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End vertices in containment interval graphs

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Abstract

An interval containment model of a graph maps vertices into intervals of a line in such a way that two vertices are adjacent if and only if the corresponding intervals are comparable under the inclusion relation. Graphs admitting an interval containment model are called containment interval graphs or CI graphs for short. A vertex v of a CI graph G is an end-vertex if there is an interval containment model of G in which the left endpoint of the interval corresponding to v is less than all other endpoints. In this work, we present a characterization of end-vertices in terms of forbidden induced subgraphs.

1 Introduction and previous results

A graph G is a *containment graph of intervals* (or CI for short) if there is a collection of intervals of the real line $(I_w)_{w\in V(G)}$ satisfying that uw is an edge of G if and only if $I_u \subset I_w$ or $I_w \subset I_u$; the collection is called

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a *CI model* of G. Without loss of generality, it can be assumed that the intervals of a CI model are closed, with positive length and no two have

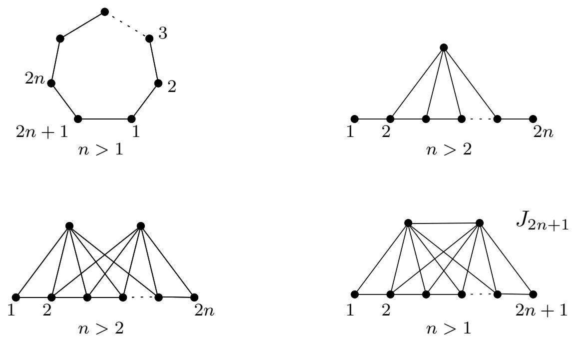

Figure 1: These graphs, together with the complement of the graphs in Figure 2, constitute a minimal family of forbidden induced subgraphs for comparability graphs.

a same endpoint [5, 8]. Let l_w and r_w denote, respectively, the left and the right endpoint of the interval I_w. A vertex v is an *end-vertex* of G if there exists a CI model of G where $l_v < l_w$ for every $w \in V(G) - \{v\}$.

In this paper, we describe end-vertices of CI graphs using a self - complementary family of forbidden induced subgraphs. In addition, in Section 3, homogeneously representable CI graphs (all vertices are end-vertices) are characterized by a simple finite family of forbidden induced subgraphs.

A *transitive orientation* $\overrightarrow{E}$ of a graph G is an assignment of one of the two possible directions, $\overrightarrow{uw}$ or $\overrightarrow{wu}$, to each edge $uw \in E(G)$, such that if $\overrightarrow{uv} \in \overrightarrow{E}$ and $\overrightarrow{vw} \in \overrightarrow{E}$ then $\overrightarrow{uw} \in \overrightarrow{E}$. Graphs admitting a transitive orientation are called *comparability graphs* [5, 8].

Next theorem, due to Gallai, provides a characterization of comparability graphs in terms of forbidden induced subgraphs. The *complement* of a graph G is the graph $\overline{G} = \big(V(G), \overline{E}(G)\big)$ such that $uv \in \overline{E}(G)$ if and only if $uv \notin E(G)$.

Theorem 1 ([3]). *A graph G is a comparability graph if and only if G contains none of the graphs depicted in Figure 1, nor the complement of* *those in Figure 2, as induced subgraphs.*

A vertex v of a comparability graph G is a *sink* (respectively, a *source*) if there exists a transitive orientation $\overrightarrow{E}$ of G such that $\overrightarrow{vw} \notin \overrightarrow{E}$ ($\overrightarrow{wv} \notin \overrightarrow{E}$) for all $w \in V(G)$. Olariu and Gimbel, almost simultaneously, obtained a characterization of sinks in terms of forbidden induced subgraphs as follows.

