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On the Laplacian coflow of invariant G_2-structures and its solitons

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Abstract. In this work, we approach the Laplacian coflow of a coclosed G_2-structure φ using the formulae for the irreducible G_2-decomposition of the Hodge Laplacian and the Lie derivative of the Hodge dual 4-form of φ. In terms of this decomposition, we characterize the conditions for a vector field as an infinitesimal symmetry of a coclosed G_2-structure, as well as the soliton condition for the Laplacian coflow. More specifically, we provide an easier proof for the absence of compact shrinking solitons of the Laplacian coflow. Moreover, we revisit the Laplacian coflow of coclosed G_2-structures on almost Abelian Lie groups addressed by Fino-Bagaglini [3]. However, our approach is based on the bracket flow point of view. Notably, by showing that the norm of the Lie bracket is strictly decreasing, we prove that we have long-time existence for any coclosed Laplacian coflow solution.

Keywords: G_2-structures, geometric flows, Lie groups.

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1 Introduction

A G_2-structure is defined by a positive 3-form φ, which, in turn, defines the metric g and the corresponding Hodge dual 4-form $\psi := *\varphi$. The main goal in G_2-geometry is the study of *torsion-free* G_2-structures, i.e. $\nabla\varphi = 0$, which is equivalent to the *closed* $d\varphi = 0$ and the *coclosed* condition $d\psi = 0$ (e.g [10]). Using Ricci flow ideas, Bryant introduced the *Laplacian flow* of closed G_2-structures [5], which is an evolution of an initial closed G_2-structure along its Hodge Laplacian, namely

$$\frac{\partial\varphi(t)}{\partial t} = \Delta_t\varphi(t), \quad \varphi(0) = \varphi. \tag{1.1}$$

The Laplacian flow is not parabolic, however, when the initial condition is closed, the flow (1.1) preserves the closed condition and it evolves as a Ricci-like flow on Ω^3. It allows to use DeTurck's trick and, then, the Laplacian flow becomes parabolic in the direction of closed forms. In [6], Bryant and Xu addressed this approach in order to prove the short-time existence of (1.1).

Motivated by Bryant and Xu ideas on the Laplacian flow of closed G_2-structures, Karigiannis, McKay and Tsui introduced the *Laplacian coflow* of coclosed G_2-structures in [16]. It means that, instead of considering the heat flow equation for φ, they deal with the flow:

$$\frac{\partial\psi(t)}{\partial t} = \Delta_t\psi(t), \quad \psi(0) = \psi. \tag{1.2}$$

Equally to the Laplacian flow, if the initial condition satisfies $d\psi = 0$, the flow (1.2) preserves the coclosed condition. On one side, the Laplacian coflow is interesting, because coclosed G_2-structures exist in any (compact and non-compact) spin and orientable 7-manifold by a parametric h-principle (see [7]). Unfortunately, the analytic approach employed for the Laplacian flow does not apply in the case (1.2), since it is not parabolic in the direction of the coclosed forms. Hence, the short-time existence of the Laplacian coflow is still an open problem. Nevertheless, in [13], Grigorian proposed a modification of (1.2) fixing the failure of the Laplacian coflow to be parabolic, specifically the *modified Laplacian coflow* of coclosed G_2-structures is the evolution given by

$$\frac{\partial\psi}{\partial t} = \Delta_t\psi(t) + 2\mathrm{d}((A - \mathrm{tr}_{g(t)} T(t))\varphi(t)), \quad \text{for} \quad A > 0. \tag{1.3}$$

However, the critical points of (1.3) are no longer torsion-free G_2-structures. For instance, if φ is a *nearly parallel* G_2-structure, i.e. $\mathrm{d}\varphi = 4\psi$, the left hand side of (1.3) vanishes for $A = 5$. So, despite the fact that the modified Laplacian coflow can be seen as a tool for improving the torsion of φ, it does not search only for the torsion-free ones.

Regardless of the absence of an analytical theory of the Laplacian coflow in the general setting, the flow (1.2) had received the attention of some authors for manifolds with either a symmetry or an additional geometrical structure. For instance:

Assuming short-time existence and uniqueness of (1.2), in [16], Karigiannis, McKay and Tsui studied soliton solutions on warped products of a circle or an interval with a compact 6-manifold N with an SU(3)-structure $(\omega, \mathrm{Re}(\Omega))$. Running the Laplacian coflow among cohomogeneity-one solutions, when $(N, \omega, \mathrm{Re}(\Omega))$ is a Calabi-Yau manifold, they proved that the unique soliton solutions on the warped product are the steady ones. In particular, in the compact case, the soliton solutions are given by translations and phase rotations of the standard torsion-free G_2-structure.

Furthermore, in [25], Manero, Otal and Villacampa consider the Laplacian coflow on a warped product of the form $M^7 = M^6 \times_f S^1$, with M^6 being a compact 6-manifold endowed with an SU(3)-structure. They provide conditions for the existence of this flow using the torsion forms related to the SU(3)-structure and the warping function f. Furthermore, they analyze the Laplacian coflow when the base is endowed with a nearly kähler, symplectic half-flat, or balanced SU(3)-structure and provide some examples of solutions of the Laplacian coflow.

In [23], Lotay, Sá Earp and Saavedra proved the existence of a family of G_2-structures on a contact Calabi-Yau manifold by solving the Laplacian coflow, choosing $\varepsilon \in \mathbb{R}^*$ and initial data $\varphi = \varepsilon\eta \wedge \omega + \mathrm{Re}(\Upsilon)$, which is coclosed and the solution exists in $t \in (-\frac{1}{10\varepsilon^2}, \infty)$. We recall that a contact Calabi-Yau manifold is a Sasakian manifolds (M, ξ, η, Φ) with a contact Calabi-Yau structure $(\omega := \mathrm{d}\eta, \mathrm{Re}(\Upsilon))$, where η is a contact form, ξ the Reeb vector field, Φ is a $(1,1)$-endomorphism and Υ is a basic holomorphic $(3,0)$-form on $\mathcal{D} = \ker \eta$ related to the almost complex structure $\Phi|_{\mathcal{D}} = J$. Hence, the solution of the Laplacian coflow is immortal with a finite singularity at $t = -\frac{1}{10\varepsilon^2}$. It was the first example of a compact solution to the Laplacian coflow which had an infinite time type IIB singularity.

On 3-Sasakian manifolds there exist two non-equivalent nearly parallel G_2-structures [12], moreover, using the natural $\mathrm{SU}(2)$-action there is a 4-parameter family of coclosed G_2-structures (up to sign), which contains the nearly parallel ones. Under a special ansatz of this family of coclosed G_2-structures, Kennon and Lotay proved that any solution of the Laplacian coflow starting at a coclosed G_2-structure converges, after rescaling, to one of the nearly parallel G_2-structures in the same family of the initial data [18]. In particular, the nearly parallel G_2-structures are both stable within their families.

On the other hand, when $M = G/H$ is a homogeneous space and the solutions of (1.2) are required to be G-invariant, the Laplacian coflow becomes an ordinary differential equation. Namely, let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ a *reductive decomposition* (i.e. $\mathrm{Ad}(H)$-invariant), any G-invariant solution of (1.2) on M is determined by an $\mathrm{Ad}(K)$-invariant 4-form $\psi(t)$ on $\mathfrak{m} \simeq T_oM$ (where $o = 1_G H$). Then, since $\Delta\psi$ is invariant by diffeomorphisms of M, the flow (1.2) restricted to G-invariant solutions is equivalent with:

$$\frac{d}{dt}\psi(t) = \Delta_{\psi(t)}\psi(t) \quad \text{for} \quad \psi(t) \in \left(\Lambda^4\mathfrak{m}^*\right)^{\mathrm{Ad}(H)}. \tag{1.4}$$

Hence, short-time existence and uniqueness of (1.4) are followed by the well known ODE arguments, since the linear map Δ on $\Lambda^4\mathfrak{m}^*$ is continuous. For instance, in [17], Kath and Lauret obtained expanding solitons and immortal solutions of the Laplacian coflow when M is the connected and simply connected Lie group with Lie algebra $\mathfrak{a} \ltimes \mathbb{R}^4$, where $\mathfrak{a}$ is any maximal

$\mathbb{R}$-split torus of $\mathfrak{sl}(\mathbb{R}^4)$. The latest have been obtained using the bracket flow approach (see [20] for a deep exposition of this method). Conversely, using a direct method, Bagaglini, Fernández and Fino obtained explicit immortal solutions of (1.4) when M is the 7-dimensional Heisenberg group [2]. In [3], Bagaglini and Fino gave explicit immortal solutions and solitons of the Laplacian coflow for a subclass of almost Abelian Lie groups.

