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Aspects of the complexity of (γ, μ)-coloring

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Abstract

In this work we consider a variation of LIST COLORING, called (γ, μ)-COLORING. We show a framework of reducibility between LIST COLORING, (γ, μ)-COLORING and PRECOLORING EXTENSION, in order to provide an parameterized complexity analysis of (γ, μ)-coloring. We remark that (γ, μ)-COLORING is FTP when parametrized by vertex cover and the maximum size of a color list, but it is W[1]-hard when parameterized by treewidth.

1 Introduction

Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges. For a graph $G = (V, E)$, an assignment $c : V \to \mathbb{N}$ is a coloring of G. Furthermore, this coloring is proper if $c(u) \neq c(v)$ for all $uv \in E$, that is, a *k-coloring* of G is an assignment of colors to the vertices of G such that no pair of adjacent vertices share the same color. The chromatic number χ_G of a graph is the minimum value of k for which G is k-colorable.

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There are several variants of the classical vertex coloring problem, involving additional constraints, in both edges and vertices of the graph. One of them is the LIST COLORING, where each vertex v of the input graph G is equipped with a list $L(v)$ of allowed colors. If it is possible to get a proper coloring of G such that each vertex v is colored with a color in $L(v)$, then G has a LIST COLORING. The problem of determining whether G has a list coloring was introduced by Paul Erdös *et al.* in 1979 [3], and independently by Vizing in 1976 [6].

LIST COLORING also has variations, such as PRECOLORING EXTENSION (i.e., given a graph with some previously colored vertices, it aims to extend such to proper coloring of$G$). Another variation called (γ, μ)-COLORING was introduced by Bonomo *et al.* [1, 2] In such a problem we are given a graph G and functions $\gamma, \mu : V(G) \to \mathbb{N}$ such that $\gamma(v) \leq \mu(v)$ and we say that G is (γ, μ)-colorable if there exists a coloring f of G such that $\gamma(v) \leq f(v) \leq \mu(v)$ for every $v \in V(G)$.

2 Reducibility in List Coloring problems

In this paper, we analyze the complexity of (γ, μ)-COLORING. It is easy to see that every instance of (γ, μ)-COLORING is an instance of LIST COLORING, where $L(v) = \{\gamma(v), \gamma(v) + 1, \ldots, \mu(v)\}$ for every $v \in V(G)$. Therefore, (γ, μ)-COLORING on $\mathcal{C}$ is not harder than LIST COLORING on $\mathcal{C}$, for any graph class $\mathcal{C}$. We show from the perspective of analogous problems that in fact the (γ, μ)-COLORING and LIST COLORING are similar in several aspects.

Denote by $Y(\Pi)$ the set of all instances I of Π yielding a yes-answer for the question "$I \in Y(\Pi)$?". The notion of *analogous problems* was introduced by Fellows *et al.* [4].

Definition 2.1. *Two decision problems Π and Π' in NP are said to be* analogous *if there exist linear-time reductions f, g such that:*

1. $\Pi \propto^f \Pi'$ *and* $\Pi' \propto^g \Pi$*;*

2. *every easily checkable certificate* $\mathcal{C}$ *for the yes-answer of the question "*$I \in Y(\Pi)$ *?" implies an easily checkable certificate* $\mathcal{C}'$ *for the yes-answer of the question "*$f(I) \in Y(\Pi')$ *?" such that size*$(\mathcal{C}) =$ *size*$(\mathcal{C}')$*, and* C' *is computable from* C *in linear time;*

3. *every easily checkable certificate* $\mathcal{C}'$ *for the yes-answer of the question "*$I' \in Y(\Pi')$ *?" implies an easily checkable certificate* $\mathcal{C}$ *for the yes-answer of the question "*$g(I') \in Y(\Pi)$ *?", such that size*$(\mathcal{C}') =$ *size*$(\mathcal{C})$*, and* C *is computable from* C' *in linear time. .*

Definition 2.2. *Let* Π *and* Π' *be analogous decision problems. The parameterized problems* $\Pi(k_1, \ldots, k_t)$ *and* $\Pi'(k'_1, \ldots, k'_t)$ *are said to be* p-analogous *if there exist linear-time reductions* f, g *and a one-to-one correspondence* $k_i \leftrightarrow k'_i$ *such that:*