Theorem 2 ([4, 6]). *Let G be a comparability graph. A vertex v of G is a sink if and only if G contains none of the graphs A, B, C, D_{2k+1} with $k \geq 2$, $\overline{E_n}$ with $n \geq 3$, in Figure 3, with v as the designated vertex.*

CI graphs have been widely studied and characterized in different ways. For instance, they are co-comparability graphs, and in [1, 8] is showed the relationship between CI graphs and the partially ordered sets (or posets) of dimension at most 2. Given a poset $\mathbf{P} = (V, \leq)$, the *comparability graph* of $\mathbf{P}$ is $G_{\mathbf{P}} = (V, E)$ with $E = \{uv : u < v \text{ or } v < u \text{ in } \mathbf{P}\}$. The *dimension* of a poset $\mathbf{P}$ is the minimal number of linear extension of $\mathbf{P}$ whose intersection is $\mathbf{P}$ [1]. The CI graphs are exactly the comparability graphs of posets with dimension at most 2.

Theorem 3 ([1]). *A graph G is CI if and only if G and its complement $\overline{G}$ are comparability graphs.*

In addition, the class of CI graphs is equivalent to the class of permutation graphs [2, 7].

2 Characterization of end-vertices of CI graphs

In the proof of the main Theorem 5, we will use the following two results.

Remark 1 ([1]). *If $\overrightarrow{E}$ is a transitive orientation of a CI graph G then there exists a CI model $(I_w)_{w \in V(G)}$ of G compatible with $\overrightarrow{E}$; ie. for each pair of vertices $u, w \in V(G)$, $I_u \subset I_w$ if and only if $\overrightarrow{uw} \in \overrightarrow{E}$.*

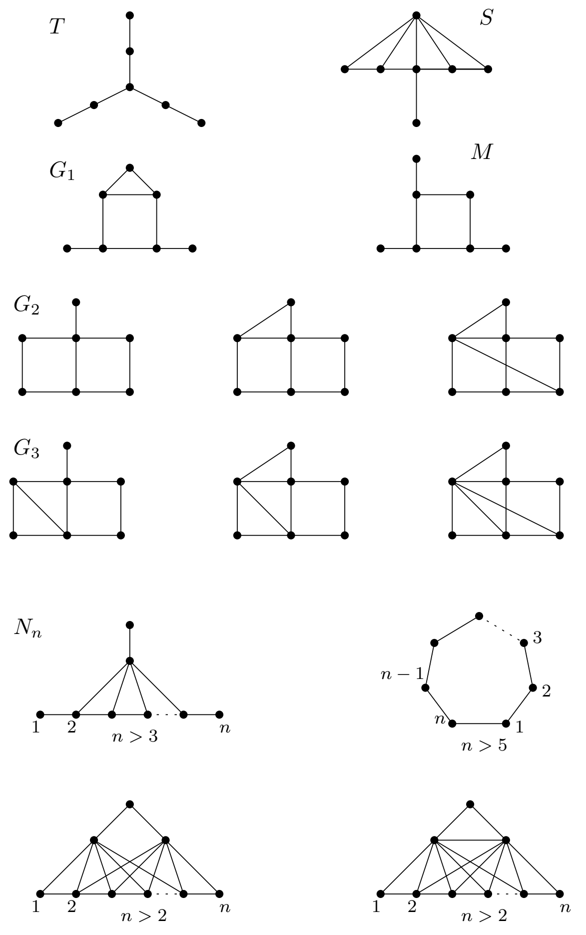

Figure 2: The complement of these graphs, together with the graphs in Figure 1, constitute a minimal family of forbidden induced subgraphs for comparability graphs.

Given a CI model of a graph G, the *right set* R_v of a vertex $v \in V(G)$ is $\{w : l_w < r_v < r_w\}$; and the *left set* L_v is $\{w : l_w < l_v < r_w\}$. Clearly, if $w \in R_v \cap L_v$ then $I_v \subset I_w$, and, consequently, w is adjacent to v.

