Matemática Contemporânea, Vol. 46, 212–220

Notes on models for distance coloring problems

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Abstract

In this paper, we explore graph coloring problems with distance constraints on the edges, following a distance geometry point of view, that is, as the positioning of the vertices on the real number line. This leads to an embedding of the input graph in 1-dimension, where the point on the line corresponds to the color to be assigned to a vertex, according to the distance between adjacent vertices. We demonstrate, for some classes of graphs, feasibility properties for each distance coloring model shown, in both senses (when there always exists at least one solution and when there cannot be any solution).

1 Introduction

Let $G = (V, E)$ be an undirected graph. In the classic vertex coloring problem (VCP) in graphs, a mapping $x : V \to \{1, 2, \dots, k\}$ such that $\forall (i, j) \in E,\ x(i) \neq x(j)$ must be found. The lowest possible value of k

*Financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001, and by Méliuz.
2000 AMS Subject Classification: 05C15, 05C12 and 51K99.
Key Words and Phrases: graph coloring, distance geometry, channel assignment.

for which x exists is called the chromatic number of G and is denoted by χ_G [2]. Finding such number is one of the most important combinatorial optimization problems and it is known to be NP-hard [7].

One of the main applications of such problems consists of assigning channels to transmitters in a mobile wireless network, where channels must be assigned to calls so interference is avoided and the spectrum usage is minimized [1, 9]. An useful generalization of VCP for channel assignment is Bandwidth Coloring Problem (BCP) [10, 12], where, for each edge $(i,j) \in E$, there is a positive integer $d_{i,j}$ such that $\forall (i,j) \in E,\ |x(i) - x(j)| \geq d_{i,j}$. BCP is further generalized to T-coloring, where, for each edge, the absolute difference between colors assigned to each vertex must not be in a given forbidden set [8].

In this work, we are interested in two cases of T-coloring:

1. For each edge $(i,j) \in E$, we have $T_{i,j} = \mathbb{Z}_{\geq 0} \backslash \{d_{i,j}\}$, where $\forall (i,j) \in E,\ d_{i,j} \in \mathbb{N}$.

2. For each edge $(i,j) \in E$, we have $T_{i,j} = \{0, 1, \ldots, d_{i,j} - 1\}$, where $\forall (i,j) \in E,\ d_{i,j} \in \mathbb{N}$.

The two cases will be explored using a distance geometry approach, henceforth, they will be called distance coloring problems. The rest of the paper is organized as follows. Section 2 defines such problems and shows some of their properties. Section 3 gives constraint and integer programming formulations for these problems. Section 4 then concludes our paper.

2 Distance coloring problems

One of the most studied distance geometry problems is the Discretizable Distance Geometry Problem (DDGP), which is defined in $\mathbb{R}^3$ as follows. Let $G = (V, E)$ be a graph where V is ordered such that there exists a subset V_0 of V such that $|V_0| = 3$, V_0 induces a clique of G and for each $i \in V \backslash V_0$, there is a subset $\{v_1, v_2, v_3\}$ of V such that $v_1 < i,\ v_2 < i,\ v_3 < i$; $\{(v_1, i),\ (v_2, i),\ (v_3, i)\} \subseteq E$ and a strict triangular inequality holds according to the order of vertices. An embedding of G in $\mathbb{R}^3$ is a mapping $x : V \to \mathbb{R}^3$ such that $||x(i) - x(j)|| = d_{i,j}$ for all $(i, j) \in E$. An important consequence of this configuration is that the position of vertex i (where $i \geq 4$) in $\mathbb{R}^3$ can be calculated using positions of the previous three vertices $i-1, i-2$ and $i-3$ by intersecting three spheres, where two points are obtained which must be checked for feasibility [11].

A similar reasoning can be used in coloring problems with distance constraints, where the space considered is actually $\mathbb{R}^1$, the position of each vertex corresponds to its color and the distances that must be respected involve the absolute difference between two values $x(i)$ and $x(j)$. The positioning of a vertex i in such space can be determined by using a neighbor j that is already positioned. Thus, we have a *0-sphere*, consisting of a projection of a 1-sphere (a circle), that is, it is a line segment with radius $d_{i,j}$, and feasible colorings consist involve intersections of these 0-spheres [3].

Based on these problems, we can define the **Minimum Equal Coloring Distance Geometry Problem (MinEQ-CDGP)**, where we are given a graph $G = (V, E)$, where, for each $(i, j) \in E$, there is a weight $d_{i,j} \in \mathbb{N}$ and we must find an embedding $x : V \to \mathbb{N}$ such that $|x(i) - x(j)| = d_{i,j}$ for each $(i, j) \in E$ whose span S, defined as $S = \max_{i \in V} x(i)$, that is, the maximum used color, is the minimum possible. A variation of this problem occurs when all weights imposed on the edges are the same (that is, for each $(i, j) \in E$, $d_{i,j} = \varphi$, where $\varphi \in \mathbb{N}$), which we call **MinEQ-CDGP with Uniform Distances (MinEQ-CDGP-Unif)**. Some feasibility properties can be stated for these problems, as defined by the following two theorems (proofs will be omitted due to space constraints in this paper).