1. $\Pi(k_1, \ldots, k_t) \propto^f \Pi'(k'_1, \ldots, k'_t)$ *and* $\Pi'(k'_1, \ldots, k'_t) \propto^g \Pi(k_1, \ldots, k_t)$*;*

2. *every easily checkable certificate* $\mathcal{C}$ *for the yes-answer of the question "*$I \in Y(\Pi(k_1, \ldots, k_t))$*?" implies an easily checkable certificate* $\mathcal{C}'$ *for the yes-answer of the question "*$f(I) \in Y(\Pi'(k'_1, \ldots, k'_t))$*?" such that* $k'_i = \varphi'_i(k_i)$ *for some linear function* φ'_i $(1 \le i \le t)$*;*

3. *every easily checkable certificate* $\mathcal{C}'$ *for the yes-answer of the question "*$I' \in Y(\Pi'(k'_1, \ldots, k'_t))$*?" implies an easily checkable certificate* $\mathcal{C}$ *for the yes-answer of the question "*$g(I') \in Y(\Pi(k_1, \ldots, k_t))$*?" such that* $k_i = \varphi_i(k'_i)$ *for some linear function* φ_i $(1 \le i \le t)$*.*

Two straightforward consequences of the above definitions are: (a) if Π and Π' are analogous problems then Π is in P (is NP-hard) if and only if Π' is in P (is NP-hard); (b) if $\Pi(k_1, \ldots, k_\ell)$ and $\Pi'(k'_1, \ldots, k'_\ell)$ are p-analogous problems then $\Pi(k_1, \ldots, k_\ell)$ is in FTP (admits a polynomial kernel/is W[1]-hard) if and only if $\Pi'(k'_1, \ldots, k'_\ell)$ is in FTP (admits a polynomial kernel/is W[1]-hard).

Now we present a general framework for reducibility from List coloring to (γ, μ)-coloring and Precoloring extension.

Definition 2.3. *Let G be a graph where each vertex $v \in V(G)$ is endowed with a list $L(v)$ of available colors and c is the largest used color. We construct a graph $\psi(G)$ from G with $\gamma(v) = 1$ and $\mu(v) = c$ for each v in $\psi(G)$. Now, for each v of G in $\psi(G)$ we add $c - |L(v)|$ pendant vertices w_i incident to v such that each of these pendant vertices forbids a different color i not in $L(v)$ to v, i.e., for each $c_i \in (\{1, \dots, c\} \backslash L(v))$ there is a distinct w_i with $\gamma(w_i) = \mu(w_i) = c_i$.*

Definition 2.4. *Let $\mathcal{C}$ be a class of graphs. Then:*

$$\psi(\mathcal{C}) = \{G \mid G = \psi(G') \text{ for some } G' \in \mathcal{C} \}.$$

A class $\mathcal{C}$ of graphs is closed under operator ψ if $\psi(\mathcal{C}) \subseteq \mathcal{C}$.

Examples of graph classes closed under ψ are chordal graphs and bipartite graphs.

Lemma 2.1. List coloring*,* (γ, μ)-coloring *and* Precoloring extension *are analogous when restricted to classes closed under operator ψ.*

Proof. First, without loss of generality, we may assume that the size of the list of each vertex is at most its degree plus one (otherwise the instance is equivalent to an instance with such a vertex removed); all the colors are present in at least two lists; the smallest color is equal to 1. Note that, in such a case, the number of colors is upper bounded by $m + n$.

Now, observe that Precoloring extension is a particular case of (γ, μ)-coloring, which is also a particular case of List coloring. Thus, by restriction, Precoloring extension $\propto^{f_1}$ (γ, μ)-coloring $\propto^{f_2}$ List coloring as required. Now, let $\mathcal{C}$ be a graph class closed under operator ψ, and let (G, L) be an instance of List coloring where $G \in \mathcal{C}$. Note that $\psi(G)$ is an instance of both (γ, μ)-coloring and Precoloring extension, because $\psi(G)$ contains only lists of size one and lists of size c. By construction, G admits a List coloring if and only if $\psi(G)$ admits a (γ, μ)-coloring, thus one can see that List coloring