In this work, we study the Laplacian coflow of invariant coclosed G_2-structures. In order to do so, in Section 2, we provide some preliminaries on coclosed G_2-structures to establish the notation that is going to be used for the rest of the paper. In Section 3, we recall the definition of the Laplacian coflow of coclosed G_2-structures and its soliton solutions. Specifically for the parameter $\lambda \in \mathbb{R}$ and the vector field $X \in \mathscr{X}(M)$, such that φ satisfies the soliton equation (3.3). In Proposition 7, we characterize the soliton condition in terms of the full torsion tensor T and the Ric tensor of φ. As a direct consequence, we give in Corollary 9 an alternative proof for the non-existence of compact shrinking solitons of the Laplacian coflow.

Finally, in Section 4, we address the Laplacian coflow of invariant coclosed G_2-structures on almost Abelian Lie groups G_A, with Lie algebra $\mathfrak{g}_A$ and Lie bracket determined by $A \in \mathfrak{gl}(\mathbb{R}^6)$. Using the bracket flow, we write the Laplacian coflow (1.4) as the ODE (4.10) of $A \in \mathfrak{gl}(\mathbb{R}^6)$. As an immediate consequence, we prove that any Laplacian coflow solution $(\mathfrak{g}_A, \varphi(t))$ starting at any coclosed (non-flat) G_2-structure is immortal (see Theorem 20). In spite of not obtaining explicit solutions of (1.4) as it has been done in [3] for a subclass of almost Abelian Lie algebras, Theorem 20 generalizes the result of long-time existence of solutions for any almost Abelian Lie algebra. Moreover, the ODE bracket flow (4.10) allows us to study the dynamical behavior of the 2-parameter family

$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & -B^t \end{array}\right] \quad \text{with} \quad B = \begin{bmatrix} 0 & x & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad x, y \in \mathbb{R}$$

showing that it is stable under the Laplacian coflow (see Example 22). To conclude, we study the invariant solitons of the Laplacian coflow in terms of the Lie bracket induced by $A \in \mathfrak{gl}(\mathbb{R}^6)$, satisfying the time independent equation (see Theorem 24)

$$[A, A^t] + S_A \circ_6 S_A = -\Big(\operatorname{tr} S_A^2 - \frac{1}{2}(\operatorname{tr} JA)^2 + 2d\Big) I_6 + (D + D^t)|_{\mathbf{R}^6},$$

where $\circ_6$ is the product on $\mathfrak{gl}(\mathbf{R}^6)$ defined in Lemma 12, D is a derivation of $\mathfrak{g}_A$ and

$$d = \frac{|[A, A^t]|^2 + \langle S_A \circ_6 S_A, [A, A^t]\rangle}{2|A|^2}.$$

As an application of the Theorem 24 , firstly we prove if A is skew-symmetric, then $(\mathfrak{g}_A, \varphi)$ defines a a semi-algebraic soliton of the Laplacian coflow (see Corollary 25). Secondly, we prove that any (non-flat) semi-algebraic soliton on an almost Abelian Lie group is an expanding one (Proposition 26). Finally, as far as we know it, we provide the first example of a semi-algebraic soliton of the Laplacian coflow, which is not algebraic (Example 28).

Note

Fino and Bagaglini [3] have substantial overlap with this paper. However, while a number of conclusions are similar, the point of view on the Laplacian coflow is different. In this paper, we use the bracket flow introduced by Lauret in [20], while in [3], a more traditional geometric flow approach is used. Both approaches are valuable and complementary, since they provide different perspectives on the same phenomenon. Since we are studying the same flow in the same space, we want to emphasise that this paper has different techniques, and both papers will give a better understanding of the Laplacian coflow.

Notation

Let (M, g) be a smooth oriented Riemannian 7-manifold. We use the Einstein summation convention throughout. We compute in a local orthonormal frame, so all indices are subscripts and any repeated indices are summed over all values from 1 to 7. A differential k-form α on M will be written as

$$\alpha = \frac{1}{k!}\alpha_{i_1 i_2 \cdots i_k}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

in local coordinates $\left(x^1, \ldots, x^7\right)$, where $\alpha_{i_1 i_2 \cdots i_k}$ is completely skew-symmetric in its indices. With this convention, the interior product $\partial_m \lrcorner\, \alpha$ of α with a coordinate vector field ∂_m is the $(k-1)$-form

$$\partial_m \lrcorner\, \alpha = \frac{1}{(k-1)!}\alpha_{m i_1 i_2 \cdots i_{k-1}}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_{k-1}}.$$

The metric g on a Riemannian manifold M induces a metric on k-forms, such that the inner product of α and β is

$$g(\alpha, \beta) = \frac{1}{k!}\alpha_{i_1 \cdots i_k}\beta_{j_1 \ldots j_k} g^{i_1 j_1} \ldots g^{i_k j_k}.$$

The Levi-Civita connection associated to g is denoted by ∇, and its Christoffel symbols by Γ^k_{ij}. We write ∇_i for covariant differentiation in the ∂_i direction. If $T_{i_1 \cdots i_k}$ is a tensor of type $(0, k)$, then $\nabla_m T_{i_1 \cdots i_k}$ always means $\left(\nabla_m T\right)_{i_1 \cdots i_k}$. We write the exterior derivative $\mathrm{d}\alpha$ of a k-form α as

$$\mathrm{d}\alpha = \frac{1}{k!}\left(\nabla_m \alpha_{i_1 \cdots i_k}\right) \mathrm{d}x^m \wedge \mathrm{d}x^{i_1} \cdots \wedge \mathrm{d}x^{i_k}$$

in terms of the covariant derivative. The metric g defines an isomorphism between TM and T^*M (raising and lowering indices.) If v is a vector field, then the metric dual 1-form $v^\flat$ is defined by $v^\flat(w) = g(v, w)$. In coordinates, $\left(\partial_i\right)^\flat = g_{ik}\, \mathrm{d}x^k$. Similarly, the 1-form α has a metric dual vector field $\alpha^\sharp$, and $\left(\mathrm{d}x^i\right)^\sharp = g^{ik}\partial_k$.

We use 'vol' to denote the volume form on M associated to the metric g and an orientation. The Hodge star operator $*$ taking k-forms to $(7-k)$-forms is defined by

$$\alpha \wedge *\beta = g(\alpha, \beta)\mathrm{vol}.$$

Our convention for labelling the Riemann curvature tensor is

$$R_{ijkm}\frac{\partial}{\partial x^m} = \left(\nabla_i \nabla_j - \nabla_j \nabla_i\right)\frac{\partial}{\partial x^k}$$

in terms of coordinate vector fields. With this convention, the Ricci tensor is $R_{jk} = R_{ljkl}$ and the first Bianchi identity of the Riemann curvature tensor is:

$$R_{abmn} + R_{amnb} + R_{anbm} = 0. \tag{1.5}$$

We use $\Gamma(E)$ to denote the space of smooth sections of a vector bundle E. As special instances, we denote the following cases as:

- $\Omega^k := \Gamma\left(\Lambda^k\left(T^*M\right)\right)$ is the space of smooth k-forms on M;
- $\mathcal{S} := \Gamma\left(\mathrm{S}^2\left(T^*M\right)\right)$ is the space of smooth symmetric 2-tensors on M.
- $\mathscr{X}(M) := \Gamma(TM)$ the space of vector fields.

With respect to the metric g on M, we use $\mathcal{S}_0$ to denote those sections h of $\mathcal{S}$ that are traceless. That is, $\mathcal{S}_0$ consists of those sections of $\mathcal{S}$, such that $\operatorname{Tr} h = g^{ij}h_{ij} = 0$ in local coordinates. Then $\mathcal{S} \simeq \Omega^0 \oplus \mathcal{S}_0$, where $h \in \mathcal{S}$ is decomposed as $h = \frac{1}{7}(\operatorname{Tr} h)g + h_0$. Then, we have $\Gamma\left(T^*M \otimes TM\right) = \Omega^0 \oplus \mathcal{S}_0 \oplus \Omega^2$, where the splitting is pointwise orthogonal with respect to the metric on $T^*M \otimes TM$ induced by g.

2 Preliminaries

In this section we collect some results related to G_2-structures that will be needed in the present paper. Any result of this section can be found in [14, 13, 5].

