Matemática Contemporânea, Vol. 46, 204–211

The unit-demand envy-free pricing problem applied to the sports entertainment industry

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Abstract

This work presents preliminary results of an investigation about the envy-free pricing problem and applications in the sports industry. Based on the literature dealing with the subject, the problem is defined and then an application of the unit-demand case to ticket sales for football matches is modeled in terms of mixed-integer nonlinear programming. Graph theory aspects for the allocation and pricing subproblems are discussed, showing the viability of these mathematical and computational treatments in the study of revenue management in the sports entertainment industry.

1 Introduction

The sudden changes that have been occurring in the market present challenges to companies demanding new innovative and productive ideas in order to maintain competitiveness and profitability.

A segment that is in the bulge of today's market discussions is the sports entertainment industry, which moves billions of dollars annually. In the

2000 AMS Subject Classification: 68-06, 68R10 and 91B32.
Key Words and Phrases: envy-free pricing, graph theory, revenue management.

process of obtaining an estimate of the volume of financial movement of this segment, a number of factors are taken into account, such as the sale of tickets, products, and sporting goods, transmission agreements, sponsorships, among others. The results presented in this article focus on the aspect of ticket sales, which is the most traditional source of income in the sports industry.

Motivated by the achievement of the biggest sporting event of 2018, the FIFA World Cup Russia, this work aims to present a mathematical formulation in terms of mixed-integer nonlinear programming of the envy-free pricing problem applied to ticket sales for football matches and show some approaches in terms of graph theory for this problem.

To achieve this goal, Section 2 states the envy-free pricing problem (EFPP). Following, Section 3 proposes a mixed-integer nonlinear programming (MINLP) formulations applied to ticket sales for a football match. Section 4 realizes an overview of strategies for solving the model by classical graph theory problems. Finally, Section 5 makes some final considerations about future work perspectives.

2 The envy-free pricing problem

Assume that there is a set I of m consumers and a set J of n different items. Each item has c_j copies, and a supply vector is given by $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{Z}^n_+$. Each consumer has a positive valuation $v_i(S)$ for each bundle $S \subseteq J$, of items, which measures how much receiving bundle S would be "worth" to consumer i; the $m \times 2^n$ matrix of valuations is denoted by V. For convenience, it is assumed that $v_i(\emptyset) = 0$ for every consumer $i \in I$.

Given a price vector $\mathbf{p} = (p_1, ..., p_n) \in I\!R^n$, the utility that consumer i derives from bundle S is $U_i(S) = v_i(S) - p_S$, where $p_S = \sum_{j \in S} p_j$; it measures the consumer's "joy" at having bought the bundle S at the given price. If consumer i's utility for the bundle S is non-negative, it is said that S is feasible for i.

User i's demand set D_i contains all bundles that would make him maximally happy, i.e., all bundles that he would most like to buy. Formally, $D_i = \{S \mid U_i(S) = \max_{S'} U_i(S')\}$. Since not buying any bundle is always an option with utility $U_i(\emptyset) = 0$, it follows that $U_i(S) \geq 0$ for all $S \in D_i$.

Applying this terminology, we can define:

Definition 2.1. *An allocation $(S_1, ..., S_m)$ of bundles to consumers is feasible if each item j is in at most c_j sets S_i.*

Definition 2.2. *Given a pricing $\mathbf{p} = (p_1, ..., p_n)$, an allocation $(S_1, ..., S_m)$ is envy-free if $S_i \in D_i$ for all i, i.e., each consumer receives a bundle from his demand set.*

Definition 2.3. *A pricing $\mathbf{p}$ is envy-free if it admits a feasible, envy-free allocation.*

Definition 2.4. *(The EFPP) Given the input $(m, n, V, \mathbf{c})$, compute an envy-free pricing $\mathbf{p}$ and a corresponding envy-free allocation $(S_1, ..., S_m)$ maximizing the seller profit $\sum_{i=1}^{m} p_{S_i}$.*

In the unit-demand case, each consumer is interested in purchasing exactly one item. Then $v_i(S) > 0$ only when $|S| = 1$. Therefore, discarding the empty set and the subsets $S \subseteq J$ such that $|S| > 1$, the dimension of the valuations matrix reduces from $m \times 2^n$ to $m \times n$, and the input $v_i(S)$ denotes the value assigned by the consumer i to the item j.

