a set of edge-disjoint subgraphs of G that cover the edge set of G. In 1968, Gallai conjectured that every connected simple graph on n vertices admits a decomposition with at most $\lceil n/2 \rceil$ elements and that contains only paths. In 1977, Chung proved a weaker version of this statement, that every graph on n vertices admits a decomposition into trees, and with size at most $\lceil n/2 \rceil$. In this paper we prove a strengthening of Chung's result, that every graph on n vertices admits a decomposition into trees with maximum degree at most $\lceil n/2 \rceil$, and with size at most $\lceil n/2 \rceil$.

1 Introduction

In this paper, every graph considered is simple, i.e., contains neither loops nor multiple edges. A *decomposition* of a graph G is a set $\mathsf{D} = \{H_1, \ldots, H_k\}$ of edge-disjoint subgraphs of G such that $E(G) =$

2000 AMS Subject Classification: 05C70 and 05C05.
Key Words and Phrases: graph, decomposition, tree.
This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, and by CNPq Project 423395/2018-1.

$\bigcup_{i=1}^{k} E(H_k)$. If H_i is a simple path for every $i \in \{1, \ldots, k\}$, then we say that D is a *path decomposition* of G. Erdős (see [9]) proposed the study of the size of a path decomposition with a minimum number of elements, which is called the *path number* of G, and is denoted by $\mathrm{pn}(G)$. In order to answer Erdős problem, Gallai posed the following conjecture.

Conjecture 1 (Gallai, 1968)**.** *If G is a connected graph on n vertices, then* $\mathrm{pn}(G) \leq \lceil n/2 \rceil$.

Conjecture 1 was listed by Bondy [2] as one of the most beautiful conjectures in Graph Theory and, although it has been widely explored [1, 3, 4, 6, 7, 8, 9, 10], it remains open. In 1977, Chung [5] studied a weak version of Conjecture 1 that considers trees instead of paths, and proved the following result.

Theorem 2 (Chung, 1977)**.** *If G is a connected graph on n vertices, then G admits a decomposition in at most $\lceil n/2 \rceil$ trees.*

Note that the bound of $\lceil n/2 \rceil$ is best possible. Indeed, since any tree in a graph G with n vertices contains at most $n-1$ edges, if $|E(G)| > \lfloor n/2 \rfloor (n-1)$, which is the case of complete graphs with an odd number of vertices, then

$$|\mathsf{D}| \geq |E(G)| / \max\{|E(T)| \colon T \in \mathsf{D}\} \geq |E(G)|/(n-1) > \lfloor n/2 \rfloor. \qquad (1)$$

In this paper we consider a strengthening of Theorem 2 in which we require a maximum degree condition on the elements of the decomposition. More specifically, given a connected graph G on n vertices, let $\varphi(G)$ denote the minimum positive integer such that G admits a decomposition D into trees with maximum degree at most $\varphi(G)$, and such that $|\mathsf{D}| \leq \lceil n/2 \rceil$. Note that, given a connected graph G on n vertices, Theorem 2 says that $\varphi(G) \leq n-1$, while Conjecture 1 states that $\varphi(G) \leq 2$. In this paper, we prove that $\varphi(G) \leq \lceil n/2 \rceil$. Our technique follows the idea in [5]. For using induction on the order of the graph, we find a special pair of vertices that can be removed without disconnecting it. Then we are able to restore these vertices while adding at most one new tree (with bounded maximum degree) to the decomposition.

This paper is organized as follows. In Section 2 we prove our main result, and in Section 3 we present some concluding remarks and variations of the problem considered in this paper. For missing definitions, we refer the reader to [3].