2.1 G_2-structures and their torsion

A G_2*-structure* on a 7-manifold M is given by a differential 3-form φ on M, which is pointwise isomorphic to the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3(\mathbb{R}^7)^*,$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $\{e^1, \dots, e^7\}$ is the dual basis of the canonical basis of $\mathbb{R}^7$. The G_2-structure φ determines a Riemannian metric g_φ and a volume form vol_φ so that

$$6g_\varphi(X, Y)\mathrm{vol}_\varphi = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi \quad \text{for} \quad X, Y \in \mathscr{X}(M).$$

In addition, φ induces a Hodge star operator $*_\varphi$ and we denote its dual 4-form by $\psi = *_\varphi \varphi$. For simplicity, we will write $g = g_\varphi$ and $* = *_\varphi$. A G_2-structure gives rise to a decomposition of the space of differential k-forms Ω^k on M into irreducible G_2-submodules. For instance,

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14} \quad \text{and} \quad \Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},$$

where Ω^k_l has (pointwise) dimension l. In [5], R. Bryant defines an injective map $\mathrm{i}_\varphi : \mathcal{S}^2 \to \Omega^3$, given in local coordinates $x^1, \dots, x^7$ by

$$\mathrm{i}_\varphi(h) = \frac{1}{3!}\,\mathrm{i}_\varphi(h)_{ijk}\mathrm{d}x^{ijk} = \frac{1}{3!}(h_i^m \varphi_{mjk} + h_j^m \varphi_{imk} + h_k^m \varphi_{ijm})\mathrm{d}x^{ijk}, \quad (2.1)$$

where $h \in \mathcal{S}^2$ is a symmetric 2-tensor field on M. Additionally, the map i_φ is surjective on $\Omega^3_1 \oplus \Omega^3_{27}$ and its Hodge dual satisfies (e.g. [14, Proposition 2.8])

$$* \,\mathrm{i}_\varphi(h) = \frac{1}{4!}(\bar{h}_i^m \psi_{mjkl} + \bar{h}_j^m \psi_{imkl} + \bar{h}_k^m \psi_{ijml} + \bar{h}_l^m \psi_{ijkm})\mathrm{d}x^{ijkl} =: \mathrm{i}_\psi(\bar{h}), \quad (2.2)$$

where $\bar{h} = \frac{1}{4}\,\mathrm{tr}(h)g - h$. In particular, for any trace-free symmetric 2-tensors $h \in S^2_0$, we have $\mathrm{i}_\varphi(h) \in \Omega^3_{27}$ and $\mathrm{i}_\psi(h) \in \Omega^4_{27} = *\left(\Omega^3_{27}\right)$. According with the G_2-decomposition of Ω^4 and Ω^5, the exterior derivative of φ and ψ are completely described in term of the *torsion forms* $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^3_{27}$, given in terms of (see [5, Proposition 1])

$$\begin{aligned} \mathrm{d}\varphi &= \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3 \in \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27} \\ \mathrm{d}\psi &= 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi \in \Omega^5_7 \oplus \Omega^5_{14}. \end{aligned} \quad (2.3)$$

Moreover, for the *full torsion tensor* is defined locally by (see [14])

$$\nabla_i \varphi_{jkl} = T_i^m \psi_{mjkl}. \quad (2.4)$$

The full torsion tensor T is given in terms of the torsion forms by

$$T = \frac{\tau_0}{4} g - \tau_{27} - \tau_1^\sharp \lrcorner \varphi - \frac{1}{2}\tau_2,$$

where τ_{27} is the trace-free symmetric 2-tensor satisfying $\tau_3 = \mathrm{i}_\varphi(\tau_{27})$ and $\tau_1^\sharp$ denotes the unique vector field induced by τ_1 and the Riemannian metric g, (i.e. $g(\tau_1^\sharp, X) = \tau_1(X)$ for any $\in \mathscr{X}(M)$). In addition, from (2.4) for the 4-form ψ, we have

$$\nabla_m \psi_{ijkl} = -(T_{mi}\varphi_{jkl} - T_{mj}\varphi_{ikl} - T_{mk}\varphi_{jil} - T_{ml}\varphi_{jki}).$$

2.2 Properties of coclosed G_2-structures

A G_2-structure φ is *coclosed* if it satisfies $\mathrm{d}\psi = 0$, in terms of (2.3) the coclosed condition is equivalent with $\tau_1 = 0$ and $\tau_2 = 0$. Hence, the full torsion tensor of a coclosed G_2-structure simplifies to the symmetric 2-tensor

$$T = \frac{\tau_0}{4} g - \tau_{27} \in \mathcal{S}^2. \tag{2.5}$$

In addition, $\mathrm{d}\varphi \in \Omega^4_1 \oplus \Omega^4_{27}$ thus, by (2.1), (2.3) and (2.5), we have

$$\mathrm{d}\varphi = *\,\mathrm{i}_\varphi\left(\frac{1}{3}(\mathrm{tr}\, T) g - T\right). \tag{2.6}$$

The following proposition includes some well known identities of coclosed G_2-structures given in [13], obtained as a consequence of a general formula of the exterior derivative of a generic 3-form. Here, we give an alternative proof of those identities, using the called G_2-*Bianchi type identity*

$$\nabla_i T_{jk} - \nabla_j T_{ik} = \left(\frac{1}{2} R_{ijmn} - T_{im}T_{jn}\right) \varphi_k{}^{mn}, \tag{2.7}$$

where T_{ij} is the coordinate of (2.5) and R_{ijmn} denotes the Riemann curvature tensor. We remark that the identity (2.7) can be read as the infinitesimal version of the diffeomorphism invariance of T as a function of φ (see [14, Section 4] for an extensive discussion in the G_2-case and [9] for any H-structure). In the statement, for any $h, k \in \mathcal{S}^2$, we denote the inner product $\langle h, k\rangle$ and the circ product $h \circ k \in \mathcal{S}^2$ by

$$\langle h, k\rangle = h_{ij}k_{ab}g^{ia}g^{jb} \quad \text{and} \quad (h \circ k)_{ab} = \varphi_{amn}\varphi_{bpq}h^{mp}k^{nq}. \tag{2.8}$$

The divergence and the curl of h are given in coordinates by

$$\mathrm{div}h_a = \nabla_b h_a^b \quad \text{and} \quad \mathrm{Curl}h_{ab} = \nabla_m h_{an}\varphi_b{}^{mn}. \tag{2.9}$$

Proposition 1. *Let* φ *be a coclosed* G_2*-structure with full torsion tensor* T*, then the divergence and the curl of* T *satisfy*

$$\mathrm{div}T_a = \nabla_a \operatorname{tr} T \quad \textit{and} \quad \mathrm{Curl}T_{ab} = \mathrm{Curl}T_{ba}. \tag{2.10}$$

In addition, the Ricci tensor and the scalar curvature are

$$\mathrm{Ric} = -\mathrm{Curl}T - T^2 + (\operatorname{tr} T)T \quad \textit{and} \quad R = (\operatorname{tr} T)^2 - |T|^2. \tag{2.11}$$

Proof. Using (1.5) and the symmetries of R_{abmn}, it is easy to prove that

$$R_{abmn}\varphi^{bmn} = 0 \quad \text{and} \quad R_{amnp}\psi_b{}^{mnp} = 0. \tag{2.12}$$

Now, since T is symmetric, using (2.7) and (2.12) for the divergence T, we have

$$\mathrm{div}T_a = \nabla_b T_a^b = \nabla_a T_b^b + \left(\frac{1}{2}R_{bamn} - T_{am}T_{bn}\right)\varphi^{bmn} = \nabla_a \operatorname{tr} T,$$

and in addition, by (A.2) for the curl of T, we get

$$\begin{aligned}
\mathrm{Curl}T_{ab} - \mathrm{Curl}T_{ba} =& \nabla_m T_{an} {\varphi_b}^{mn} - \nabla_m T_{bn} {\varphi_a}^{mn} \\
=& \left(\frac{1}{2}R_{mapq} - T_{mp}T_{aq}\right) {\varphi_n}^{pq} {\varphi_b}^{mn} \\
& - \left(\frac{1}{2}R_{mbpq} - T_{mp}T_{bq}\right) {\varphi_n}^{pq} {\varphi_a}^{mn} \\
=& \left(\frac{1}{2}R_{mapq} - T_{mp}T_{aq}\right) \left(g^p_b g^{qm} - g^q_b g^{pm} + {\psi_b}^{mpq}\right) \\
& - \left(\frac{1}{2}R_{mbpq} - T_{mp}T_{bq}\right) \left(g^p_a g^{qm} - g^q_a g^{pm} + {\psi_a}^{mpq}\right) \\
=& \frac{1}{2}R_{mabq}g^{mq} - \frac{1}{2}R_{mapb}g^{mp} - T_{am}T^m_b + \mathrm{tr}(T)T_{ab} \\
& - \frac{1}{2}R_{mbaq}g^{mq} + \frac{1}{2}R_{mbpa}g^{mp} + T_{bm}T^m_a - \mathrm{tr}(T)T_{ba} \\
=& -\mathrm{Ric}_{ab} + \mathrm{Ric}_{ba} = 0.
\end{aligned}$$

The formula for Ric can be derived from the computation above and for the scalar curvature, it follows from the observation

$$\mathrm{Curl}T_{aa} = \nabla_m T_{an} {\varphi_a}^{mn} = 0.$$

□

Similar to [5]*Corollary 2 for the case of closed G_2-structures, we can characterize the Einstein metrics induced by a coclosed G_2-structure:

Corollary 2. *A coclosed G_2-structure φ induces an Einstein metric if and only if the full torsion tensor satisfies*

$$\mathrm{i}_\varphi(\mathrm{Curl}T) = \frac{3}{7}|T|^2\varphi - (\mathrm{tr}\,T)\tau_3 - \mathrm{i}_\varphi(T^2). \tag{2.13}$$

Proof. The result follows by applying the map i_φ in (2.11). □

Remark 3. Using the expression of the full torsion tensor in terms of the torsion forms (2.5), the equation (2.13) becomes

$$\mathrm{i}_\varphi(\mathrm{Curl}\tau_{27}) = \frac{3}{7}|\tau_{27}|^2\varphi - \frac{5\tau_0}{4}\tau_3 - \mathrm{i}_\varphi(\tau_{27}^2). \tag{2.14}$$

It is well know that a metric induced by the nearly parallel G_2-structure (i.e. $\tau_3 = \mathrm{i}_\varphi(\tau_{27}) = 0$) is Einstein. It is easy to check that (2.14) is satisfied trivially for a nearly G_2-structure.

