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Optimal Edge Fault-Tolerant Embedding of a Star over a Cycle.

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Abstract

An embedding of a guest graph G over a host graph H is an injective map Φ from the vertices of G to the vertices of H and a mapping ρ, which associates every edge $e = \{x, y\}$ in G to a $\Phi(x)$-$\Phi(y)$ path $\rho(e)$ in H. Given an edge f in H, if ρ^{-1} is the set of those edges that cross f, i.e., $\{e : f \in \rho(e)\}$, then the cardinality of $\rho^{-1}(f)$ is the (edge) congestion $cong_\rho(f)$ of f. The length of $\rho(e)$ is called the dilatation $dil(e)$ of e. The sum of all the dilatations is the cost of the embedding. The removal of an edge f of H gives rise to a *surviving graph* $G_f = G \backslash \rho^{-1}(f)$. Given positive integers n and b, and a fixed vertex v of the n-cycle C_n, we are facing the problem of finding a guest graph G of n vertices with an embedding (Φ, ρ) over C_n of minimum cost, such that for any surviving graph G_f there is an embedding of the star $S_n = K_{1,n-1}$ over G_f that associates the center of the star to $\Phi^{-1}(v)$, with congestions not greater than b. This work presents the optimal cost as well as a family of optimal solutions.

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1 Introduction

A typical network design problem Internet-Service-Providers have to deal with is: given a set of "n" nodes to interconnect with known traffic demands among them, and a network (so called optical or physical) to which leasing point-to-point connections of certain capacity "b" between any two of those points, what is the minimum cost resilient data (a.k.a. logical) network capable of delivering the customers' traffic without logical congestion. The general problem is called *Free Routing Protection Multi-Overlay Resilient Network Design Problem*, or FRP-MORNDP for short. For further information of FRP-MORNDP please refer to [2]. Although FRP-MORNDP is NP-Hard in general, there are analytical solutions for some particular instances. Those with a cycle by physical network are of remarkable practical interest, and [1] presents a family of optimal solutions for each pair of integers (n, b), when there are demands between each pair of nodes and all of them are equal to 1. In addition, the present work tackles down ONE-TO-ALL-CYCLE-FRP-MORNDP, another family of FRP-MORNDP instances with cycles as physical topologies, although in this case all demands (still unitary) are from/to a unique hub node, so called center.

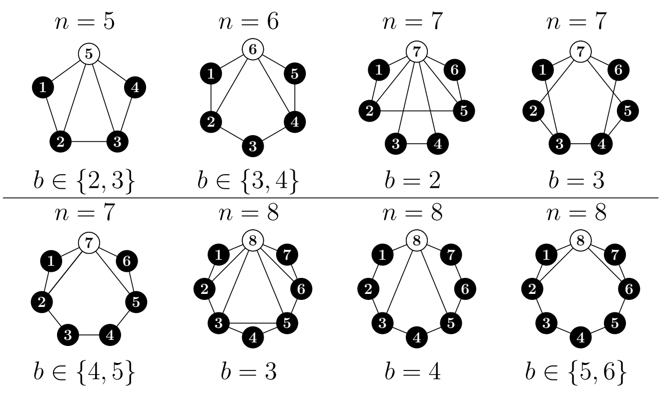

Figure 1: Numerical solutions for some instances of ONE-TO-ALL-CYCLE-FRP-MORNDP.

The graphs in Fig.-1 are computer-aided constructions of ONE-TO-ALL-CYCLE-FRP-MORNDP, found with IBM CPLEX up from the general MIP formulation of FRP-MORNDP. The present work not only confirms the optimality of those solutions, but also presents an optimal family of graphs for every (n, b).