2 Trees with bounded maximum degree

In this section, we prove the main result of this paper. Given a vertex x of a graph G, we denote by $d_G(x)$ and $N_G(x)$, respectively, the degree and the set of neighbors of x in G. The distance between two vertices x and y in a graph G is the length of a shortest path that joins x and y, and it is denoted by $\text{dist}_G(x, y)$. The *diameter* of a graph G is the maximum distance between two of its vertices. We say that a decomposition D of a graph G is a (k, d)-*decomposition* if each element of D is a tree with diameter at most k and maximum degree at most d. Thus, Conjecture 1 and Theorem 2 state, respectively, that every graph admits an $(n-1, 2)$-decomposition and an $(n-1, n-1)$-decomposition of size at most '$\lceil n/2 \rceil$. In this section, we prove that every connected graph on n vertices admits a $\big(4, \lceil n/2 \rceil\big)$-decomposition with size at most $\lceil n/2 \rceil$. We use the following lemma, which is a strengthening of a lemma presented by Chung [5, Lemma 2.1].

Lemma 3. *Let G be a connected graph with at least three vertices. Then, there are vertices x, y such that $G - x - y$ is connected and either (i) $d_G(x) = d_G(y) = 1$ and $\text{dist}_G(x, y) = 2$; or (ii) $xy \in E(G)$.*

Proof. Let G be as in the statement. Given a spanning tree T of G and a vertex $v \in V(G)$, we denote by $l_T(v)$ the number of leaves of T that are neighbors of v in T. Let T be a spanning tree of G that minimizes $\delta_l(T) = \min\{l_T(v) \colon l_T(v) \geq 1 \text{ and } v \in V(G)\}$, and let v be a vertex of G such that $l_T(v) = \delta_l(T)$. If $l_T(v) = 1$, then let $x = v$ and let y be the leaf of T adjacent to v. In this case, we have $xy \in E(G)$ and $T - x - y$ is a spanning tree of $G - x - y$, hence $G - x - y$ is connected. This proves case (ii). Thus, we may suppose that $l_T(v) \geq 2$. Let u be a leaf of T that is a neighbor of v. Suppose that $d_G(u) > 1$. Thus u has a neighbor v' different from v. Put $T' = T - uv + uv'$. If v' is a leaf that is a neighbor of v, then $\delta_l(T') = 1$, a contradiction. Thus, we have $\delta_l(T') \leq l_{T'}(v) = l_T(v) - 1 = \delta_l(T) - 1$, a contradiction to the minimality of $\delta_l(T)$. Thus, we have $d_G(u) = 1$ for every leaf of T that is a neighbor of v. Since $l_T(v) \geq 2$, there are two leaves, say x, y, of T that are neighbors of v. Since $d_G(x) = d_G(y) = 1$, the graph $G - x - y$ is connected, and xvy is a path in G that joins x and y, hence $\text{dist}_G(x, y) = 2$. This proves case (i). □

For the proof of our result we use the following concept. A *double-rooted decomposition* of a graph G is a pair (D, ρ) in which D is a decomposition of G and $\rho\colon V(G) \to \mathsf{D}$ is a function such that, for every $T \in \mathsf{D}$, the following hold.

(a) if $\rho(v) = T$, then $v \in V(T)$;

(b) $|\rho^{-1}(T)| \leq 2$; and

(c) if $\rho^{-1}(T) = \{x, y\}$, then $\text{dist}_T(x, y) \leq 2$.

Given a tree $T \in \mathsf{D}$, the vertices in $\rho^{-1}(T)$ are called the *roots* of T.

Theorem 4. *Let G be a connected graph on n vertices. Then G admits a double-rooted decomposition (D, ρ) with $|\mathsf{D}| \leq \lceil n/2 \rceil$ and such that D is a $\big(4, \lceil n/2 \rceil\big)$-decomposition.*

