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On the Helly number in the P_3 and related convexities for $(q, q-4)$ graphs

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Abstract

Let $G = (V, E)$ be a graph. The P_3-convex hull (resp. P_3^*-convex hull) of a set $C \subseteq V$ is obtained by the iteratively addition of vertices with at least two neighbors (resp. non-adjacent neighbors) in C. A P_3-Helly-independent (resp. P_3^*-Helly-independent) of G is a set $S \subseteq V$ such that the intersection of the P_3-convex hulls (resp. P_3^*-convex hulls) of $S \setminus \{v\}$ ($\forall v \in S$) is empty. The P_3-Helly number (resp. P_3^*-Helly number) is the size of a maximum P_3-Helly-independent (resp. P_3^*-Helly-independent). The edge versions of these two P_3-Helly-independent follow the same restrictions applied to its edges. The VP3HI, VSP3HI, EP3HI, and ESP3HI problems aim to determine the P_3-Helly number, P_3^*-Helly number, edge P_3-Helly number, and edge P_3^*-Helly number of a graph, respectively. A graph G is $(q, q-4)$ when every induced subgraph of G with q vertices has at most $q-4$ paths of size four as induced

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subgraphs. We.establish polynomial time algorithms to VP3HI, VSP3HI, EP3HI, and ESP3HI for $(q, q-4)$ graphs with fixed q.

1 Introduction

Several concepts concerning variations of convexities on graphs has been established [3, 4]. The interest in these convexities comes from both their central role in many applications and purely theoretical questions. Among such applications, there are those related to distributed systems [8], social networks, and marketing strategies [6]. Moreover, various problems have been dealt considering the Helly property in the past [5]. This paper studies the Helly number in the P_3-convexity, P_3^*-convexity, edge P_3-convexity, and edge P_3^*-convexity.

Let $G = (V, E)$ be a graph. The P_3*-convex hull* (resp. P_3^**-convex hull*) of a set $C \subseteq V$ is obtained by the iteratively addition of vertices with at least two neighbors (resp. non-adjacent neighbors) in C. A P_3*-Helly-independent* (resp. P_3^**-Helly-independent*) of G is a set $S \subseteq V$ such that the intersection of the P_3-convex hulls (resp. P_3^*-convex hulls) of $S \setminus \{v\}$ $(\forall v \in S)$ is empty. The P_3*-Helly number* (resp. P_3^**-Helly number*) is the size of a maximum P_3-Helly-independent (resp. P_3^*-Helly-independent). A natural variation of the P_3 convexities occurs when we consider the same corresponding concepts of the vertices in the edges of the graph. The VP3HI, VSP3HI, EP3HI, and ESP3HI problems aim to determine the P_3-Helly number (h_{P_3}), P_3^*-Helly number ($h_{P_3^*}$), edge P_3-Helly number (h'_{P_3}), and edge P_3^*-Helly number ($h'_{P_3^*}$) of a graph, respectively.

A graph G is $(q, q-4)$ when every induced subgraph of size q has at most $q-4$ induced subgraphs of paths of size four (P_4). A graph $G = (V, E)$ with $V = K \cup I \cup R$ is a *thin spider* (resp. *thick spider*) when $|K| = |I| \geq 2$ (resp., $|K| = |I| \geq 3$), K induces a clique, I is a stable set, there is a join between vertices of K and R, and there is a one-to-one relation between vertices of K and I which gives the $|K|$ edges between I and K (resp. which gives the $|K|$ non-edges between I and K). Babel and Olariu [1] showed that a $(q, q-4)$ graph G is structurally quite rich in the sense that G is always: (i) the union or the join of two $(q, q-4)$ graphs; (ii) a spider such that $G[R]$ is a $(q, q-4)$ graph; (iii) a graph with a separable p-component $H \subseteq V(G)$ ($H = H_1 \cup H_2$) with $|V(H)| \leq q$ such that $G \setminus H$ is a $(q, q-4)$ graph, there is a join between vertices of $G \setminus H$ and H_1 and there is no edges between vertices of $G \setminus H$ and H_2 or; (iv) a graph with at most q vertices (which can be $V(G) = \emptyset$).

Our contributions in the present work concern to settle the computational complexity of the VP3HI, VSP3HI, EP3HI, and ESP3HI for $(q, q-4)$ graphs. We succeed to show an FPT (fixed parameter tractable) polynomial time algorithms for these problems when q is a constant.