3 Laplacian coflow of G_2-structures

In this section, we recall the definition of the Laplacian coflow and we also study soliton solutions and symmetries of coclosed G_2-structure. Here we follow [16, 13].

Definition 4. A time-dependent family of G_2-structures $\{\varphi(t)\}_{t\in(\varepsilon_1,\varepsilon_2)}$ on a 7-manifold M, satisfy the Laplacian coflow of coclosed G_2-structures, if for any $t \in (\varepsilon_1, \varepsilon_2)$ we have

$$\frac{\partial}{\partial t}\psi(t) = \Delta_t\psi(t) \quad \text{and} \quad \mathrm{d}\psi(t) = 0, \tag{3.1}$$

where $\psi(t) = *_t\varphi(t)$ and $\Delta_t = \mathrm{dd}^{*_t} + \mathrm{d}^{*_t}\mathrm{d}$ is the Hodge Laplacian with respect to the metric $g(t) = g_{\varphi(t)}$.

As for many geometric flows, we are interested in considering *self-similar solutions*,

$$\varphi(t) = \lambda(t)f(t)^*\varphi \quad \text{where} \quad \lambda(t) \in C^\infty(M) \quad \text{and} \quad f(t) \in \mathrm{Diff}(M), \tag{3.2}$$

This means that the solution $\varphi(t)$ evolves from the initial data φ, by a scaling with the function $\lambda(t)$ and by pullback with the diffeomorphism $f(t)$. Since this kind of solutions are expected to be related to singularities of the flow. In particular, self-similar solutions with initial condition φ are equivalent with a time independent equation of $\psi = *\varphi$, called the *soliton equation*, namely, φ is called a *soliton* for the Laplacian coflow (3.1), if ψ satisfies the soliton equation:

$$\Delta_\psi\psi = \mathcal{L}_X\psi + \lambda\psi \tag{3.3}$$

where $\lambda \in \mathbb{R}$ and X is a complete vector field on M. Moreover, the soliton (φ, λ, X) is called *expanding, steady*, or *shrinking*, if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

The following lemma, decomposes the Hodge Laplacian of ψ according to the G_2-irreducible decomposition of Ω^2, it appeared originally in [13]*Proposition 4.6. Here, we provide the computations in detail so that the work is self-contained. We follow the computation given in [24] for $\Delta_\varphi \varphi$ in the closed case.

Lemma 5. *Let φ be a coclosed G_2-structure on a manifold M with associated metric g. Then,*

$$\Delta_\psi \psi = \frac{2}{7}((\operatorname{tr} T)^2 + |T|^2)\psi \oplus (\mathrm{d} \operatorname{tr} T) \wedge \varphi$$
$$\oplus *_\varphi \mathrm{i}_\varphi \Big(\mathrm{Ric} - \frac{1}{2} T \circ T - (\operatorname{tr} T) T + \frac{1}{14} \left((\operatorname{tr} T)^2 + |T|^2 \right) g \Big) \in \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}.$$

Proof. Since $\mathrm{d}\psi = 0$, by (2.6) we have

$$\begin{aligned} \Delta_\psi \psi = \mathrm{d}\mathrm{d}^* \psi = \mathrm{d} * \mathrm{d}\varphi = \mathrm{d}\beta \quad \text{where} \quad \beta :=& \mathrm{i}_\varphi(h) \\ =& \mathrm{i}_\varphi \left(\frac{1}{3} (\operatorname{tr} T) g - T \right) \in \Omega^3_{27}. \end{aligned} \tag{3.4}$$

In local coordinates, we can write (3.4) as

$$\Delta_\psi \psi = \frac{1}{4!} (\Delta_\psi \psi)_{ijkl} dx^{ijkl},$$

where

$$(\Delta_\psi \psi)_{ijkl} = \nabla_i \beta_{jkl} - \nabla_j \beta_{ikl} + \nabla_k \beta_{ijl} - \nabla_l \beta_{ijk}. \tag{3.5}$$

We can decompose $\Delta_\psi \psi$ into irreducible summands as

$$\Delta_\psi \psi = a\psi + X^\flat \wedge \varphi + * \mathrm{i}_\varphi(s),$$

where $a \in C^\infty(M)$, X a vector field and s is a trace-less symmetric 2-tensor. Now, we compute the expression of a, X and s in terms of the full torsion tensor of φ. For a, using (3.4), (3.5), (A.3) and (A.5), we have

$$\begin{aligned}
a = \frac{1}{7}\langle \Delta_\psi, \psi\rangle =& \frac{1}{168}(\nabla_i\beta_{jkl} - \nabla_j\beta_{ikl} + \nabla_k\beta_{ijl} - \nabla_l\beta_{ijk})\psi^{ijkl} \\
=& \frac{1}{42}\nabla_i(h_j^m\varphi_{mkl} + h_k^m\varphi_{jml} + h_l^m\varphi_{jkm})\psi^{ijkl} \\
=& \frac{1}{14}(\nabla_i h_j^m\varphi_{mkl} + h_j^m T_i^n\psi_{nmkl})\psi^{ijkl} \\
=& \frac{2}{7}(\nabla_i h_j^m {\varphi_m}^{ij} + \operatorname{tr} h \operatorname{tr} T - \langle h, T\rangle) \\
=& \frac{2}{7}\left((\operatorname{tr} T)^2 + |T|^2\right),
\end{aligned}$$

where h is the symmetric 2-tensor given in (3.4). For the vector field X, we have

$$\langle \Delta_\psi\psi, e^m\wedge\varphi\rangle = *(X^\flat\wedge\varphi\wedge *(e^m\wedge\varphi)) = 4\langle X^\flat, e^m\rangle = 4X_n g^{nm}.$$

Thus, using (3.4), (A.2), (A.1), (A.3) and (2.10), we get

$$\begin{aligned}
X_m =& \frac{1}{4}\langle \Delta_\psi\psi, e^n\wedge\varphi\rangle g_{mn} \\
=& \frac{1}{98}(\nabla_i\beta_{jkl} - \nabla_j\beta_{ikl} + \nabla_k\beta_{ijl} - \nabla_l\beta_{ijk})(e^n\wedge\varphi)^{ijkl} g_{mn} \\
=& \frac{1}{4!}(\nabla_m\beta_{jkl}\varphi^{jkl} - 3\nabla_j\beta_{mkl}\varphi^{jkl}) \\
=& \frac{1}{4!}(\nabla_m(\beta_{jkl}\varphi^{jkl}) - \beta_{jkl}\nabla_m\varphi^{jkl} - 3\nabla_j(\beta_{mkl}\varphi^{jkl}) + 3\beta_{mkl}\nabla_j\varphi^{jkl}) \\
=& \frac{1}{4!}\Big(3\nabla_m(h_j^n\varphi_{njk}\varphi^{jkl}) - 3h_j^n\varphi_{nkl}T_{mp}\psi^{pjkl} - 3\nabla_j(h_m^n\varphi_{nkl}\varphi^{jkl} \\
&+ 2h_k^n\varphi_{mnl}\varphi^{jkl})\Big) \\
=& \frac{1}{8}\left(6\nabla_m h_j^n g_n^j - 4h_j^n T_{mp}{\varphi_n}^{pj} - 6\nabla_j h_m^j - 2\nabla_m h_k^n g_n^k + 2\nabla_n h_m^n\right) \\
=& \frac{1}{2}\left(\frac{4}{3}\nabla_m(\operatorname{tr} T) - \frac{1}{3}\nabla_m(\operatorname{tr} T) + \nabla_j T_m^j\right) = (\operatorname{div} T)_m.
\end{aligned}$$