Lemma 4. *Let $(I_{v_i})_{0\leq i\leq k}$ be a CI model of a chordless path $P = [v_0v_1 \dots v_k]$ with $k \geq 2$. If $I_{v_1} \subset I_{v_0}$ then exactly one of the following statements holds: (i) $v_2 \in R_{v_0}$ and $r_{v_0} < r_{v_i}$ for $2 \leq i \leq k$. (ii) $v_2 \in L_{v_0}$ and $l_{v_i} < l_{v_0}$ for $2 \leq i \leq k$.*

Proof. We proceed by induction. The proposition holds trivially when $k = 2$. Let $k > 2$ and assume $I_{v_1} \subset I_{v_0}$. Since v_2 is adjacent to v_1 and nonadjacent to v_0, we have that $v_2 \in R_{v_0} \setminus L_{v_0}$ or $v_2 \in L_{v_0} \setminus R_{v_0}$.

Assume, w.l.g., that $v_2 \in R_{v_0}$. Thus, $r_{v_0} < r_{v_i}$ for $2 \leq i \leq k-1$. To complete the proof, we will show that $r_{v_0} < r_{v_k}$.

Since v_k is adjacent to v_{k-1}, it follows that $I_{v_{k-1}} \subset I_{v_k}$ or $I_{v_k} \subset I_{v_{k-1}}$. In the former case, it is clear that $r_{v_0} < r_{v_{k-1}} < r_{v_k}$. Thus, assume $I_{v_k} \subset I_{v_{k-1}}$; and, in order to derive a contradiction, suppose that $r_{v_k} < r_{v_0}$.

Since $r_{v_0} < r_{v_{k-1}}$ and v_{k-1} is not adjacent to v_0, we have that $l_{v_0} < l_{v_{k-1}}$. Therefore $l_{v_0} < l_{v_{k-1}} < l_{v_k}$, and so $I_{v_k} \subset I_{v_0}$, which contradicts the fact that v_k and v_0 are nonadjacent. ■

Theorem 5. *Let G be a connected CI graph. A vertex z is an end-vertex of G if and only if G contains none of the graphs A, $\overline{A}$, B, $\overline{B}$, C, $\overline{C}$, D_{2k+1}, $\overline{D_{2k+1}}$, E_3, E_{2k}, $\overline{E_{2k}}$, for $k \geq 2$, as induced subgraphs with z as the vertex highlighted in Figure 3.*

Proof. Let $(I_w)_{w\in V(G)}$ be a CI model of G with $l_z < l_w$ for all $w \in V(G) - \{z\}$. Let $\overrightarrow{E}$ be the orientation of G obtained by orienting u to v ($\overrightarrow{uv}$) whenever $I_u \subset I_v$. Recall that two vertices are adjacent if and only if the interval corresponding to one of them is contained in the interval corresponding to the other. Clearly, $\overrightarrow{E}$ is a transitive orientation of G and z is a sink; thus, by Theorem 2, G does not contain the graphs A, B, C, D_{2k+1} with $k \geq 2$ and $\overline{E_n}$ with $n \geq 3$ as induced subgraph with z as the designated vertex. Observe that $\overline{E_3} = E_3$.

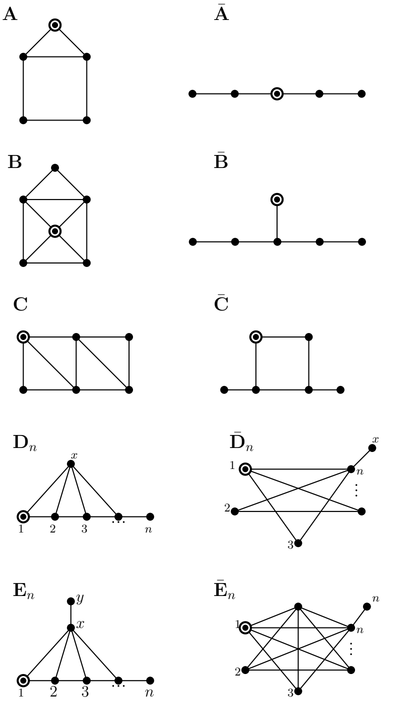

Figure 3: The highlighted vertices in the graphs A, B, C, D_{2k+1} with $k \geq 2$, $\overline{E_n}$ with $n \geq 3$, cannot be a sink of a comparability graph. The highlighted vertices in the graphs A, $\overline{A}$, B, $\overline{B}$, C, $\overline{C}$, D_{2k+1}, $\overline{D_{2k+1}}$, E_3, E_{2k}, $\overline{E_{2k}}$, for $k \geq 2$, cannot be an end-vertex of a CI graph.