Proof. The proof follows by induction on n. Since every graph with $n \leq 4$ vertices has path number at most $\lceil n/2 \rceil$, we may assume $n \geq 5$. Let G be a connected graph on n vertices. By Lemma 3, there are vertices x and y such that $G' = G - x - y$ is connected and either (i) $d_G(x) = d_G(y) = 1$ and $\text{dist}_G(x, y) = 2$; or (ii) $xy \in E(G)$. By the induction hypothesis, there is a double-rooted decomposition (D', ρ') of G' with $|\mathsf{D}'| \leq \big\lceil (n-2)/2 \big\rceil$ and such that every tree in D' has diameter at most 4 and maximum degree at most $\lceil(n-2)/2\rceil$. First, suppose that case (i) holds. Then, there is a vertex $v \in V(G)$ such that $N_G(x) = N_G(y) = \{v\}$. Put $T^* = xvy$, and put $\rho(v) = \rho'(v)$ for every $v \in V(G) \setminus \{x,y\}$, and $\rho(x) = \rho(y) = T^*$. Let $\mathsf{D} = \mathsf{D}' \cup \{T^*\}$, and note that (D, ρ) is a double-rooted decomposition with $|\mathsf{D}| \leq \lceil n/2 \rceil$ and such that D is a $\big(4, \lceil(n-2)/2\rceil\big)$-decomposition. This concludes the proof when case (i) holds.

Now, let us analyze case (ii), i.e. suppose that $xy \in E(G)$. Let $Z = N_G(x) \cap N_G(y)$, and let X (resp. Y) be the set of neighbors of x (resp. y) that are not neighbors of y (resp. x), i.e., $X = N_G(x) \setminus Z$ and $Y = N_G(y) \setminus Z$. Put $W = X \cup Y \cup Z$. In what follows, we partition D' according to the distribution of the roots of each element in W. There are three types of subsets of D' as follows. Let D'_0 be the set of elements of D' that have no roots in W; for $S \in \{X, Y, Z\}$, let D'_S be the set of elements of D' containing precisely one root in W, and such that this root is in S; for $S, R \in \{X, Y, Z\}$ (possibly, $S = R$), let D'_{SR} be the set of elements of D' containing precisely two roots, say x, y, in W, and such that $x \in S$, $y \in R$. It is clear that these sets partition D', i.e., we have

$$\mathsf{D}' = \mathsf{D}'_0 \cup \mathsf{D}'_X \cup \mathsf{D}'_Y \cup \mathsf{D}'_Z \cup \mathsf{D}'_{XX} \cup \mathsf{D}'_{YY} \cup \mathsf{D}'_{ZZ} \cup \mathsf{D}'_{XY} \cup \mathsf{D}'_{XZ} \cup \mathsf{D}'_{YZ}.$$

Given a tree $T' \in \mathsf{D}'$ and $S \subset V(G)$, we denote by ${\rho'}_S^{-1}(T')$ the set ${\rho'}^{-1}(T') \cap S$. In what follows, for each tree $T' \in \mathsf{D}'$ we construct a new tree T by adding at most two edges of $E(G) \setminus E(G')$. This operation is performed in such a way that each new edge is added at a vertex $z \in {\rho'}^{-1}(T')$, which implies that T is connected and has diameter at most 4. Moreover, if T is obtained from T' by adding exactly two edges, then (a) these edges are added at distinct vertices of ${\rho'}^{-1}(T')$, which implies that $\Delta(T) \leq \lceil(n-2)/2\rceil + 1 = \lceil n/2 \rceil$; and (b) one of these edges is incident to x and the other is incident to y, which implies that T has no cycles. The construction of T consists of the following seven rules.

1. If $T' \in \mathsf{D}'_0$, then let $T = T'$;

2. If $T' \in \mathsf{D}'_X \cup \mathsf{D}'_{XX} \cup \mathsf{D}'_Z$, then let $z \in {\rho'}^{-1}_{X\cup Z}(T')$ and put $T = T' + zx$. In this case, we choose precisely one vertex z in ${\rho'}^{-1}_{X\cup Z}(T')$ to add zx to T';

3. If $T' \in \mathsf{D}'_Y \cup \mathsf{D}'_{YY}$, then let $z \in {\rho'}^{-1}_{Y\cup Z}(T')$ and put $T = T' + zy$. As above, we choose precisely one vertex z in ${\rho'}^{-1}_{Y\cup Z}(T')$ to add zy to T';

4. If $T' \in \mathsf{D}'_{XY}$, then let $z \in {\rho'}^{-1}_X(T'), z' \in {\rho'}^{-1}_Y(T')$ and put $T = T' + zx + z'y$;

