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On the Helly Number in P_3-Convexity in Graphs

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Abstract

A subset of vertices $S \subseteq V(G)$ of a graph G is P_3-*convex* if all vertices with at least two neighbors on S lie on S. The *Helly number* is the least integer k such that any subfamily of k-intersecting P_3-convex sets of G contains a common vertex. In this work, we determine the Helly number for path graphs, cycles, and threshold graphs. We show that complete k-partite graphs satisfies the Helly property and we present bounds for the Helly number of a graph in the P_3-convexity.

1 Introduction

Given a finite set V, a family $\mathcal{C}$ of subsets of V is a *convexity* in V if:

(1) The empty set and V belong to $\mathcal{C}$.

(2) $\mathcal{C}$ is closed under intersection.

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The subsets of V in $\mathcal{C}$ are called *convex sets*.

In this work, we consider a convexity in graphs known as P_3-*convexity* [3]. Let G be a graph and S a set of vertices of $V(G)$, S is P_3-*convex* if every vertex with at least two neighbors on S belongs to S. The *closed interval* between two vertices u and v is the set $I_{P_3}[u,v]$ (or $I[u,v]$, if there is no ambiguity) that contains u and v and all vertices of paths of length two between the pair of vertices u, v, and the *open interval* $I(u,v) = I[u,v] \setminus \{u,v\}$. The set $\langle S \rangle$ is the smallest convex set of G containing S and is called the *convex hull* of S in G [6].

The *core* of a family of sets $\mathcal{C}$ is defined as $core(\mathcal{C}) = E_1 \cap E_2 \cap \ldots \cap E_m$ where $E_i \subseteq V(G)$, for $i = 1, \ldots, m$. A family of sets is k-*intersecting* if each k of their sets has a non-empty core. A family $\mathcal{C}$ is k-*Helly* when every k-intersecting subfamily of $\mathcal{C}$ has a non-empty core. The *Helly number* of $\mathcal{C}$ is the smallest integer p for which $\mathcal{C}$ is p-Helly. When $p = 2$, we say that $\mathcal{C}$ satisfies the *Helly property* [2]. The Helly property got its name thanks to the mathematician Eduard Helly [5]. We denote by $\mathcal{M}_G$ the set with all convex sets in the graph G and denote by $G[A]$ a subgraph of G induced by A, where $A \subseteq V(G)$.

In this paper we study the Helly number in P_3-convexity in graphs, denoted by $h(G)$.

The next Theorem by Berge and Duchet [1] and the Lemma by Berge, Duchet and Calder [4] will be very useful in our proofs.

Theorem 1 (Berge and Duchet [1])**.** *A family of sets $\mathcal{C}$ is k-Helly if and only if for every set $A \subseteq V(\mathcal{C})$ with $|A| = k+1$, the intersection of the sets S_j, with $|S_j \cap A| \geq k$, is non-empty.*

Lemma 2 (Berge, Duchet and Calder [4])**.** *In any convexity, the Helly number is the least integer k such that every $(k+1)$-element set $S \subseteq V$ has the property $\bigcap_{a \in S} \langle S \setminus \{a\} \rangle \neq \emptyset$.*

In the graph shown in Figure 1, we have two different sets formed by black vertices. The first one is a convex set and the second one is not a convex set.

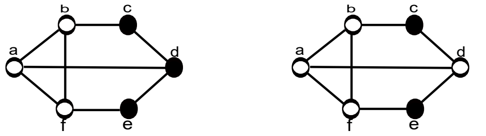

Figure 1: Example and counterexample of P_3-convex set.

2 The Helly number in P_3-convexity

In this section, we present a theorem about universal vertices, and results to some graph classes. In all proofs, the trivial graph is not considered since its Helly number is equal to one. Also we only consider connected graphs.

Lemma 3. *Let G be a graph. If all non-empty convex sets S_i of $\mathcal{M}_G$ are $S_i = \{v_i\}$ (vertices of G), $S_i = \{u, v\}$ (induces an edge in G) or $S_i = V(G)$, then $h(G) = 2$.*

Proof. If G is isomorphic to K_3, it is easy to see that $h(G) = 2$.