2 Union and Join of graphs

Since, in the considered convexities, a vertex of a connected component cannot be in the convex hull of a set of vertices contained in a different connected component, the parameters are given as the sum of the parameters of its connected components. Moreover, when the parameters are bounded by a constant, we may test which possible combinations of sets of vertices (or edges) respect the P_3-convexity with the Helly property and determine the largest size among them in polynomial time. As a remark, when $|V(G)|$ is a constant, we may also obtain the parameters in polynomial time by brute force. Hereinafter we only consider connected graphs without trivial small values of the parameter.

Carvalho et al. [2] established the computational complexity of VP3HI, VSP3HI, EP3HI, and ESP3HI for subclasses of bipartite graphs, split graphs, and join of graphs. The following property about the edge P_3^* Helly number of a graph G plays a central role in our proofs: $h'_{P_3^*} = |V(G)| - s_t(G)$, where $s_t(G)$ is the least number of vertex disjoint stars subgraphs to partition $V(G)$ such that the centers of the stars with at least three vertices are non-adjacent [2]. Carvalho et al. [2] also point out two useful forbidden configurations to a P_3-Helly-independent S of a graph G: **(Forbidden 1)** three vertices of S adjacent to a same vertex of G; and **(Forbidden 2)** three vertices x, y and z of S such that xyz is a P_3 subgraph of G. Note that **(Forbidden 1)** and **(Forbidden 2)** are also forbidden configurations for a P_3^*-Helly-independent if we consider the vertices to be an induced star of size four or induced path of size three. Lastly, in order to establish the values of the parameters for the join graph $G = G_1 \wedge G_2$ of $(q, q-4)$ graphs, we refer the following results of Carvalho et al [2] for any join of graph G: (i) $h_{P_3}(G) \leq 2$; (ii) $h_{P_3^*}(G) = \max\{\omega(G_1) + \omega(G_2), h_{P_3^*}(G_1 \wedge K_1), h_{P_3^*}(G_2 \wedge K_1)\}$; (iii) $h'_{P_3}(G) = \max\{\beta^*(G_1), \beta^*(G_2)\}$; (iv) $h'_{P_3^*}(G) = \max\{|V(G_1)| + h'_{P_3^*}(G_2), |V(G_2)| + h'_{P_3^*}(G_1)\}$. Note that $h_{P_3^*}(H \wedge K_1)$ of a $(q, q-4)$ graph H is given by the size of a maximum induced complete bipartite graph in H^c (which are also a $(q, q-4)$ graph). Using the polynomial-time algorithms to determine the size of a maximum complete subgraph ω and the size of a maximum induced complete bipartite subgraph for $(q, q-4)$ graphs (which can be trivially constructed using its structural decomposition [1]) and the algorithm to determine $\beta^*(G)$ for $(q, q-4)$ graphs [7], we are able to obtain the P_3-Helly parameters in polynomial time for join of graphs.

3 Spider graphs

Let $G = (V, E)$ be a spider graph with $V = I \cup K \cup R$.

Thin spiders

(P_3-Helly-independent S) If there are three vertices of $K \cup R$ in S, then we have a **Forbidden 1** or **Forbidden 2** configuration. Moreover, if there are two vertices of $K \cup R$ in S and a vertex of I in S, then we have a **Forbidden 1** or **Forbidden 2** configuration. Therefore, $h_{P_3}(G) = |I| + 1$ where S is composed by all vertices of I and one vertex of $K \cup R$.

(P_3^*-Helly-independent S^*) If there is no vertex of $K \cup I$ in S^*, then $h_{P_3^*}(G) = h_{P_3^*}(G[R] \wedge K_1)$, which is the size of a maximum induced complete bipartite subgraph on the complement graph of $G[R]$. If there are two non-adjacent vertices of R and a vertex of $I \cup K$ in S^*, then we have a