$\propto^{\psi}$ (γ,μ)-COLORING as we require. Then LIST COLORING and (γ,μ)-COLORING are analogous on $\mathcal{C}$. Since $\psi(G)$ contains only lists of size one and lists of size c then LIST COLORING, (γ,μ)-COLORING, and precoloring extension are analogous on $\mathcal{C}$. □

Definition 2.5. *A* tree decomposition *of a graph* $G = (V, E)$ *is a pair* (T, χ) *where* $T = (V(T), E(T))$ *is a tree with* $V(T) = \chi$ *is a family* $(X_i)_{i=1}^r$ *of subsets of* V*, called* bags *or* nodes *such that*

- $\bigcup_{i=1}^r X_i = V$;
- $\forall uv \in E; \exists i \in \{1, 2, \cdots, r\} | \{u, v\} \subset X_i$;
- $\forall u \in V$*, the set* $T_u = \{X_i \in V(T) | u \in X_i\}$ *induces a connected subtree of* T*.*

The width *of the tree decomposition* (T, χ) *is* $\max_{1 \leq i \leq r} |X_i| - 1$*. The* treewidth *of* G*, denoted by* $tw(G)$*, is the minimum width over all its tree decompositions.*

Lemma 2.2. LIST COLORING, (γ,μ)-COLORING *and* PRECOLORING EXTENSION *on classes closed under operator* ψ *are p-analogous when parameterized by treewidth and feedback vertex set.*

Proof. It follows from Lemma 2.1, the fact that the treewidth and the size of a minimum feedback vertex set of $\psi(G)$ and G have the same size. □

Theorem 2.1. (γ,μ)-COLORING *and* PRECOLORING EXTENSION *parameterized by the feedback vertex set remains* $W[1]$*-hard even when restricted to bipartite graphs.*

Proof. It follows from Lemma 2.2 and the $W[1]$-hardness of LIST COLORING on bipartite graphs parameterized by the feedback vertex set (see [5]).

□

In [5], it was shown that List coloring parameterized by the vertex cover number is $W[1]$-hard even when the input graph G is bipartite. Now, we consider the vertex cover number and the maximum size of a list as agregated parameters.

Theorem 2.2. List coloring *is FPT when parameterized by the vertex cover number and the maximum size of a list.*

Proof. Let ℓ be the size of the largest list of the input, let k be the size of the vertex cover number of the input graph G, and let S be a minimum vertex cover of G ($|S| = k$). Assume $I = V(G) \setminus S$ (these vertices form an independent set of G). We can exhaustively analyze all possible ways of coloring the vertex cover S in $O(\ell^k)$ time. After that, for each possibility one can use a greedy algorithm to check whether such a coloring of S can be extended to a List coloring of G. This algorithm can be performed in $O(\ell^k.n)$ time. □

Corollary 2.1. List coloring *parameterized by the vertex cover number is FPT when $|L(v)| = |L(w)|$ for each $v, w \in V(G)$.*

Proof. Let S be a minimum vertex cover of G and $|S| = k$. First, apply the following reduction rule: For each $v \in V(G)$, if $|L(v)| > |N(v)|$ then remove v. It is easy to see that the previous reduction rule is safe, because if $|L(v)| > |N(v)|$, after coloring $N(v)$ one can always choose a viable color for v. Now, since $|L(v)| = |L(w)|$ for each pair $v, w \in V(G)$, we have the following cases: case 1: $|L(v)| > k$, in this case every vertex of $V(G) \setminus S$ was removed by rule 1. After that, $G[S]$ is an instance where its vertices v has $|L(v)| > |S| = k$. Thus, by applying a greedy algorithm that first visit vertices in S, and after visit the vertices in $V(G) \setminus S$, we obtain a (γ, μ)-coloring for G in polynomial time; case 2: $|L(v)| \leq k$, in this case the size of the lists are bounded by k. Therefore we can apply the FPT-algorithm present in the proof of Theorem 2.2. □

3 Concluding Remarks

In this work is presented a study of (γ, μ)-COLORING in graphs under the perspective of analogous problems and parameterized complexity. We show that (γ, μ)-COLORING is $W[1]$-hard parameterized by treewidth, but it is FPT when parameterized by the vertex cover number and the maximum size of a color list.

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