5. If $T' \in \mathsf{D}'_{XZ}$, then let $z \in {\rho'}^{-1}_X(T'), z' \in {\rho'}^{-1}_Z(T')$ and put $T = T' + zx + z'y$;

6. If $T' \in \mathsf{D}'_{YZ}$, then let $z \in {\rho'}^{-1}_Y(T'), z' \in {\rho'}^{-1}_Z(T')$ and put $T = T' + zy + z'x$;

7. If $T' \in \mathsf{D}'_{ZZ}$, then let $z, z' \in {\rho'}^{-1}_Z(T')$ with $z \neq z'$ and put $T = T' + zx + z'y$.

Now, let T^* be the tree induced by the remaining edges, i.e., the edges in $E(G) \setminus E(G')$ not present in T, for every $T' \in \mathsf{D}'$. We claim that $\Delta(T^*) \leq \lceil n/2 \rceil$. It is not hard to check that the only neighbors of x in T^* are y; one, but not two, of the roots of each tree in D'_{XX}; the roots in Z of the trees in D'_{XZ}; and one, but not two, of the roots of each tree in D'_{ZZ}. Thus, we have $d_{T^*}(x) \leq 1 + |\mathsf{D}'_{XX}| + |\mathsf{D}'_{XZ}| + |\mathsf{D}'_{ZZ}| \leq 1 + |\mathsf{D}'| \leq 1 + \lceil (n-2)/2 \rceil \leq \lceil n/2 \rceil$. Analogously, the only neighbors of y in T^* are x; the roots in Z of the trees in D'_Z; one, but not two, of the roots of each tree in D'_{YY}; the roots in Z of the trees in D'_{YZ}; and one, but not two, of the roots of each tree in D'_{ZZ}. Thus, we have $d_{T^*}(x) \leq 1 + |\mathsf{D}'_Z| + |\mathsf{D}'_{YY}| + |\mathsf{D}'_{YZ}| + |\mathsf{D}'_{ZZ}| \leq 1 + |\mathsf{D}'| \leq 1 + \lceil (n-2)/2 \rceil \leq \lceil n/2 \rceil$.

Finally, by the remarks above, we have $\Delta(T) \leq \lceil n/2 \rceil$ for every $T' \in \mathsf{D}'$, and hence $\mathsf{D} = \{T \colon T' \in \mathsf{D}'\} \cup \{T^*\}$ is a $\big(4, \lceil n/2 \rceil\big)$-decomposition of G such that $|\mathsf{D}| \leq \lceil n/2 \rceil$. Moreover, let $\rho(v) \colon V(G) \to \mathsf{D}$ be such that $\rho(v) = \rho'(v)$ for every $v \in V(G) \setminus \{x, y\}$; and $\rho(x) = \rho(y) = T^*$. Clearly, (D, ρ) is a double-rooted decomposition of G, as desired. $\square$

3 Concluding remarks

In this paper, we give a first step in a novel approach for dealing with Gallai's Conjecture by proving an intermediate statement between Conjecture 1 and Theorem 2. In this section, we present some variations of the problem studied in this paper, that explore decompositions into subgraphs with constraints on their degrees or diameter. Let $e_{k,d}$ denote the maximum number of edges in a tree with diameter at most k and maximum degree at most d. By induction on k, one can check that $e_{k,d} = d\big(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{k/2-1}\big)$ if k is even; and $e_{k,d} = 1 + 2\big((d-1) + \cdots + (d-1)^{\lfloor k/2 \rfloor}\big)$ if k is odd. The following statement generalizes both Conjecture 1 and Theorem 4.

Conjecture 5. *Let k, d be positive integers such that $k \geq 3$, $d \geq 2$ and $e_{k,d} \geq n-1$. If G is a connected graph on n vertices, then G admits a (k,d)-decomposition with size at most $\lceil n/2 \rceil$.*

Note that, by Equation 1, the condition $e_{k,d} \geq n-1$ is necessary. Moreover, Conjecture 1 does not imply Conjecture 5 since a path decomposition may contain paths of arbitrary lengths, and hence arbitrary diameter. For example, any minimum path decomposition of the complete graph K_{2k} consists of Hamilton paths.