Forbidden 1 or **Forbidden 2** configuration. Moreover, if there is a vertex $x \in R$ in S^* and two adjacent vertices $y \in K$ and $z \in I$ in S^*, then we have a **Forbidden 2** configuration. Therefore, if there are one or more pairwise adjacent vertices of R in S^*, then we can only have at most K vertices of $K \cup I$ in S^* and such S^* exists with $|S^*| = |K| + \omega(G[R])$, where it is composed by the vertices of a maximum clique of $G[R]$ and the vertices of K. Lastly, if there is no vertex of R in S^*, then $|S^*| \geq |I|+1$ and such S^* exists, where it is composed by the vertices of I and one vertex of K. For the sake of contradiction, assume there is no vertex of R in S^* and $|S^*| \geq |I| + 2$, then there is two adjacent vertices $x \in K$ and $y \in I$ in S^* and at least another vertex $z \in K$ in S^* which form a **Forbidden 2** configuration contradiction. Therefore, $h_{P_3^*} = \max\{|K| + \omega(G[R]), |I| + 1, h_{P_3^*}(G[R] \wedge K_1)\}$.
(edge P_3-Helly-independent M) Due to nature of thin spiders, the edge P_3-convex hull of a set reaches all edges when: (i) there are two edges in the set with both endpoints in vertices of $K \cup I$; (ii) there are an edge with both endpoints in $K \cup I$ and other edge with both endpoints in R in the set or; (iii) there are two edges with both endpoints in vertices of R in the set and other edge (not necessarily in the set) sharing endpoints with both these edges. Therefore, $h'_{P_3}(G) = \max\{2, \beta^*(G[R])\}$, where M is any two edges or a maximum induced matching of $G[R]$ (the induced subgraph of G by the vertices of R). Note that we obtain β^* for the $(q, q-4)$ graph $G[R]$ [7] in polynomial time.
(edge P_3^*-Helly-independent M^*) Recall that $h'_{P_3^*} = |V(G)| - s_t(G)$. To partition the vertices of G in $|I|$ vertex disjoint star subgraphs we take a vertex x of K as the center of a star with all other vertices of $K \cup R$ and one vertex of I as their leaves, and others $|I| - 1$ vertices of I as one vertex stars. This is the best possible, since we have the additional restriction to forbid two centers of stars in $K \cup R$ with degree two or more. Therefore, $h'_{P_3^*} = |V(G)| - s_t(G) = |V(G)| - |I|$ where M^* is composed by the edges of the star centered in the vertex x.
Thick spiders
(P_3-Helly-independent S) Due to the nature of a thick spider, any

three vertices of S imply a **Forbidden 1** or **Forbidden 2** configuration. Therefore, $h_{P_3}(G) = 2$ where S is composed by any two vertices of G.

(P_3^*-Helly-independent S^*) By a similar argument of thin spiders, we have $h_{P_3^*} = \max\{|K| + \omega(G[R]), |I|, h_{P_3^*}(G[R] \wedge K_1)\}$. The proof only differs when there is no vertex of R in S^*. In this case, $|S^*| = |I|$ instead of $|I| + 1$

(edge P_3-Helly-independent M) The same argument for thin spiders holds.

(edge P_3^*-Helly-independent M^*) Recall that $h'_{P_3^*} = |V(G)| - s_t(G)$. Since G has no universal vertex, $s_t(G) \geq 2$. To partition the vertices of G in two vertex disjoint star subgraphs we take a vertex x of K as a star with $|K| + |R| + |I| - 2$ leaves and the vertex y of I no adjacent to x as an one vertex star. Therefore, $h'_{P_3^*}(G) = |V(G)| - 2$ where M^* is composed by the edges incidents to x.

4 $(q,q-4)$ graphs with separable p-component

(P_3-Helly-independent S) Since there is a join between the vertices of $G \setminus H$ and H_1, there are at most two vertices of $G \setminus H_2$ in S. Moreover, $|H| = |H_1 \cup H_2|$ is a constant q. Thus, there are $O(|V(G)|^2)$ combinations of at most two vertices in $G \setminus H_2$ and $O(2^q) = O(1)$ subsets of vertices of H_2. One can combine these two sets obtaining a new one with $O(2^q|V(G)|^2)$ elements. For each one, we test if the resulting combinations are indeed a P_3-Helly-independent. The size of the valid combination servers as a witness to a lower bound of the parameter $h_{P_3}(G)$. At the end, $h_{P_3}(G)$ is the largest size among these valid combinations.