Theorem 1 *A graph G admits at least one feasible solution for MinEQ-CDGP-Unif if and only if G is bipartite [3].*

Theorem 2 *If a graph G is a tree and, for each $(i, j) \in E$, there is a weight $d_{i,j} \in \mathbb{N}$, then G admits at least one feasible solution for MinEQ-*

CDGP [3].

Another distance coloring problem can be defined by changing the type of adjacency constraint. In the **Minimum Greater than or Equal Coloring Distance Geometry Problem (MinGEQ-CDGP)**, we have a graph $G = (V, E)$, where, for each $(i, j) \in E$, there is a weight $d_{i,j} \in \mathbb{N}$ and we must find an embedding $x : V \to \mathbb{N}$ such that $|x(i) - x(j)| \geq d_{i,j}$ for each $(i, j) \in E$ with minimum span. We can also define a variation when all weights of the edges are the same, which is called **MinGEQ-CDGP with Uniform Distances (MinGEQ-CDGP-Unif)**. These two problems are equivalent to existing coloring problems: MinGEQ-CDGP is equivalent to BCP, and MinGEQ-CDGP-Unif is to VCP (even if the weights are not 1). We remark that these two problems always admit feasible solutions, since the set of possible values for $|x(i) - x(j)|$ that satisfy the inequality constraint ($\geq$) is infinite.

3 Constraint and integer programming models

In order to obtain solutions for the distance coloring problems, we can employ mathematical programming approaches. The first formulation we explore is based on constraint programming (CP). Let x_i be an integer variable consisting of the color assigned to vertex i. The CP model is then:

$$\text{Minimize} \quad \max_{i \in V} x(i) \tag{1}$$

$$\text{Subject to} \quad |x(i) - x(j)| \circledast d_{i,j} \quad (\forall (i,j) \in E) \tag{2}$$

$$x(i) \in \mathbb{N} \quad (\forall i \in V) \tag{3}$$

Where $\circledast$ is $=$ for MinEQ-CDGP and $\geq$ for MinGEQ-CDGP. This model follows the definition of distance coloring problems, as shown in Section 2. In [6], empirical results BCP (that is, MinGEQ-CDGP) instances are presented for this formulation.

For MinGEQ-CDGP (or BCP), integer programming (IP) models have also been developed. The standard formulation depends on a given upper bound U for the span and uses two sets of variables: x_{ik} (for each $i \in V$ and $1 \leq k \leq U$), which has value 1 if i uses color k and 0 otherwise; and z_{max}, which indicates the span. The formulation is defined as follows [6, 9]:

$$\text{Minimize} \quad z_{max} \tag{4}$$

$$\text{Subject to} \quad \sum_{k=1}^{U} x_{ik} = 1 \qquad (\forall i \in V) \tag{5}$$

$$x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ 1 \leq k, m \leq U \mid |k - m| < d_{i,j}) \tag{6}$$

$$z_{max} \geq k x_{ik} \qquad (\forall i \in V;\ 1 \leq k \leq U) \tag{7}$$

$$x_{ik} \in \{0, 1\} \qquad (\forall i \in V;\ 1 \leq k \leq U) \tag{8}$$

Constraint set (5) ensures that all vertices must be colored. Constraint set (6) require that distances between colors of adjacent vertices are respected. Constraints (7) require that variable z_{max} be greater than or equal to any color used, so it will be the maximum color used. Constraints (8) are type and bound constraints for x variables.

Another IP model for MinGEQ-CDGP is based on orientations of the input graph. Such formulation uses three sets of variables: x_i, which are integer and indicate the color assigned to vertex i; y_{ij}, which has value 1 if $x_i < x_j$ and 0 otherwise (inducing an orientation of G), and the same z_{max} variable of the previous model. The **orientation-based formulation** is defined as follows [4].

$$
\begin{array}{llll}
\text{Minimize} & z_{max} & & (9) \\
\text{Subject to} & x_i + d_{i,j} \le x_j + s(1 - y_{ij}) & \forall (i,j) \in E,\ i < j & (10) \\
& x_j + d_{i,j} \le x_j + s y_{ij} & \forall (i,j) \in E,\ i < j & (11) \\
& z_{max} \ge x_i & \forall i \in V & (12) \\
& x_i \in \mathbb{N} & \forall i \in V & (13) \\
& y_{ij} \in \{0,1\} & \forall (i,j) \in E,\ i < j & (14)
\end{array}
$$

In the above formulation, (10) ensures that $y_{ij} = 0$ if $x_i < x_j$, for $(i,j) \in E$, while (11) does the opposite when $x_i > x_j$. Both ensure that $|x_i - x_j| \ge d_{i,j}$. Constraints (12) impose z_{max} to take a value greater than or equal to every used color, and in an optimal solution this bound will be tight. Finally, constraints (13)-(14) define variable types and bounds. The model is full-dimensional when $s \ge z_{max}^{opt} + 2d_{max}$, where z_{max}^{opt} is the optimal value of z_{max} and $d_{max} = \max_{(i,j)\in E} d_{ij}$ [4].