Finally, to find the symmetric 2-tensor s, we have:

$$\begin{aligned}
&(\Delta_\psi\psi)_{imnp}{\psi_j}^{mnp} + (\Delta_\psi\psi)_{jmnp}{\psi_i}^{mnp} \\
&= a(\psi_{imnp}{\psi_j}^{mnp} + \psi_{jmnp}{\psi_i}^{mnp}) + (*\mathrm{i}_\varphi(s))_{imnp}{\psi_j}^{mnp} + (*\mathrm{i}_\varphi(s))_{jmnp}{\psi_i}^{mnp},
\end{aligned} \tag{3.6}$$

Then, using (2.2),(A.5) and (A.6), we get

$$\begin{aligned}(*\mathrm{i}_\varphi(s))_{imnp}\psi_j{}^{mnp} =& - s_i^q\psi_{qmnp}\psi_j{}^{mnp} - 3s_m^q\psi_{iqnp}\psi_j{}^{mnp}\\ =& - 24s_i^q g_{qj} - 3s_m^q(4g_{ij}g_q^m - 4g_i^m g_{qj} + 2\psi_{iqj}{}^m) = -12s_{ij}.\end{aligned}$$

By symmetry, the right hand side of (3.6) becomes

$$(\Delta_\psi\psi)_{imnp}\psi_j{}^{mnp} + (\Delta_\psi\psi)_{jmnp}\psi_i{}^{mnp} = 24\,(2ag_{ij} - s_{ij})\,. \tag{3.7}$$

Now, using (3.5), (A.3), (A.4) and (A.5), we have

$$\begin{aligned}&(\Delta_\psi\psi)_{imnp}\psi_j{}^{mnp} = (\nabla_i\beta_{mnp} - 3\nabla_m\beta_{inp})\psi_j{}^{mnp}\\ =&3(\nabla_i h_m^q\varphi_{qnp} + h_m^q\nabla_i\varphi_{qnp})\psi_j{}^{mnp} - 3\nabla_m((h_i^q\varphi_{qnp} + 2h_n^q\varphi_{iqp})\psi_j{}^{mnp})\\ &+ 3(h_i^q\varphi_{qnp} + 2h_n^q\varphi_{iqp})\nabla_m\psi_j{}^{mnp}\\ =&3\left(4\nabla_i h_m^q\varphi_{qj}{}^m + h_m^q T_i^l(4g_{lj}g_q^m - 4g_l^m g_{qj} + 2\psi_{lqj}{}^m) - 4\nabla_m(h_i^q\varphi_{qj}{}^m)\right.\\ &+2\nabla_m(h_i^q\varphi_{qj}{}^m - h_{nj}\varphi_i{}^{mn} - h_n^m\varphi_{ji}{}^n - \operatorname{tr}h\varphi_{ij}{}^m)\\ &\left.+(h_i^q\varphi_{qnp} + 2h_n^q\varphi_{iqp})(-T_{mj}\varphi^{mnp} + \operatorname{tr}T\varphi_j{}^{np} - T_m^n\varphi_j{}^{mp} + T_m^p\varphi_j{}^{mn})\right)\\ =&6\,(2(\operatorname{tr}hT_{ij} - T_i^m h_{mj}) - \nabla_m h_i^q\varphi_{qj}{}^m - \nabla_m h_{nj}\varphi_i{}^{mn} - \nabla_m(h_n^m\varphi_{ji}{}^n + \operatorname{tr}h\varphi_{ij}{}^m)\\ &-3h_i^m T_{mj} - \operatorname{tr}hT_{ij} + h_i^m T_{mj} + 3\operatorname{tr}Th_{ij} + \operatorname{tr}T\operatorname{tr}hg_{ij} - \operatorname{tr}Th_{ij} - \operatorname{tr}Th_{ij}\\ &+h_i^m T_{mj} - T_m^n h_n^m g_{ij} + T_i^m h_{mj} - (T\circ h)_{ij})\\ =&6\,(\operatorname{tr}hT_{ij} - T_i^m h_{mj} - (\mathrm{Curl}h)_{ij} - (\mathrm{Curl}h)_{ji} - \nabla_m(h_n^m\varphi_{ji}{}^n + \operatorname{tr}h\varphi_{ij}{}^m)\\ &-h_i^m T_{mj} + \operatorname{tr}Th_{ij} + (\operatorname{tr}T\operatorname{tr}h - \langle T,h\rangle)g_{ij} - (T\circ h)_{ij})\,.\end{aligned}$$

Thus, replacing $h = \frac{1}{3}(\operatorname{tr}T)g - T$ in the above expression and using (2.10), the left hand side of (3.6) becomes

$$\begin{aligned}&(\Delta_\psi\psi)_{imnp}\psi_j{}^{mnp} + (\Delta_\psi\psi)_{jmnp}\psi_i{}^{mnp}\\ &= 24\left(T_i^m T_{mj} + (\mathrm{Curl}T)_{ij} + \frac{1}{2}((\operatorname{tr}T)^2 + |T|^2)g_{ij} + \frac{1}{2}(T\circ T)_{ij}\right).\end{aligned}$$

Finally, from (3.7), we obtain

$$s_{ij} = -(\mathrm{Curl}T)_{ij} - T_i^m T_{mj} - \frac{1}{2}(T\circ T)_{ij} + \frac{1}{14}((\operatorname{tr}T)^2 + |T|^2)g_{ij}.$$

□

Similar to the Laplacian of ψ, we can compute the decomposition of the Lie derivative with respect to any vector field. We recall that the vector field X is called an *infinitesimal symmetry* of ψ, if $\mathcal{L}_X\psi = 0$. The next result was done in [8] for the 3-form φ.

Proposition 6. *Let φ be a coclosed G_2-structure on M^7, with associated metric g, and let X be a vector field on M. Then, if $\psi = *\varphi$,*

$$\mathcal{L}_X\psi = \frac{4}{7}(\mathrm{div}X)\psi \oplus (-\frac{1}{2}\mathrm{Curl}X + X\lrcorner T)^\flat \wedge\varphi \oplus *\mathrm{i}_\varphi\Big(\frac{1}{7}(\mathrm{div}X)g - \frac{1}{2}(\mathcal{L}_X g)\Big) \in \Omega^4_1\oplus\Omega^4_7\oplus\Omega^4_{27}. \tag{3.8}$$

In particular, X is an infinitesimal symmetry of ψ if and only if X is a Killing vector field of g and satisfies $\mathrm{Curl}(X) = 2X\lrcorner T$.

Proof. Since φ is coclosed, i.e. $\mathrm{d}\psi = 0$, we have

$$\mathcal{L}_X\psi = \mathrm{d}(X\lrcorner\psi) + X\lrcorner\mathrm{d}\psi = \mathrm{d}(X\lrcorner\psi).$$

Let $\alpha = X\lrcorner\psi$, so that locally $\alpha_{ijk} = X^l\psi_{lijk}$ and

$$(\mathcal{L}_X\psi)_{ijkl} = (\mathrm{d}\alpha)_{ijkl} = \nabla_i\alpha_{jkl} - \nabla_j\alpha_{ikl} + \nabla_k\alpha_{ijl} - \nabla_l\alpha_{ijk}.$$

Denoting by $\pi^k_l : \Omega^k \to \Omega^k_l$ the orthogonal projections, we decompose $\mathcal{L}_X\psi$ as

$$\mathcal{L}_X\psi = \pi^4_1(\mathcal{L}_X\psi) + \pi^4_7(\mathcal{L}_X\psi) + \pi^4_{27}(\mathcal{L}_X\psi) = a\psi + W^\flat\wedge\varphi + *i_\varphi(h), \tag{3.9}$$

where $a \in \Omega^0$, and h is a trace-free symmetric 2-tensor on M. We compute a as follows:

$$\begin{aligned} a &= \frac{1}{7}\langle\mathcal{L}_X\psi,\psi\rangle = \frac{1}{168}(\nabla_i\alpha_{jkl} - \nabla_j\alpha_{ikl} + \nabla_k\alpha_{ijl} - \nabla_l\alpha_{ijk})\psi^{ijkl} \\ &= \frac{1}{42}\nabla_i\alpha_{jkl}\psi^{ijkl} = \frac{1}{42}\nabla_i(\alpha_{jkl}\psi^{ijkl}) - \frac{1}{42}\alpha_{jkl}\nabla_i\psi^{ijkl} \\ &= \frac{24}{42}\nabla_i(X^m g_{mi}) - \frac{1}{42}X^m\psi_{mjkl}(\nabla_i\psi^{ijkl}) = \frac{4}{7}\nabla_i X_i = \frac{4}{7}\mathrm{div}X, \end{aligned} \tag{3.10}$$

where we used (A.3) and because T is symmetric. To compute $W^\flat$, note that

$$\langle *((*\mathcal{L}_X\psi)\wedge\varphi), e^m\rangle = 4\langle W^\flat, e^m\rangle,$$

thus

$$4W^m = *\big((*\mathcal{L}_X\psi)\wedge\varphi\wedge e^m\big) = \langle\varphi\wedge e^m, \mathcal{L}_X\psi\rangle = \langle\varphi\wedge e^m, d\alpha\rangle.$$