2 Basic Definitions

Given $n \geq 3$, let $\mathbb{N}_n$ be the set $\{1, \ldots, n\}$, S_n=$(\mathbb{N}_n, \{\{i, n\} : i \in \mathbb{N}_{n-1}\})$ the star graph (*demands graph*) with n vertices and center n, and $C_n = (\mathbb{N}_n, \{\{n, 1\}, \{1, 2\}, \ldots, \{n-1, n\}\})$ the n-cycle. Given integers n and b, a *feasible solution* consists of a *logical graph* G^L=$(\mathbb{N}_n, E^L)$, and an embedding (Φ, ρ) of G^L over C_n, such that for each edge f in C_n there exists in turn an embedding (Ξ, σ_f) of S_n over $G^L_f = G^L \backslash \rho^{-1}(f)$, whose maximum congestion is b at most, i.e., $cong_{\sigma_f}(e) = |\sigma_f^{-1}(e)| \leq b$ for each edge e in G^L_f while $\Xi(n) = n$. Since the cardinalities of all vertex sets match, we assume that Φ and Ξ are the identity function. Hence, a solution is given by the pair (G^L, ρ). The *cost* of a ligthpath $\rho(e)$ is its length $|\rho(e)|$ or $dil(e)$, and the cost of a solution is $cost(G^L, \rho)$=$\sum_{e \in E^L} dil(e)$. It holds that $cost(G^L, \rho)$=$\sum_{f \in C_n} cong_\rho(f)$. The problem consists in finding $\mathfrak{c}_{n,b}$, the minimum $cost(G^L, \rho)$ over the set of all the feasible solutions. For example purposes, when (n, b)=$(7, 3)$ consider G^L as in Fig.-1 where ρ always follows the shortest path (shortest lightpaths), so: $dil(e)$=1 for e in $\{\{1, 7\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{6, 7\}\}$ while $dil(e)$=2 for e in $\{\{1, 3\}, \{2, 7\}, \{4, 6\}, \{5, 7\}\}$, from where $\mathfrak{c}_{7,3}$=13. Regarding physical congestions: $cong_\rho(f)$=2 for f in $\{\{1, 2\}, \{1, 7\}, \{2, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7\}\}$ and $cong_\rho(\{3, 4\})$=1, whose sum also adds up to $6 \times 2 + 1$=13=$\mathfrak{c}_{7,3}$. In addition the failure of $f = \{4, 5\}$ gives rise to $E(G^L_{\{4,5\}})$=$\{\{1, 3\}, \{1, 7\}, \{2, 3\}, \{2, 7\}, \{3, 4\}, \{5, 7\}, \{6, 7\}\}$. A possible arrangement of demands over $G^L_{\{4,5\}}$ (see Figure 2) could be: $\sigma_f(\{x, y\}) = x, y$ for $\{x, y\}$ in $\{\{1, 7\}, \{2, 7\}, \{5, 7\}, \{6, 7\}\}$, $\sigma_f(\{3, 7\}) = 3, 1, 7$, and $\sigma_f(\{4, 7\}) = 4, 3, 2, 7$, whose associated logical congestions $|\sigma_f^{-1}(e)|$ always are lower or equal to 2, therefore viable, since b=3.

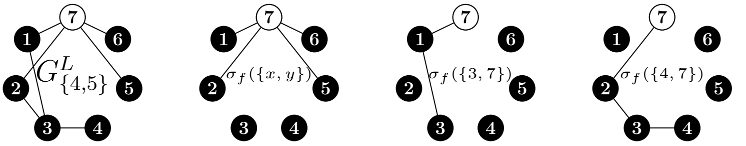

Figure 2: A possible arrangement of demands over $G^L_{\{4,5\}}$.

3 Lower Bounds

This section presents lower bounds for $\mathfrak{c}_{n,b}$, which allow us to prove the optimality of both: the graphs of Fig.-1 and the family $T_{n,b}$ introduced later on. It is easy to see that when $b \geq n-1$ the optimal solution is C_n, so $\mathfrak{c}_{n,b} = n$. Moreover, depending upon the parity of n it can be proven that $b \geq \{2,3\}$ is necessary for the existence of feasible solutions, for n odd or even. For (n,b) in general, let (G^L, ρ) be a solution and consider the failure of f=$\{n,1\}$ in C_n.