Assume G is not isomorphic to K_3. Suppose for contradiction that $h(G) \geq 3$. Then there is $A \subseteq V(G)$, $A = \{a_1, \ldots, a_k\}$, $|A| \geq 3$ such that $\bigcap_{a_i \in A} \langle A \setminus \{a_i\} \rangle = \emptyset$. We can assume that $S_j = \langle A \setminus \{a_j\} \rangle \neq V(G)$ for every $a_j \in A$, because if $S_j = V(G)$, $\bigcap_{a_i \in A'} \langle A' \setminus \{a_i\} \rangle = \emptyset$ for $A' = A \setminus \{a_j\}$ then S_j is an edge of G. This implies that $|A| = 3$ and $G[A]$ is isomorphic to K_3 and A is a convex set, a contradiction.

Then, $h(G) = 2$. ■

Theorem 4. *If a graph G contains a universal vertex, then $h(G) = 2$.*

Proof. Let G be a graph and $v_1, \ldots, v_k$, $k \geq 1$, universal vertices in G.

Let A be a subset of $V(G)$, where $|A| = 3$, and $\mathcal{C}$ be the family of the convex sets containing at least two vertices of A. This implies that each $S_i \in \mathcal{C}$ has at least two distinct vertices, so every convex set S_i contains all the universal vertices of G, hence they belong to the intersection of the convex sets S_i. Then, by Theorem 1, G is 2-Helly. Since G is not trivial graph, the result follows.

Hence, $h(G) = 2$. ■

A graph is a *threshold graph* if there is a real number r and for every vertex v there is a real weight a_v such that: vw is an edge if and only if $a_v + a_w \geq r$.

Corollary 5. *If G is a threshold graph, then $h(G) = 2$.*

This result is a direct consequence of Theorem 4.

2.1 Path Graphs

The Helly number in a path P_n is directly related to the order of the graph.

Theorem 6. *If $G = P_n$ is a path, then $h(P_n) = \lceil \frac{2}{3}n \rceil$.*

Proof. Let $G = P_n$ be a graph with n vertices. We will label the vertices of G as follows: $w_1, w_2, \ldots, w_n$, where $d(w_1) = d(w_n) = 1$ and $d(w_i) = 2$, for $i = 2, \ldots, n-1$; thus, w_i is adjacent to the vertices w_{i-1} and w_{i+1}.

Taking $V_k = \{v_1, v_2, v_3, \ldots, v_k\}$, where $v_j = w_i$, for $j = \lfloor \frac{2}{3}i \rfloor + 1$, and $i \not\equiv 0 \pmod 3$. Thus, given any three vertices in V_k, such vertices do not induce a subgraph P_3, and since in every P_3 induced in G there are exactly two vertices of V_k, then V_k has maximum cardinality.

By Lemma 2 there exists in G a family $\mathcal{C}_G$ of convex sets, $(k-1)$-intersecting, with the empty core, i.e., by Theorem 1, the family $\mathcal{C}_G = \{S_1, \ldots, S_k\}$ is such that:

$S_1 \supseteq \{v_2, v_3, \ldots, v_k\} \not\supseteq \{v_1\}$, $S_2 \supseteq \{v_1, v_3, \ldots, v_k\} \not\supseteq \{v_2\}$, $\ldots$, $S_k \supseteq \{v_1, v_2, v_3 \ldots, v_k\} \not\supseteq \{v_k\}$ and $\bigcap_{j=1}^{k} S_j = \emptyset$.

It is easy to see that the inclusion of any other vertex of $V(G)$ in V_k would imply, for some p, in the vertex sequence v_{p-1}, v_p and v_{p+1}, that is, it would imply $S_p \supseteq \{v_p\}$, a contradiction. In general, if every induced subgraph P_3 in G has exactly two vertices of V_k, we have that $|V_k|$ is maximum. Thus, $h(G) = k$.

In fact, for every three vertices in $V(G)$, exactly two vertices belong to V_k, then we considered three cases as a function of the number of vertices of G. Then we have the cases $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$.

Case 1: $n \equiv 0 \pmod 3$.

Clearly, for every three vertices w_p, w_{p+1} and w_{p+2} in G, with $p \equiv 1 \pmod 3$, we have exactly the first two of these vertices in V_k (w_p and w_{p+1}), and in particular, $v_k = w_{n-1}$, i.e., $k = \frac{2}{3}n$.

Case 2: $n \equiv 1 \pmod 3$.