Therefore, we obtain

$$\begin{aligned}
W^m &= \frac{1}{4}\langle\varphi\wedge e^m, d\alpha\rangle = \frac{1}{4!}(\nabla^i\alpha^{jkm} - \nabla^j\alpha^{ikm} + \nabla^k\alpha^{ijm} - \nabla^m\alpha^{ijk})\varphi_{ijk} \\
&= \frac{1}{4!}(3\nabla^i\alpha^{jkm}\varphi_{ijk} - \nabla^m\alpha^{ijk}\varphi_{ijk}) \\
&= \frac{3}{4!}\nabla^i(\alpha^{jkm}\varphi_{ijk}) - \frac{3}{4!}\alpha^{jkm}\nabla^i\varphi_{ijk} - \frac{1}{4!}\nabla^m(\alpha^{ijk}\varphi_{ijk}) + \frac{1}{4!}\alpha^{ijk}\nabla^m\varphi_{ijk} \\
&= \frac{3}{4!}\nabla^i(X_l\psi^{ljkm}\varphi_{ijk}) - \frac{3}{4!}X_l\psi^{ljkm}T^i_n\psi^n{}_{ijk} \\
&\quad - \frac{1}{4!}\nabla^m(X_l\psi^{lijk}\varphi_{ijk}) + \frac{1}{4!}X_l\psi^{lijk}T^m_n\psi^n{}_{ijk} \\
&= -\frac{1}{2}\nabla^i(X_l\varphi_i{}^{lm}) + X_lT^{ml} = -\frac{1}{2}(\nabla^iX_l\varphi_i{}^{lm} + X_l\nabla_i\varphi_i{}^{lm}) + X_lT^{ml} \\
&= -\frac{1}{2}\mathrm{Curl}X^m - \frac{1}{2}X_lT_i{}^n\psi_{ni}{}^{lm} + (X\lrcorner T)^m = -\frac{1}{2}(\mathrm{Curl}X)^m + (X\lrcorner T)^m.
\end{aligned} \tag{3.11}$$

Finally, to compute h, observe that

$$\begin{aligned}
&(\mathcal{L}_X\psi)_{imnp}\psi_j{}^{mnp} + (\mathcal{L}_X\psi)_{jmnp}\psi_i{}^{mnp} \\
&= a(\psi_{imnp}\psi_j{}^{mnp} + \psi_{jmnp}\psi_i{}^{mnp}) + (*\mathrm{i}_\varphi(h))_{imnp}\psi_j{}^{mnp} + (*\mathrm{i}_\varphi(h))_{jmnp}\psi_i{}^{mnp},
\end{aligned} \tag{3.12}$$

where

$$(*\mathrm{i}_\varphi(h))_{imnp} = -(h^q_i\psi_{qmnp} + h^q_m\psi_{iqnp} + h^q_n\psi_{imqp} + h^q_p\psi_{imnq}).$$

Using (A.5) and (A.6), we get

$$\begin{aligned}
(*\mathrm{i}_\varphi(h))_{imnp}\psi_j{}^{mnp} &= -\, h^q_i\psi_{qmnp}\psi_j{}^{mnp} - 3h^q_m\psi_{iqnp}\psi_j{}^{mnp} \\
&= -\, 24h^q_ig_{qj} - 3h^q_m(4g_{ij}g^m_q - 4g^m_ig_{qj} + 2\psi_{iqj}{}^m) \\
&= -12h_{ij}.
\end{aligned}$$

By symmetry, the right hand side of (3.12) becomes

$$(\mathcal{L}_X\psi)_{imnp}\psi_j{}^{mnp} + (\mathcal{L}_X\psi)_{jmnp}\psi_i{}^{mnp} = 24\left(\frac{8}{7}(\mathrm{div}X)g_{ij} - h_{ij}\right). \tag{3.13}$$

For the left-hand side of (3.13), using the identities (A.4),(A.3),(A.5) and (A.6), we have:

$$
\begin{aligned}
&(\mathcal{L}_X\psi)_{imnp}{\psi_j}^{mnp} = \nabla_i\alpha_{mnp}{\psi_j}^{mnp} - 3\nabla_m\alpha_{inp}{\psi_j}^{mnp}\\
=&\nabla_i(\alpha_{mnp}{\psi_j}^{mnp}) - \alpha_{mnp}\nabla_i{\psi_j}^{mnp} - 3\nabla_m(\alpha_{inp}{\psi_j}^{mnp}) + 3\alpha_{inp}\nabla_m{\psi_j}^{mnp}\\
=&24\nabla_iX_j - 12T_i^m(X\lrcorner\varphi)_{mj} + 12(\mathrm{div}X)g_{ij} - 12\nabla_iX_j - 6{(\nabla_mX\lrcorner\psi)_{ij}}^m\\
&- 6\,\mathrm{tr}(T)(X\lrcorner\varphi)_{ij} + 6(X\lrcorner T)_m{\varphi^m}_{ij} + 6T_i^m(X\lrcorner\varphi)_{mj} - 6T_j^m(X\lrcorner\varphi)_{mi}\\
&- 12\,\mathrm{tr}(T)(X\lrcorner\varphi)_{ij} + 6T_j^m(X\lrcorner\varphi)_{mi} + 6T_i^m(X\lrcorner\varphi)_{mj} + 6(X\lrcorner T)_m{\varphi^m}_{ij}\\
=&12\nabla_iX_j + 12(\mathrm{div}X)g_{ij} - 6{(\nabla_mX\lrcorner\psi)_{ij}}^m - 18\,\mathrm{tr}(T)(X\lrcorner\varphi)_{ij}\\
&+ 12(X\lrcorner T)_m{\varphi^m}_{ij}.
\end{aligned}
$$

By symmetry, we get

$$(\mathcal{L}_X\psi)_{imnp}{\psi_j}^{mnp} + (\mathcal{L}_X\psi)_{jmnp}{\psi_i}^{mnp} = 12(\nabla_iX_j + \nabla_jX_i) + 24(\mathrm{div}X)g_{ij}.$$

So, using (3.10), (3.13) and the above expressions, we obtain

$$\frac{1}{2}(\nabla_iX_j + \nabla_jX_i) + (\mathrm{div}X)g_{ij} = \frac{8}{7}(\mathrm{div}X)h_{ij} - h_{ij},$$

which, upon re-arranging gives

$$h_{ij} = \frac{1}{7}(\mathrm{div}X)g_{ij} - \frac{1}{2}(\mathcal{L}_Xg)_{ij}. \tag{3.14}$$

Hence, substituting (3.10), (3.11) and (3.14) into (3.9) we obtain (3.8). □

Proposition 7. *Let* φ *be a coclosed* G_2*-structure on* M *with associated metric* g*. If* (φ, X, λ) *is a soliton of the Laplacian coflow as in* (3.3)*, then its full torsion tensor* T *satisfies*

$$
\begin{aligned}
\mathrm{div}T =& -\frac{1}{2}(\mathrm{Curl}X)^\flat + X\lrcorner T,\\
-\mathrm{Ric} + \frac{1}{2}T\circ T + (\mathrm{tr}\,T)T =& \frac{\lambda}{4}g + \frac{1}{2}\mathcal{L}_Xg.
\end{aligned}
\tag{3.15}
$$

Proof. Using (2.10), (2.2) and Lemma 5, we obtain

$$\Delta_\psi\psi = (\mathrm{div}T)^\flat\wedge\varphi + \mathrm{i}_\psi\Big(-\mathrm{Ric} + \frac{1}{2}T\circ T + (\mathrm{tr}\,T)T\Big).$$

On the other hand, by (2.2) and Proposition 6 we have

$$\lambda\psi + \mathcal{L}_X\psi = (-\frac{1}{2}\mathrm{Curl}X + X\lrcorner T)^\flat \wedge \varphi + \mathrm{i}_\psi\Big(\frac{\lambda}{4}g + \frac{1}{2}(\mathcal{L}_X g)\Big)$$

and thus we get (3.15). □

Remark 8.
- We notice that (3.15) coincides with the soliton equation for a general geometric flow given in [9, Definition 1.52].
- The tuple (g, X, λ) is called a *Ricci soliton* if it satisfies $\mathrm{Ric} = \lambda g + \mathcal{L}_X g$. The second equation of (3.15) can be viewed as a perturbation of the Ricci soliton equation using the torsion tensor T. A similar remark was done by Lotay-Wei for the Laplacian flow [24], but in contrast, the first equation of (3.15) coincides with one of the equation of the isometric soliton condition of the harmonic flow of G_2-structures [8, Definition 2.16].
- From the second equation of (3.15) is natural to ask for solitons of the Laplacian coflow, inducing Ricci solitons, aside from the nearly parallel case where $\Delta\psi = \lambda^2\psi$ and $\mathrm{Ric} = \frac{3}{8}\tau_0^2 g$. For instance, in [26] the authors obtain an example of a Laplacian coflow soliton inducing a Ricci soliton on a solvable Lie group.