The existence of an embedding σ_f is equivalent to the feasibility of a flow problem for the network G^L_f, with the nodes in $\mathbb{N}_{n-1}$ injecting one unit of flow towards n (the sink), and b as the unique capacity. Because of the min-cut theorem every cut-set $\mathcal{A} = (V, V^c)$ of G^L_f must hold: $b|\mathcal{A}| \geq |V|$, being V the set of nodes in the partition not containing n. Particularly the condition must hold for those cut-sets $\mathcal{A}_{f_i} \subset \rho^{-1}(f_i)$, with $f_i = \{i, i+1\}$, i in $\mathbb{N}_{n-1}$, for which $|V| = i$. Thus, a feasible solution implies $i \leq b \cdot cong_\rho(f_i)$. It is inferred: $n-i-1 \leq b \cdot cong_\rho(f_i)$ with a similar reasoning for the failure $\{n-1, n\}$, so:

$$cong_\rho(f_i) \geq B_{n,b,i} = \max\left\{\left\lceil \frac{i}{b} \right\rceil, \left\lceil \frac{n-i-1}{b} \right\rceil\right\}, 0 \leq i \leq n-1. \quad (1)$$

The expression (1) assumes that $f_0 = \{n,1\}$. It is of relevance to keep in mind that $\lceil \frac{i}{b} \rceil$ comes paired with a cut-set $\mathcal{A}_{f_i}$ for the failure $\{n,1\}$, while $\lceil \frac{n-i-1}{b} \rceil$ arises from a failure $\{n-1, n\}$.

Theorem 1. *For all n and b*

$$\mathfrak{c}_{n,b} \geq \left(\sum_{i=0}^{n-1} B_{n,b,i}\right) + \begin{cases} 2 & n \equiv 2 \pmod{b}, \\ 0 & n \not\equiv 2 \pmod{b}. \end{cases} \tag{2}$$

■

Proof. By (1) and since $\mathfrak{c}_{n,b} = \sum_{f \in C_n} cong_\rho(f)$, we only need to prove the inequality for $n \equiv 2 \pmod{b}$. Let k be so that $n = kb + 2$. It holds that $cong_\rho(f_0) \geq B_{n,b,0} = k+1$. Assume that both $cong_\rho(f_0)$ and $cong_\rho(f_1)$ match their bound, which is $B_{n,b,1} = k$ for the later. These bounds come paired with a cut-set induced over the survival graph $G^L_{f_{n-1}}$, so its links cannot traverse n. We can assert that the $k+1$ lightpaths that are passing over the physical edge f_0 are coming from n; and one of them should be copying the physical edge f_0, otherwise, the congestion of f_1 will be $k+1$, which violates the hypothesis.

Furthermore, none of the lightpaths that congest f_1 can be mapped down from a logical link $\{1, j\}$, because they must have origin in n (not 1). As a result, the only connection of the logical node 1 in $G^L_{f_{n-1}}$ is with the logical node n. It is a premise of the problem that G^L is resilient so it should be 2-edge-connected, and the logical node 1 must have at least another link in G^L. The only possibility for such links are not in $G^L_{f_{n-1}}$ is that their lightpaths pass through n, so all lightpaths having an endpoint in 1 use f_0. Hence, a failure in f_0 disconnects 1 in $G^L_{f_0}$ which is absurd. Therefore, either $cong_\rho(f_0)$ or $cong_\rho(f_1)$ cannot match their bound, and should be at least one unit higher. A symmetric argument allows to prove the same for $cong_\rho(f_{n-1})$ or $cong_\rho(f_{n-2})$. ■

It is immediate to compute the costs of the solutions in Fig.-1 to verify they always match bound (2). This observation closes the optimality of those solutions assuming their feasibility, which in turn is given by the set of constraints, detailed in [2], with which was fed the MIP optimizer used to find them.