Guruswami *et al.* [2] proved that the unit demand envy-free pricing problem is APX-hard, even if each item exists in unlimited supply, and each consumer has equal valuations (of either 1 or 2) for all the items he has any interest in.

3 A MINLP formulation for ticket sales

Suppose the company responsible for selling tickets for a football match wants to maximize its revenue, leaving consumers satisfied with the price to be paid. A mathematical formulation aiming at this end can take into account the following assumptions: the tickets are typified and valued according to the proximity and visibility of the seats in relation to the field; market analyses that observe the purchasing behavior of potential consumers provide an estimation of the valuations that consumers assign to the various ticket types; consumers with similar purchasing behaviors are grouped into segments; there is a maximum number of available units for each ticket type, according to the maximum capacity of the stadium; each consumer can buy only one ticket; consumers who want the same ticket type pays the same price for it; the seller establishes a minimum price for each ticket type.

A MINLP formulation that models this situation is given as follows:

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} N_i x_{ij} p_j \tag{1}$$

$$\text{s. t.} \quad \sum_{j \neq k} (v_{ij} - p_j) x_{ij} \geq v_{ik} \sum_{j \neq k} x_{ij} - p_k \quad \forall i \in I,\ \forall k \in J \tag{2}$$

$$(v_{ij} - p_j) x_{ij} \geq 0 \quad \forall i \in I,\ \forall j \in J \tag{3}$$

$$\sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall i \in I \tag{4}$$

$$\sum_{i=1}^{m} N_i x_{ij} \leq c_j \quad \forall j \in J \tag{5}$$

$$p_j \geq \underline{p}_j \quad \forall j \in J \tag{6}$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in I,\ \forall j \in J \tag{7}$$

$$p_j \geq 0 \quad \forall j \in J \tag{8}$$

where $I = \{1, ..., m\}$ is the set of consumers' segments, $J = \{1, ..., n\}$ is the set of ticket types, x_{ij} is the decision variable such that $x_{ij} = 1$ if the item j is allocated to consumers in segment i and $x_{ij} = 0$ otherwise, N_i is the number of consumers in segment i, v_{ij} is the valuation that consumers of segment i assign to type j tickets, c_j is the maximum availability of type j tickets, p_j is the price of type$j$ tickets to be determined and $\underline{p}_j$ is the minimum price for type j tickets.

Note, then, that formulation (1) – (8) is a valid formulation for the unit-demand envy-free pricing, where a solution $(\mathbf{x}, \mathbf{p})$ maximizes both the seller revenue and the buyers' (winners) utilities $u_{ij} = v_{ij} - p_j$.

4 Graph theory perspectives

4.1 The allocation subproblem

If inequality in constraint (5) becomes an equality, consider the following integer linear program:

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} N_i v_{ij} x_{ij} \tag{9}$$

$$\text{s. t.} \quad \sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall i \in I \tag{10}$$

$$\sum_{i=1}^{m} N_i x_{ij} = c_j \quad \forall j \in J \tag{11}$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in I, \ \forall j \in J \tag{12}$$

The *First Social Welfare Theorem* ensures that an allocation that maximizes the social welfare also maximizes the seller's revenue with envy-free prices, i.e., a solution for (9) – (12) is also a solution for (1) – (8). Thus, finding an optimal allocation is equivalent to find a maximum weight many-to-one matching with upper bounds [1] in the graph associated to matrix V.