Using (3.15), we can give an alternative proof for the non-existence of shrinking solitons in the compact case [16, Proposition 4.3], and we extend this result to non-compact cases with X divergence free:

Corollary 9.
1. *There are no compact shrinking solitons of the Laplacian coflow.*
2. *The only compact steady solitons of the Laplacian coflow are given by torsion-free* G_2*-structures.*
3. *There do not exist steady (non-trivial i.e.* $X \neq 0$*) and shrinking solitons of the Laplacian coflow with* $\mathrm{div}X = 0$*.*

Proof. Taking the trace on the second equation of (3.15), we obtain

$$\frac{1}{2}\left((\operatorname{tr} T)^2 + |T|^2\right) = \frac{7}{4}\lambda + \operatorname{div} X, \tag{3.16}$$

since $\operatorname{tr}(T \circ T) = (\operatorname{tr} T)^2 - |T|^2$ and $\operatorname{tr}\operatorname{Ric} = R$ (see (2.11)). If $\operatorname{div} X = 0$ then $\lambda \geq 0$. When the manifold M is compact, we have

$$\lambda \operatorname{vol}(M) = \frac{2}{7}\int_M \left((\operatorname{tr} T)^2 + |T|^2\right) \operatorname{vol} \geq 0.$$

Hence, $\lambda > 0$ or $\lambda = 0$ if and only if $T = 0$. $\square$

Remark 10. In [16, Proposition 4.3], the flow equation (3.1) was defined with a minus sign on the right-hand side, by analogy with the heat equation. However, as pointed out in [15, Theorem 5.3], the definition of (3.1) agrees with its definition as the gradient flow of the volume functional. For this reason, the result in Corollary 9 1. is stated in terms of shrinking instead of expanding.

4 Almost Abelian Lie groups revisited

We study in this section the Laplacian coflow and its solitons in a class of solvable Lie groups, which have a codimension one Abelian ideal using the bracket flow as described in [20].

Let G be a Lie group, it is called *almost Abelian* if its Lie algebra $\mathfrak{g}$ admits an Abelian ideal $\mathfrak{h}$ of codimension 1. For $\dim G = 7$, any invariant G_2-structure is completely determined by a G_2-structure on $\mathfrak{g}$. Moreover, since G_2 acts transitively on the 6-sphere, thus, for any orthonormal basis $\{e_1, ..., e_7\}$, we can suppose that $e_7 \perp \mathfrak{h}$ and the G_2-structure has the form

$$\varphi = \omega \wedge e^7 + \rho^+ = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{245} - e^{236},$$

where $\omega = e^{12} + e^{34} + e^{56}$ and $\rho^+ = e^{135} - e^{146} - e^{245} - e^{236}$ are the canonical $\mathrm{SU}(3)$–structure of $\mathfrak{h} \cong \mathbb{R}^6$. Additionally, the induced dual 4-form is

$$\psi := *\varphi = \frac{1}{2}\omega^2 + \rho^- \wedge e^7 = e^{1234} + e^{1256} + e^{3456} - e^{2467} + e^{2357} + e^{1457} + e^{1367},$$

where $\rho^- = J^*\rho^+$ and J is the canonical complex structure on $\mathbb{R}^6$ defined by $\omega := \langle J\cdot, \cdot\rangle$. Moreover, the Lie bracket of $\mathfrak{g}$ is encoded by $A \in \mathfrak{gl}(\mathbb{R}^6)$ where $A := \mathrm{ad}(e_7)|_{\mathfrak{h}}$. To emphasize the role of this matrix, we will usually denote the Lie algebra $\mathfrak{g}$ by $\mathfrak{g}_A$.

The transitive action of $\mathrm{GL}(\mathfrak{g})$ on the space of G_2-structures, defined by $h \cdot \varphi := (h^{-1})^*\varphi$ (for $h \in \mathrm{GL}(\mathfrak{g})$), yields an infinitesimal representation of the alternating 3-form

$$\Lambda^3(\mathfrak{g})^* = \theta(\mathfrak{gl}(\mathfrak{g}))\varphi, \tag{4.1}$$

where $\theta : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{End}(\Lambda^3\mathfrak{g}^*)$ is defined by

$$\theta(B)\varphi := \frac{d}{dt}\Big|_{t=0} e^{tB} \cdot \varphi = -\varphi(B\cdot, \cdot, \cdot) - \varphi(\cdot, B\cdot, \cdot) - \varphi(\cdot, \cdot, B\cdot).$$

Since the orbit $\mathrm{GL}(\mathfrak{g})\psi$ is also open in $\Lambda^4\mathfrak{g}^*$, the relation (4.1) also holds for the 4-form ψ, namely $\Lambda^4(\mathfrak{g})^* = \theta(\mathfrak{gl}(\mathfrak{g}))\psi$. Coclosed G_2-structures on almost Abelian Lie algebras are equivalent with the Lie bracket constrain $A \in \mathfrak{sp}(\mathbb{R}^6)$ [11], where

$$\mathfrak{sp}(\mathbb{R}^6) = \{A \in \mathfrak{gl}(\mathbb{R}^6) : \quad AJ + JA^t = 0 \quad \Leftrightarrow \quad \theta(A)\omega = 0\}.$$

In particular, the non-vanishing torsion forms τ_0 and τ_3 can be described in terms of the Lie bracket of $\mathfrak{g}_A$ induced by A:

Proposition 11. *[27, Prop. 3.2 & Cor 3.3] Let $\mathfrak{g}_A$ be an almost Abelian Lie algebra with coclosed G_2-structure φ. Hence, the torsion forms of φ are*

$$\tau_0 = \frac{2}{7}\,\mathrm{tr}(JA) \quad \textit{and} \quad \tau_{27} = \left(\begin{array}{c|c} \frac{1}{14}\,\mathrm{tr}(JA)\mathrm{I}_{6\times 6} - \frac{1}{2}[J, A] & 0 \\ \hline 0 & -\frac{3}{7}\,\mathrm{tr}(JA) \end{array}\right).$$

And its full torsion tensor is

$$T = \left(\begin{array}{c|c} \frac{1}{2}[J, A] & 0 \\ \hline 0 & \frac{1}{2}\,\mathrm{tr}(JA) \end{array}\right). \tag{4.2}$$

Moreover, we can describe the Hodge Laplacian $\Delta\psi$ in function of $A \in \mathfrak{sp}(\mathbb{R}^6)$, hence according with Lemma 5, we first compute the tensor $T \circ T$ given in (2.8):

Lemma 12. *Let* $\mathfrak{g}_A$ *be an almost Abelian Lie algebra with coclosed* G_2*-structure* φ*. Denote by* $S_A := \frac{1}{2}(A + A^t)$ *the symmetric part of* A*, then we have*

$$T \circ T = \left(\begin{array}{c|c} -\frac{1}{2}(\operatorname{tr} JA)[J,A] - S_A \circ_6 S_A & 0 \\ \hline 0 & -\operatorname{tr} S_A^2 \end{array} \right), \tag{4.3}$$

where $\circ_6$ *is the product on* $\mathfrak{gl}(\mathbb{R}^6)$*, defined by*

$$(S_A \circ_6 S_A)_{ab} := (S_A)_{mn}(S_A)_{pq}\rho^+_{mpa}\rho^+_{nqb}.$$

Proof. We first compute the entry $(T \circ T)_{77}$, thus by (4.2) we have

$$\begin{aligned}(T \circ T)_{77} =& T_{mn}T_{pq}\varphi_{mp7}\varphi_{nq7} = \frac{1}{4}[J,A]_{mn}[J,A]_{pq}\omega_{mp}\omega_{nq} \\ =& \frac{1}{4}(J[J,A])_{np}([J,A]J)_{np} \\ =& \frac{1}{4}\langle J[J,A],[J,A]J\rangle = -\operatorname{tr} S_A^2,\end{aligned}$$

for the last equality we used $A = JA^tJ$ (i.e. $A \in \mathfrak{sp}(\mathbb{R}^6)$). Now, for $i \neq 7$ and $j = 7$, we have

$$\begin{aligned}(T \circ T)_{i7} =& T_{mn}T_{pq}\varphi_{mpi}\omega_{nq} = \frac{1}{4}[J,A]_{mn}[J,A]_{pq}\rho^+_{mpi}\omega_{nq} \\ =& \frac{1}{4}([J,A]J[J,A])_{mp}\rho^+_{mpi} = \frac{1}{4}([J,A]^2)_{mn}J_{np}\rho^+_{pmi} \\ =& -\frac{1}{4}([J,A]^2)_{mn}\rho^-_{nmi} = 0.\end{aligned}$$