4 A Family of Optimal Networks

We present here a family $(T_{n,b}, \rho_{n,b})$ of feasible solutions that match the lower bound of Theorem 1, thus optimal. In order to define graphs $T_{n,b}$ we introduce three embeddings over the n-path with vertex set $V = \mathbb{N}_n \cup \{0\}$. Let $T^+_{n,b}$, $T^-_{n,b}$ and $T^*_{n,b}$ be guest graphs with edges defined as follows, where $i \sim j$ iff $\{i, j\}$ is an edge of the corresponding graph.

In $T^+_{n,b}$ we have $i \sim \begin{cases} n & i \equiv 0 \pmod b, \\ i+1 & i \not\equiv 0 \pmod b, \end{cases}$ for all $i \in \mathbb{N}_{n-1}$.

In $T^-_{n,b}$ we have $i \sim j$ iff $n - i \sim n - j$ is in $T^+_{n,b}$, see Fig.-3 a mode of reference.

The edges of $T^*_{n,b}$ are defined as follows, where we call an edge *short* (respectively *long*) if its dilatation is smaller (respectively greater or equal) than $\lceil n/2 \rceil$:

- if $i \sim i+1$ is in $T^+_{n,b}$ or $T^-_{n,b}$, then $i \sim i+1$ is in $T^*_{n,b}$.
- if $e = \{n, j\}$ is a short edge of $T^+_{n,b}$, then
  - if $j \sim j+1$ is not in $T^-_{n,b}$, then $n \sim j$ is in $T^*_{n,b}$.
  - if $j \sim j+1$ is in $T^-_{n,b}$, then $n \sim j+1$ and $j+1 \sim j$ are in $T^*_{n,b}$. If $n \equiv 2 \pmod b$, we also add $n \sim j$ in $T^*_{n,b}$.
- if $e = \{0, j\}$ is a short edge of $T^-_{n,b}$, then
  - if $j-1 \sim j$ is not in $T^+_{n,b}$, then $0 \sim j$ is in $T^*_{n,b}$.
  - if $j-1 \sim j$ is in $T^+_{n,b}$, then $0 \sim j-1$ and $j-1 \sim j$ are in $T^*_{n,b}$. If $n \equiv 2 \pmod b$, we also add $0 \sim j$ in $T^*_{n,b}$.
- if $e^- = \{0, j\}$ and $e^+ = \{n, n-j\}$ are long edges of $T^-_{n,b}$ and $T^+_{n,b}$ respectively, then we add edges $\{0, n-j\}, \{n-j, j\}$ and $\{n, j\}$ to $T^*_{n,b}$.

Finally, we are in position to define $T_{n,b}$, simply by identifying in $T^*_{n,b}$ vertex 0 with vertex n, and keeping the (unique) embedding $\rho_{n,b}$ as it was in the n-path, which, of course, might not be the shortest in C_n.

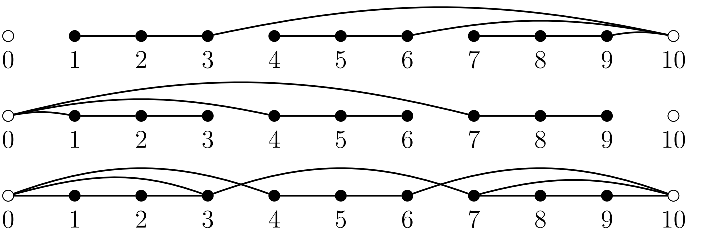

Figure 3: $T^+_{10,3}$, $T^-_{10,3}$ and $T^*_{10,3}$.

Proposition 1. *$(T_{n,b}, \rho_{n,b})$ is feasible for any n and b.*

Proof. First of all, notice that *if (G^L, ρ) is feasible, and $\rho(\{x,y\})$ is a lightpath passing through vertex z such that neither $\{x,z\}$ nor $\{z,y\}$ are in G^L, then $(G^L - \{x,y\} + \{x,z\} + \{z,y\}, \rho')$ is feasible*, with:

$$\rho'(e) = \begin{cases} \rho(\{x,y\})[x:z] & e = \{x,z\}, \\ \rho(\{x,y\})[z:y] & e = \{z,y\}, \\ \rho(e) & \text{otherwise}, \end{cases}$$

where $\rho(\{x,y\})[a:b]$ is the subpath from vertex a to vertex b.