(P_3^*-Helly-independent S^*) When there is no vertex of $G \setminus H$ in S^* we obtain M_1, the largest size of a subset of vertices of $H_1 \cup H_2$ that are P_3^*-Helly-independent, by testing all $O(2^q) = O(1)$ possible subsets. When there is no vertex of $H_1 \cup H_2$ in S^* we obtain M_2 as $h_{P_3^*}((G \setminus H) \wedge K_1)$ that are the size of a maximum induced complete bipartite graph of the $(q,q-4)$ graph $(G \setminus H)^c$ (the complement graph of $G \setminus H$). When there is no vertex of H_1 in S^*, since every vertex of H_2 is adjacent to a vertex of H_1, two non-adjacent vertices of $(G \setminus H) \in S^*$ and a vertex of $H_2 \in S^*$ or two non-adjacent vertices of $H_2 \in S^*$ and a vertex of $(G \setminus H) \in S^*$ would imply a **Forbidden 1** configuration. Then, we obtain $M_3 = \omega(G \wedge H) + \omega(H_2)$. Otherwise, we test all possible subsets of H with at least one vertex of H_1, for each of them we add a maximum clique of $G \wedge H$ in S^* and verify if they are a valid P_3^*-Helly-independent, in the end, we obtain M_4 as the maximum among their sizes. Therefore, $h_{P_3^*}(G) = \max\{M_1, M_2, M_3, M_4\}$.

(edge P_3-Helly-independent M) If the edges of M have no endpoint in $G \setminus H$, it is easy to test which $O(2^{q^2})$ subsets of edges of H are edge P_3-Helly-independent of G and take the largest size among them as M_5. Otherwise, due to the nature of the p-separable components, the edge P_3-convex hull of the following sets of edges reaches all edges of G: (i) sets that contains two edges e_1 and e_2 of M, where $e_1 = uv$ with $u \in G \setminus H$ and $v \in G \setminus H_2$, and $e_2 = xy$ with $x \in H_1$ and $y \in H$; (ii) sets that contains edges e_1 and e_2 with both endpoints in $G \setminus H$ and there is another edge (not necessarily in the set) which shares one endpoint with e_1 and the other with e_2; (iii) sets that contains an edge e_1 with an endpoint in $G \setminus H$ and two edges e_2 and e_3 with both endpoints in H_2 such that there exits another edge e_4 (not necessarily in the set) which shares one endpoint with e_2 and the other with e_3. Therefore, $h'_{P_3}(G) = \max\{2, \beta^*(G \wedge H) + \beta^*(G[H_2]), M_5\}$.

(edge P_3^*-Helly-independent M^*) There are $O(2^{q^2}) = O(1)$ subsets of edges of H. For each one of these, we can test if they are a partition of the vertices in stars. When there is a star centered in H_1 with degree more than two or a star with degree one or zero in H_1 not adjacent to a star with degree more than two in H_2, we can extend this star and add all edges between its center and the vertices of $G \setminus H$, considering the sum of these two values as a lower bound to $h'_{P_3^*}(G)$. Otherwise, the lower bound is given by the sum of the number of edges of this set and $h'_{P_3^*}(G \setminus H)$. At the end, $h'_{P_3^*}(G)$ is given by the largest size among these sets.

5 Final Remarks

In this work we manage to show that VP3HI, VSP3HI, EP3HI, and ESP3HI are in $\mathcal{P}$ for $(q, q-4)$ graphs with fixed q. Our approach to accomplish this directly lies on the structural characterization of $(q, q-4)$ graphs given by Babel and Olariu [1]. Particularly, $(4, 0)$ graphs are also known as *cographs* and $(5, 1)$ are the P_4*-sparse* [1]. As a consequence, VP3HI, VSP3HI, EP3HI, and ESP3HI are in $\mathcal{P}$ for cographs and P_4-sparse. We invite the readers to check the modifications required to adapt our algorithm for $(q, q-4)$ graphs to P_4-tidy, a superclass of P_4-sparse. Informally, we only need to deal with one new case, the *quasi-spiders* (that are spiders for which we can add one true twin or false twin to one vertex of K or I). Such adaptation is quite natural for P_4-tidy, but it is not so obvious for others superclasses of P_4-lite (the P_4-tidy which are also perfect graphs). Therefore, further investigations are required for the following hierarchy of nested superclasses of P_4-lite: P_4-laden, split-perfect, and brittle.

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