In this case, w_1 and w_n, vertices of degree one, belong to V_k and $w_{n-1} \notin V_k$. Thus, $k = \frac{2}{3}(n-1) + 1 = \frac{2}{3}n + \frac{1}{3}$, $n \in \mathbb{N}$, then $k = \lceil \frac{2}{3}n \rceil$.

Case 3: $n \equiv 2 \pmod 3$.

In this case, w_{n-1} and w_n belong to V_k. Then, $k = \frac{2}{3}(n-2)+2 = \frac{2}{3}n+\frac{2}{3}$, since $k, n \in \mathbb{N}$, then $k = \lceil \frac{2}{3}n \rceil$.

Since for $n \equiv 0 \pmod 3$ we have $\frac{2}{3}n = \lceil \frac{2}{3}n \rceil$, then $k = \lceil \frac{2}{3}n \rceil$, for $n \in \mathbb{N}$.

Hence, $h(P_n) = \lceil \frac{2}{3}n \rceil$. ■

2.2 Cycles

Theorem 7. *If $G = C_n$ is a cycle, then $h(C_n) = \lfloor \frac{2}{3}n \rfloor$.*

Proof. Let G be a cycle C_n with n vertices. We will label the n vertices as w_1, w_2, $\ldots$, w_n, where w_1 and w_n are adjacent vertices and for each vertex w_i, for $i = 2, \ldots, n-1$; w_i is adjacent to the vertices w_{i-1} and w_{i+1}.

For the proof we consider the cases: $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$.

Case 1: $n \equiv 0 \pmod 3$

Take $V_k = \{v_1, v_2, \ldots, v_k\}$, where $v_j = w_i$, for $j = \lfloor \frac{2}{3}i \rfloor + 1$, and $i \not\equiv 0 \pmod 3$. Thus, V_k has maximum cardinality, since every induced subgraph P_3 in G has exactly two vertices of V_k. Then, by Lemma 2, there exists a $(k-1)$-intersecting family $\mathcal{C}_G$ containing k convex sets with empty core, which implies, by Theorem 1, that the family $\mathcal{C}_G = \{S_1, \ldots, S_k\}$ is such that:

$S_1 \supseteq \{v_2, v_3, \ldots, v_k\} \not\supseteq \{v_1\}$, $S_2 \supseteq \{v_1, v_3, \ldots, v_k\} \not\supseteq \{v_2\}$, $\ldots$, $S_k \supseteq \{v_1, v_2, v_3 \ldots, v_k\} \not\supseteq \{v_k\}$ and $\bigcap_{j=1}^{k} S_j = \emptyset$.

Hence, $h(G) \geq k$. Since $|V_k|$ is maximum, then $h(G) = k$.

Thus, for every three vertices w_p, w_{p+1} and w_{p+2} in $V(G)$, with $p \equiv 1 \pmod 3$, we will have exactly the first and two of these vertices in V_k (w_p and w_{p+1}), and in particular, $v_k = w_{n-1}$ and $w_n \notin V_k$, i.e., $k = \frac{2}{3}n$.

Case 2: $n \equiv 1 \pmod 3$

Since $n \equiv 1 \pmod 3$, $w_{n-1} \notin V_k$ and $w_n \in V_k$, then $G[\{w_n, w_1, w_2\}] = P_3$, a contradiction. Therefore, w_n cannot belong to V_k. Take $V_k = \{v_1, v_2, \ldots, v_k\}$, where $v_j = w_i$, for $j = \lfloor \frac{2}{3}i \rfloor + 1$, $i \not\equiv 0 \pmod 3$ and $i < n$. In this way, V_k has maximum cardinality, since every induced subgraph P_3 in $G \setminus \{w_n\}$ has exactly two vertices of V_k and for every vertex $w_t \in V(G) \setminus V_k$, there will exist in the set $V_k \cup \{w_t\}$ a P_3 induced subgraph, then $|V_k|$ has maximum cardinality.

By Lemma 2 there exists in G a family $\mathcal{C}_G$ of k convex sets, $(k-1)$-intersecting, with the empty core, then $h(G) \geq k$. Since $|V_k|$ is maximum, then $h(G) = k$.

Thus, $k = \frac{2}{3}(n-1) = \frac{2}{3}n - \frac{2}{3}$, and since $k, n \in \mathbb{N}$, we have that $k = \lfloor \frac{2}{3}n \rfloor$.