Let H be any induced subgraph of G containing vertex z. Denote by H' the graph H plus a vertex z' adjacent to z, that is, $H' = (V(H) \cup z', E(H) \cup z'z)$ and denote by H'' the graph H' plus a vertex z'' adjacent to z', that is, $H'' = (V(H'') \cup z'', E(H) \cup z''z')$. We claim that H' and H'' are CI graphs .

Indeed, CI models of H' and H'' can be obtained by adding to $(I_w)_{w \in V(H)}$ the intervals $I_{z'} = [l_z + \epsilon, l_z + 2\epsilon]$ and $I_{z''} = [l_z - \epsilon, l_z + 3\epsilon]$ with ϵ small enough.

Now, in order to derive a contradiction, assume that H is any one of the graphs $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D_{2k+1}}$, E_{2k}, with $k \geq 2$, and z is the vertex of H highlighted in Figure 3. Notice that $(\overline{A})'' = (\overline{B})' = T$ in Figure 2; $(\overline{C})' = M$ in Figure 2; $(\overline{D_{2k+1}})' = \overline{J_{2k+1}}$ in Figure 1 and ${E_{2k}}' = N_{2k+1}$ in Figure 2.

Thus, by the previous claim (2), these graphs are CI, which contradicts the fact that neither the graphs depicted in Figures 1 and 2 nor their complements are CI graphs.

To prove the converse, notice first that, by Theorem 2, the vertex z must be a sink of G, which means there exists a transitive orientation $\overrightarrow{E}$ of G in which all edges incident in z are oriented towards z.

Let G' be the graph G plus a pendent vertex z' adjacent to z. Since a transitive orientation $\overrightarrow{E'}$ of G' can be obtained adding $\overrightarrow{z'z}$ to $\overrightarrow{E}$, it follows that G' is a comparability graph and z is a sink of G'.

Assume, in order to derive a contradiction, that G' is not CI. Thus, G' contains an induced subgraph H which is either a graph in Figure 2 or the complement of a graph in Figure 1; even more, H must contain z', so H must have a vertex of degree one and z must be the only neighbor of that vertex.

An inspection of those figures reveals that H has to be one of the graphs T, S, M, G_1, G_2, G_3 or N_n for some $n \geq 4$, in Figure 2; or the complement of the graph J_{2k+1} for some $k \geq 2$ in Figure 1. In the following paragraph, we will show that each case implies a contradiction, therefore G' is a CI graph.

* If H is any one of the graphs S, G_1, G_2 or G_3, then G contains the graph $\overline{A}$ in Figure 3 with z as the highlighted vertex.

* If H is the graph T, then G contains the graph $\overline{B}$ in Figure 3 with z as the highlighted vertex.

* If H is the graph M, then we have to consider two cases depending on which vertex of degree one is z': its neighbor z is adjacent to two vertices with degree 3, or its neighbor z is adjacent to a vertex with degree 3 and other with degree 2. In the first case, G contains the graph $\overline{A}$ in Figure 3 with z as the highlighted vertex; and, in the latter, G contains the graph $\overline{C}$ in Figure 3 with z as the highlighted vertex.

* If H is the graph N_4, then G contains the graph E_3 in Figure 3 with z as the highlighted vertex.