Now, let $\mathbb{Z}_{i,j} = \{k \in \mathbb{Z} : i \leq k \leq j\}$. By construction, the tree $T^+_{n,b}$ with its corresponding embedding over $P_{n+1} = (\mathbb{Z}_{0,n}, \{\{i,j\} : j = i+1, i \in \mathbb{Z}_{0,n-1}\})$, supports an embedding of $S^+_i = (\mathbb{Z}_{i+1,n}, \{\{j,n\} : j \in \mathbb{Z}_{i+1,n-1}\})$ over $(T^+_{n,b})_{f_i}$, a kind of rightwards embedding. Symmetrically, there is a leftwards embedding of $S^-_i = (\mathbb{Z}_{0,i}, \{\{0,j\} : j \in \mathbb{Z}_{1,i}\})$ over $(T^-_{n,b})_{f_i}$. But, if an edge $\{x,y\}$ in $T^+_{n,b}$ or in $T^-_{n,b}$, then either $\{x,y\}$ is in $T^*_{n,b}$ or exists z such that edges $\{x,z\}$ and $\{z,x\}$ are in $T^*_{n,b}$. Therefore, by the observation made at the beginning of the proof, when a physical edge f_i fails, the surviving logical graph $(T^*_{n,b})_{f_i}$ is able to support an embedding of both S^+_i and S^-_i, so $(T^*_{n,b})_{f_i}$ can support the full embedding of S_n. So do $(T_{n,b})_{f_i}$. ■

Corollary 1. *$(T_{n,b}, \rho_{n,b})$ is optimal for any n and b.*

Proof. By Proposition 1, it is enough to prove that the cost of $(T^*_{n,b}, \rho_{n,b})$ reaches the bound in Theorem 1. Let $std(G, H)$ be the standard embedding of G over H: that where lightpaths always follow the minimum path. Notice that the congestion for $(T^*_{n,b}, \rho_{n,b})$ is as in $std(T^*_{n,b}, P_n)$, since there are no edges joining 0 with n in $T^*_{n,b}$. Thus, we must prove that $cost(std(T^*_{n,b}, P_n))$ reaches the bound in Theorem 1. On the other hand, by definition of $std(T^*_{n,b}, P_n)$, the congestion in an edge of P_n is the maximum between the corresponding congestions in $std(T^+_{n,b}, P_n)$ and $std(T^-_{n,b}, P_n)$ except when $n \equiv 2 \pmod{b}$. In that case, we have to add logical edges $\{1, 2\}$ and $\{n-2, n-1\}$ with the standard routing, what increases in 2 the total cost of the embedding. But the congestions in $std(T^+_{n,b}, P_n)$ and $std(T^-_{n,b}, P_n)$ are $\lceil i/b \rceil$ and $\lceil (n-i-1)/b \rceil$ respectively, so the congestion for $std(T^*_{n,b}, P_n)$ also verifies the formula in Theorem 1 and what we obtain is the optimal cost. ■

5 Conclusions and Future Work

This work presents an algorithm of polynomial complexity to generate solutions for a subfamily within an NP-Hard problem. Thus, it constitutes yet another example of how particular instances of NP-Hard problems, can tear down the general complexity till the point of making the problem analytically tractable. It is worth pointing out that often along this work, different solutions were found for the same instance, which evidences the non-unicity of the problem ONE-TO-ALL-CYCLE-FRP-MORNDP. Some of these solutions were asymmetrical.

This article elaborates over those solutions simplest to express, whose physical mappings not always match the shortest paths. However, till now we did find alternative constructions with hop-optimal mapping, so we are confident of finding such a general family in the near future. Finally, up from the rule that most real world optical networks are in fact composition of optical fiber rings (i.e. cycles), a promising line of work we are upon, explores the usage of suboptimal constructions to synthesize solutions to real world instances.

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