For the model, two families of valid inequalities have been also identified. Let $\delta_K^i(j) := \min_{k\in K\cup\{i\}\setminus\{j\}} d_{j,k}$. In the first family, for a vertex $i \in V$, a clique $K \subseteq N(i)$ and $j \in K$, we define $x_i \ge \sum_{j\in K} \delta_K^i(j)\, y_{ji}$, as the **clique inequality** associated with the vertex i and the clique K. It induces a facet of the orientation polyhedron if $s \ge z_{max}^{opt} + 3d_{max}$, $d_{i,j} = \delta_K^i(j)$ for every $j \in K$ and for every $t \in N(i)\setminus K$ there exists $j \in K$ with $(j,t) \notin E$ and $d_{i,t} \le d_{i,j}$ [4].

For the second family of inequalities, let $\delta_K^{ij}(k) = \min_{\ell\in K\cup\{i,j\}\setminus\{k\}} d_{k,\ell}$; $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$ and $\gamma_k = \max\{0, \delta_K^{ij}(k) - d_{i,j}\}$ for $k \in K\setminus\{p\}$. Then, for an edge $(i,j) \in E$, a clique $K \subseteq N(i) \cap N(j)$ and a vertex $p \in K$, we define $x_i + d_{i,j} + \sum_{k\in K} \gamma_k (y_{ik} - y_{jk}) \ \le\ x_j + (s + d_{i,j} - \gamma(K)) y_{ji}$ as the **double clique inequality** associated with (i,j), K and p. This inequality is facet-inducing if $s \ge \chi(G,d) + 4d_{max}$; $d_{i,k} = d_{j,k} = \delta_K^{ij}(k)$ for every $k \in K$; $d_{p,k} = d_{p,j}$ for every $k \in K\setminus\{p\}$; $d_{i,j} \le \delta_K^{ij}(k)$ for every $k \in K\setminus\{p\}$ and $(t,p) \notin E$ and $d_{i,t} + d_{t,j} \le d_{i,j}$ for every $t \in [N(i) \cap N(j)]\setminus K$ [4].

A third formulation can be derived from the orientation-based model for VCP, in which variables represent the difference between colors assigned to different vertices of G. Let, for each $i, j \in V$, q_{ij} be a variable corresponding to the difference between colors of i and j, that is, the distance between them. Using the same y variables from the orientation model, we have the following **distance-based formulation** for VCP [5]:

$$q_{ik} = q_{ij} + q_{jk} \quad \forall i, j, k \in V,\ i < j < k \tag{15}$$

$$q_{ij} \geq 1 - |C| y_{ij} \quad \forall (i, j) \in E,\ i < j \tag{16}$$

$$q_{ij} \leq -1 + |C|(1 - y_{ij}) \quad \forall (i, j) \in E,\ i < j \tag{17}$$

$$q_{ij} \in \{-|C| + 1, \ldots, |V| - 1\} \quad \forall i, j \in V,\ i < j \tag{18}$$

$$y_{ij} \in \{0, 1\} \quad \forall (i, j) \in E,\ i < j \tag{19}$$

In the above model, constraints (15) correspond to the separation between colors of different vertices. Constraint sets (16) and (17) induce to graph orientations in a similar manner to the orientation model. The last two sets state integrality and bounds for variables. If $V = \{1, \ldots, |V|\}$, the set (15) can be replaced for the two following sets of constraints, which make the formulation asymptotically smaller [5]:

$$x_{i,i+1} + x_{i+1,i+2} = x_{i,i+2} \quad \forall i \in V, i \leq |V| - 2 \tag{20}$$

$$x_{ij} + x_{i+1,j-1} = x_{i,j-1} + x_{i+1,j} \quad \forall i, j \in V, i \leq |V| - 3, i + 3 \leq j \tag{21}$$

Due to the similarity between the orientation-based and distance-based models, we can use the valid inequalities of the first one in the latter one, which will be facet-inducing under the same conditions for both [5].

4 Concluding remarks

In this work, we summarized results for graph coloring problems involving distances. We explored two types of adjacency constraints, using equality and inequality, and gave results concerning feasibility according to the input graph. We also proposed constraint and integer programming formulations for these problems. For the orientation-based IP model, we also provide two families of valid inequalities which are facet-inducing under certain conditions.

Ongoing works include using the distance-based IP model in other coloring problems, such as BCP, and developing exact methods based on the mathematical programming formulations to solve distance coloring problems.

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