4.2 The pricing subproblem

Once the binary variables are found, then considering $C_j = \{i : \ x_{ij} = 1\}$, $B = \{j : \ C_j \neq \emptyset\}$ and $M_j = \sum_{i \in C_j} N_i$, model (1) – (8) becomes:

$$\max \sum_{j=1}^{n} M_j p_j \tag{13}$$

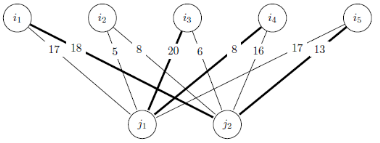

Figure 1: A maximum weight many-to-one matching with upper bounds based on model (9)–(12), where the bipartite graph is $G = (I \cup J, E)$, the weights are the consumers valuations, $N = (2, 3, 4, 3, 4)$ and $c = (7, 6)$. Source: Authors.

$$\text{s. t.} \quad v_{ij} - p_j \geq v_{ik} - p_k \quad \forall j \in B,\ \forall k \in J \setminus \{j\},\ \forall i \in C_j \tag{14}$$

$$v_{ij} - p_j \geq 0 \quad \forall j \in B,\ \forall i \in C_j \tag{15}$$

$$p_j \geq \underline{p}_j \quad \forall j \in J \tag{16}$$

$$p_j \geq 0 \quad \forall j \in J \tag{17}$$

Assuming that $v_{ij} \geq \underline{p}_j$ $\forall i \in I,\ \forall j \in J$, model (13) – (17) is simplified to

$$\max \sum_{j=1}^{n} M_j p_j \tag{18}$$

$$\text{s. t.} \quad p_j - p_k \leq \min_{i \in C_j} \{v_{ij} - v_{ik}\} \quad \forall j \in B,\ \forall k \in J \setminus \{j\} \tag{19}$$

$$p_j \leq \min_{i \in C_j} \{v_{ij}\} \quad \forall j \in J \tag{20}$$

$$p_j \geq 0 \quad \forall j \in J \tag{21}$$

The dual of this last formulation is

$$\min \sum_{j,k} \alpha_{jk} y_{jk} + \sum_{j,k} \beta_j z_j \tag{22}$$

$$\text{s. t.} \quad \sum_{j \neq k} y_{jk} - \sum_{j \neq k} y_{kj} + z_j = M_j \quad \forall j \in B \tag{23}$$

$$\sum_{j \neq k} y_{kj} = 0 \quad \forall j \notin B \tag{24}$$

$$y_{jk}, z_J \geq 0 \quad \forall j, k \in J \tag{25}$$

where $\alpha_{jk} = \min_{i \in C_j}\{v_{ij} - v_{ik}\}$ and $\beta_j = \min_{i \in C_j}\{v_{ij}\}$.

Removing all $y_{kj},\ j \notin B$ and adding a single redundant constraint, the model becomes

$$\min \sum_{j,k} \alpha_{jk} y_{jk} + \sum_{j,k} \beta_j z_j \tag{26}$$

$$\text{s. t.} \quad \sum_{j \neq k} y_{jk} - \sum_{j \neq k} y_{kj} + z_j = M_j \quad \forall j \in B \tag{27}$$

$$-\sum_j z_j = -\sum_j M_j \tag{28}$$

$$y_{jk}, z_J \geq 0 \quad \forall j, k \in B \tag{29}$$

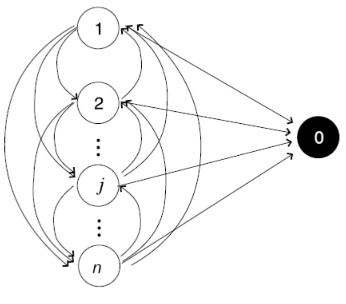

Figure 2: A graph representing formulation (26) – (29). The arcs costs are the objective function parameters. The optimal price of type j ticket is the shortest path length from node j to node 0. Source: Shioda *et al.* [4].

which corresponds to a formulation of $|B|$ shortest path problems in the digraph of Figure 2.

5 Conclusions

This work exposed an application of the unit-demand envy-free pricing problem to ticket sales of sporting events, proposing a model in terms of mixed-integer nonlinear programming that treats the consumers in segments and take into account the reserve prices of both the seller and the consumers. This initial study shows that pricing problems form a fertile field for investigations that seek to establish connections between game theory, integer programming and graph theory, and the formulations presented provides support for the development of approximation algorithms, as well as make it possible to extend the unit-demand case to other demand cases that arise in real problems.

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