* If H is the graph N_n for some $n > 4$, then, again, we consider two cases depending on which vertex of degree one is z': if z' is the vertex labelled 1 (or n) then G contains the graphs E_{n-1} and D_{n-1} in Figure 3 with z as the highlighted vertex. Notice that E_{n-1} is forbidden when n is odd and D_{n-1} is forbidden when n is even. If z' is neither the vertex labelled 1 nor the vertex labelled n, then G contains the graph $\overline{A}$ in Figure 3 with z as the highlighted vertex.

Thus, we have proved that G' is CI. By Remark 1, there exist a CI model $(I_w)_{w \in V(G')}$ compatible with $\overrightarrow{E'}$, i.e. $\overrightarrow{uv} \in \overrightarrow{E'}$ implies $I_u \subset I_v$.

Clearly, if I_z is an end interval in this model, the vertex z is an end-vertex of G, and the proof follows.

So, we assume that the left set L_z and the right set R_z of z are non empty. The fact that z is the only neighbor of z' implies $L_z \cap R_z = \emptyset$. Let x be a vertex of L_z minimizing the distance to z in G'; and $P_{zx} = [v_0 = z, v_1, v_2, ...v_{k-1}, v_k = x]$ be a shortest path joining z with x. Since z is a sink, we have that $\overrightarrow{v_1 z} \in \overrightarrow{E'}$, which implies that $I_{v_1} \subset I_z$. Therefore, by Lemma 4, $v_2 \in R_z$ and $r_z < r_{v_i}$ for every i, or $v_2 \in L_z$ and $l_{v_i} < l_z$ for every i. Since $v_k = x \in L_z$, the former is not possible; thus, $v_2 \in L_z$ which implies $k = 2$ and the existence of the induced path $[z, v_1, x]$. Analogously, there exist vertices v'_1 and $y \in R_z$ inducing the path $[z, v'_1, y]$.

The proof will be complete showing that the concatenation of both paths induces the graph $\overline{A}$ with z as the highlighted vertex.

Clearly, x and y are non-adjacent. We claim that x is non-adjacent to v'_1. Indeed, if x is adjacent to v'_1, we have that $r_{v'_1} < r_x$; on the other hand, since z' has degree 1 and $I_{z'} \subset I_z$, it follows that $r_x < r_{z'}$ and $l_{z'} < l_y$. Therefore, $l_{z'} < l_y < l_{v'_1} < r_{v'_1} < r_x < r_{z'}$ which implies $I_{v'_1} \subset I_{z'}$, contrary to the fact that z' is adjacent only to z. Analogously, y is non-adjacent to v_1. ■

As a consequence of Theorems 2 and 5, we have the following result.

Corollary 6. *Let G be a CI graph. A vertex z is an end-vertex of G if and only if z is a sink of G and $\overline{G}$.*

3 Characterization of CI homogeneously representable graphs

A CI graph G is *homogeneously representable* if every one of its vertices is an end-vertex.

Theorem 7. *Let G be CI. The graph G is homogeneously representable if and only if it contains none of the graphs A, $\overline{A}$, C, $\overline{C}$, E_3, depicted in Figure 3, as induced subgraphs.*

Proof. It is a straightforward consequence of Theorem 5 and the facts: A is an induced subgraph of B and of D_{2k+1}, for $k \geq 2$; and E_3 is an induced subgraph of E_{2k}, for $k \geq 2$. ■

In the previous theorem, the condition that G is a CI graph can be relaxed by adding C_5 to the family of forbidden induced subgraph.

Theorem 8. *A graph G is CI homogeneously representable if and only if it contains none of the graphs C_5, A, $\overline{A}$, C, $\overline{C}$, E_3, depicted in Figure 3, as induced subgraphs.*

Proof. The direct implication follows from Theorem 7 and the fact that C_5 is not a comparability graph.

To prove the converse, keep in mind that the graphs depicted in Figures 1 and 2 and their complements are a family of forbidden induced subgraphs of a CI graph; and notice that C_5 and E_3 are self-complementary. ■

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