We believe that a natural first step to improve the result in this paper is to explore the case $k = 6$ and $d = \lceil n/4 \rceil$. Following the idea in [5], our proof consisted in presenting a lemma (Lemma 3) that allows us to remove a pair of special vertices without disconnecting the graph. For improving Theorem 4, one may generalize Lemma 3 even further, perhaps by introducing new ingredients.

Another possible direction to tackle Conjecture 1 by using trees with weaker maximum degree conditions is to omit the diameter condition and aim the following statements.

Conjecture 6.

1. *There is a positive constant* $c_1 < 1/2$ *such that every connected graph on* n *vertices admits an* $(n-1, c_1 n)$*-decomposition with size at most* $\lceil n/2 \rceil$*.*

2. *There is a positive integer* c_2 *such that every connected graph on* n *vertices admits an* $(n-1, c_2)$*-decomposition with size at most* $\lceil n/2 \rceil$*.*

The statements above are significant improvements on the result of this paper. Clearly, proving that $c_2 = 2$ implies Conjecture 1. To prove these statements one may easily avoid the cases for which Conjecture 1 has been already verified. For example, one may assume that the given graph has maximum degree at least 6 [1], and that it contains a cycle consisting only of vertices with even degree [10].

Let T be a tree. We say that T is *internally even* if the only vertices of odd degree in T are its leafs. Clearly, every path decomposition is a decomposition into internally even trees. The following is a natural weakening of Conjecture 1 for internally even trees.

Conjecture 7. *Every connected graph on* n *vertices admits a decomposition into internally even trees with size at most* $\lceil n/2 \rceil$*.*

Finally, one may weaken the acyclic condition on the elements of the decomposition as follows. We say that a decomposition D is a *d-decomposition* if every element of D is connected and has maximum degree d. The following conjecture generalizes Lovász's result [9].

Conjecture 8. *Let* $d \geq 2$ *be a positive integer. Then every connected graph on* n *vertices admits a* d*-decomposition with size at most* $\lceil (n-1)/d \rceil$*.*

References

[1] M. Bonamy and T. J. Perrett, *Gallai's path decomposition conjecture for graphs of small maximum degree*, Discrete Mathematics **342** (2019), no. 5, 1293–1299.

[2] A. Bondy, *Beautiful conjectures in graph theory*, European Journal of Combinatorics **37** (2014), 4–23.

[3] F. Botler, A. Jiménez, and M. Sambinelli, *Gallai's path decomposition conjecture for triangle-free planar graphs*, Discrete Mathematics **342** (2019), no. 5, 1403–1414.

[4] F. Botler, M. Sambinelli, R. S. Coelho, and O. Lee, *On Gallai's and Hajós' conjectures for graphs with treewidth at most 3*, arXiv preprint arXiv:1706.04334 (2017).

[5] F. R. K. Chung, *On partitions of graphs into trees*, Discrete Mathematics **23** (1978), no. 1, 23–30.

[6] N. Dean and M. Kouider, *Gallai's conjecture for disconnected graphs*, Discrete Mathematics **213** (2000), no. 1-3, 43–54. Selected topics in discrete mathematics (Warsaw, 1996).

[7] A. Donald, *An upper bound for the path number of a graph*, Journal of Graph Theory **4** (1980), no. 2, 189–201.

[8] G. Fan, *Path decompositions and gallai's conjecture*, Journal of Combinatorial Theory, Series B **93** (2005), no. 2, 117–125.

[9] L. Lovász, *On covering of graphs*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press New York (1968), 231–236.

[10] L. Pyber, *Covering the edges of a connected graph by paths*, Journal of Combinatorial Theory, Series B **66** (1996), no. 1, 152–159.

Fábio Botler
Programa de Engenharia de
Sistemas e Computação
Universidade Federal do Rio de Janeiro
Rio de Janeiro, Brazil
fbotler@cos.ufrj.br