Case 3: $n \equiv 2 \pmod 3$

Since $w_1, w_2 \in V_k$ and $w_{n-1} \in V_k$, if $w_n \in V_k$, then $G[\{w_{n-1}, w_n, w_1, w_2\}] = P_4$, a contradiction. Hence, w_n cannot belong to V_k. Take $V_k = \{v_1, v_2, \ldots, v_k\}$, where $v_j = w_i$, for $j = \lfloor \frac{2}{3}i \rfloor + 1$, $i \not\equiv 0 \pmod 3$ and $i < n$. In this way, V_k has maximum cardinality, since every induced subgraph P_3 in $G \setminus \{w_n\}$ has exactly two vertices of V_k and for every vertex $w_t \in V(G) \setminus V_k$, there will exist in the set $V_k \cup \{w_t\}$ a subgraph P_3 induced, then $|V_k|$ has maximum cardinality.

By Lemma 2, there exists in G a family $\mathcal{C}_G$ of k convex sets, $(k-1)$-intersecting, with the empty core, then $h(G) \geq k$. Since $|V_k|$ is maximum, we have that $h(G) = k$.

Thus, $k = \frac{2}{3}(n-2)+1 = \frac{2}{3}n - \frac{1}{3}$, such as $k, n \in \mathbb{N}$, we have that $k = \lfloor \frac{2}{3}n \rfloor$.

Since for $n \equiv 0 \pmod 3$, we have that $\frac{2}{3}n = \lfloor \frac{2}{3}n \rfloor$, thus $k = \lfloor \frac{2}{3}n \rfloor$, for $n \in \mathbb{N}$.

Hence, $h(C_n) = \lfloor \frac{2}{3}n \rfloor$. ■

2.3 Complete k-partite graphs

A graph G is k-partite if $V(G)$ can be partitioned in k independent sets and every possible edge that could connect vertices in different subsets is part of the graph.

Theorem 8. *If G is a complete k-partite graph, then $h(G) = 2$.*

Proof. Let G be a complete k-partite graph and S a proper subset of $V(G)$ with at least 2 vertices. It is clear that if $k = 2$, then S is a convex set if and only if S is an edge, and if $k \neq 2$, then S is not a convex set, hence, by Lemma 3, the result follows. ■

Corollary 9. *If $G = K_n$ is a complete graph, then $h(K_n) = 2$.*

Since complete graphs K_n are also complete k-partite graphs, such that each independent set contains only one vertex, then $h(K_n) = 2$.

3 Bounds

The bounds in Theorems 10 and 11 hold for all graph convexities [2].

Theorem 10 ([2])**.** *Let G be a graph. Then $h(G) \leq |V(G)|$.*

If S is a convex set in a graph G, $h(G[S])$ is a lower bound of $h(G)$.

Theorem 11 ([2])**.** *Let S be a convex set of a graph G. Then $h(G[S]) \leq h(G)$.*

Definition 12. *A (k, l)-stable set $\mathcal{S}$ is a set of vertices in a graph where:*

a) The elements of $\mathcal{S}$ induces maximal cliques of the size less or equal to k.

b) If u and v are two vertices in distinct cliques of $\mathcal{S}$, $d(u, v) \geq l$.

Corollary 13. *If $\mathcal{S}$ is a maximum $(2, 3)$-stable set of G, then, $h(G) \geq |\mathcal{S}|$.*

Proof. Let $\mathcal{S}$ be a maximum $(2, 3)$-stable set with k vertices, such that $\mathcal{S} = \{v_1, v_2, \ldots, v_k\}$.

The family of the convex sets $\mathcal{C}_G = \{S_1, S_2, \ldots, S_k\}$:

$S_1 = \{v_2, v_3, \ldots, v_k\}$, $S_2 = \{v_1, v_3, \ldots, v_k\}$, $\ldots$, $S_{k-1} = \{v_1, \ldots, v_{k-2}, v_k\}$, $S_k = \{v_1, v_2, \ldots, v_{k-1}\}$ is k-intersecting with empty core, i.e., $\bigcap_{i=1}^{k} S_i = \emptyset$.

Thus, G is not $(k-1)$-Helly. Since $|\mathcal{S}|$ is maximum, the result follows. Hence, $h(G) \geq |\mathcal{S}|$. ∎

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