For the above computation, we used that $[J,A] \in \mathfrak{sp}(\mathbb{R}^6)$ is symmetric and (A.7). Finally, for $i \neq 7$ and $j \neq 7$ we have

$$\begin{aligned}(T \circ T)_{ij} =& 2T_{mn}T_{77}\omega_{mi}\omega_{nj} + T_{mn}T_{pq}\rho^+_{mpi}\rho^+_{nqj} \\ =& \frac{1}{2}(\operatorname{tr} JA)[J,A]_{mn}J_{mi}J_{nj} + \frac{1}{4}[J,A]_{mn}[J,A]_{pq}\rho^+_{mpi}\rho^+_{nqj} \\ =& -\frac{1}{2}(\operatorname{tr} JA)(J[J,A]J)_{ij} + (JS_A)_{mn}(JS_A)_{pq}\rho^+_{mpi}\rho^+_{nqj} \\ =& -\frac{1}{2}(\operatorname{tr} JA)([J,A])_{ij} + J_{mk}(S_A)_{kn}J_{pl}(S_A)_{lq}\rho^+_{mpi}\rho^+_{nqj} \\ =& -\frac{1}{2}(\operatorname{tr} JA)([J,A])_{ij} - (S_A)_{kn}(S_A)_{lq}\rho^+_{kli}\rho^+_{nqj}.\end{aligned}$$

Once again, we used the identities (A.7) Finally, combining each case of i and j, we get the expression for $T \circ T$. □

Now, for almost Abelian Lie algebras $\mathfrak{g}_A$, the Ricci curvature is [1]*Eq (8):

$$\mathrm{Ric}_A = \left(\begin{array}{c|c} \frac{1}{2}[A, A^t] & 0 \\ \hline 0 & -\operatorname{tr} S_A^2 \end{array}\right) \quad \text{for} \quad A \in \mathfrak{sp}(\mathbb{R}^6), \tag{4.4}$$

and since $\mathrm{tr}(T)$ is constant, we have $\mathrm{div} T = 0$. Therefore, we can write Lemma 5 in function of the Lie bracket induced by A:

Proposition 13. *Let $\mathfrak{g}_A$ be an almost Abelian Lie algebra with coclosed G_2-structure φ. Thus, the Hodge Laplacian of ψ is $\Delta_A\psi = \theta(Q_A)\psi$ where*

$$Q_A = \left(\begin{array}{c|c} Q_A^{\mathfrak{h}} & 0 \\ \hline 0 & q_A \end{array}\right) = \left(\begin{array}{c|c} \frac{1}{2}[A, A^t] + \frac{1}{2} S_A \circ_6 S_A & 0 \\ \hline 0 & -\frac{1}{2}\operatorname{tr}(S_A)^2 - \frac{1}{4}(\operatorname{tr} JA)^2 \end{array}\right). \tag{4.5}$$

In particular, $Q_A \in \mathfrak{gl}(\mathfrak{g}_A)$ is symmetric.

Proof. The result follows by applying equations (4.2), (4.4) and (4.3) into Lemma 5. □

Notice that the closed condition on ψ implies that $\Delta_\psi\psi = dd^*\psi$ is also closed. Similarly, it is interpreted as $Q_A^{\mathfrak{h}} \in \mathfrak{sp}(\mathbb{R}^6)$ for $A \in \mathfrak{sp}(\mathbb{R}^6)$. Indeed:

Lemma 14. *If $A \in \mathfrak{sp}(\mathbb{R}^6)$ is symmetric then $A \circ_6 A \in \mathfrak{sp}(\mathbb{R}^6)$.*

Proof. Notice that $B := A \circ_6 A$ is symmetric, thus it is enough to prove the equality $JBJ = B$. Hence

$$\begin{aligned}(JBJ)_{ij} =& J_{ik}(A \circ_6 A)_{kl} J_{lj} = J_{ik} A_{mn} A_{pq} \rho^+_{mpk} \rho^+_{nql} J_{lj} \\ =& -(JAJ)_{mn} A_{pq} \rho^-_{mpi} \rho^-_{nqj} = -J_{mr} A_{rs} J_{sn} A_{pq} \rho^-_{mpi} \rho^-_{nqj} \\ =& A_{rs} A_{pq} \rho^+_{rpi} \rho^+_{sqj} = B_{ij}.\end{aligned}$$

Here, we used the identities (A.7) time and again, as well as the symmetry of A. □

4.1 The bracket flow

In this section we adapt the general approach of geometric flows of homogeneous geometric structures, proposed by J. Lauret, to the framework of the Laplacian coflow (3.1) on almost Abelian Lie algebras with coclosed G_2-structures, for a broad exposition see [20].

Let $\{\varphi(t)\}_{t\in(\varepsilon_1,\varepsilon_2)}$ be a solution of the Laplacian coflow on $\mathfrak{g}_A$ with initial condition $\varphi(0)=\varphi_0$. Since $\varphi(t)\in \mathrm{GL}(\mathfrak{g}_A)\varphi_0$, we can write $\varphi(t)=h(t)^*\varphi_0$ for $h(t)\in\mathrm{GL}(\mathfrak{g}_A)$ satisfying $h(0)=I$. Since $*_{\varphi(t)}=(h^{-1})^* *_{\varphi_0} h^*$ (see [26, Lemma 3.1]), we can write $\psi(t)=h(t)^*\psi_0$ for $\psi_0=*_{\varphi_0}\varphi_0$ and by Proposition 13, we have

$$\Delta_A\psi(t)=\theta(Q_A(t))\psi(t),$$

hence, the Laplacian coflow is equivalent with

$$\frac{d}{dt}h(t)=-h(t)Q_A(t). \tag{4.6}$$

Definition 15. Let (G_1,φ_1) and (G_2,φ_2) be Lie groups with G_2-structure φ_i (for $i=1,2$). An isomorphism $f:(G_1,\varphi_1)\to(G_2,\varphi_2)$ is called an *equivariant isomorphism*, if it is a Lie group isomorphism such that $\varphi_1=f^*\varphi_2$, and in this case, (G_1,φ_1) and (G_2,φ_2) are called *equivariant equivalent*.

Since $\varphi(t)=*_t\psi(t)$ induces a SU(3)-structure on $\mathfrak{h}$ for each t, we can write

$$h(t)=k(t)+a(t)e^7\otimes e_7 \quad \text{where} \quad k(t)\in \mathrm{Gl}(\mathbb{R}^6) \quad \text{and} \quad a(t)\in\mathbb{R}^*. \tag{4.7}$$

We can define a time-depending Lie bracket on $\mathfrak{g}_{A(t)}$ determined by $A(t)=a(t)^{-1}k(t)Ak(t)^{-1}$, such that (4.7) becomes a Lie algebra isomorphism between $(\mathfrak{g}_A,\varphi(t))$ and $(\mathfrak{g}_{A(t)},\varphi)$ with $\varphi(t)=h(t)^*\varphi$. Moreover, since $\Delta_A\psi(t)=h(t)^*\Delta_{A(t)}\psi$, we get the relation $Q_{A(t)}=h(t)Q_A(t)h(t)^{-1}$ and consequently, the equation (4.6) becomes an ODE on $(\mathfrak{g}_{A(t)},\varphi)$

$$\frac{d}{dt}h(t)=-Q_{A(t)}h(t). \tag{4.8}$$

In particular, under the flow (4.8) the matrix $A(t)$ evolves by:

$$\frac{d}{dt}A(t) = q_{A(t)}A(t) - [Q^{\mathfrak{h}}_{A(t)}, A(t)], \tag{4.9}$$

where $q_A(t)$ and $Q^{\mathfrak{h}}_{A(t)}$ are defined in (4.5) for each $t \in (\varepsilon_1, \varepsilon_2)$. Since the Lie bracket of $\mathfrak{g}_{A(t)}$ is completely encoded by $A(t)$, the ODE (4.9) is named the *bracket flow* and it provides an equivalent analysis of the geometric flow of homogeneous geometric structures, varying the Lie bracket instead of the geometric structure:

Theorem 16. *[20, Theorem 5] Let $\{\varphi(t)\}_{t\in(\varepsilon_1,\varepsilon_2)}$ be a solution of the Laplacian coflow on $(\mathfrak{g}_A)$ with initial condition $\varphi(0) = \varphi_0$. Then, there exist an equivariant isomorphism $f(t) : (G_A, \varphi(t)) \to (G_{A(t)}, \varphi)$, such that $h(t) = df(t)_1$ solves either (4.6) or (4.8) for all $t \in (\varepsilon_1, \varepsilon_2)$. In addition, the solutions of (3.1) and (4.9) are*

$$\varphi(t) = h(t)^*\varphi \quad \textit{and} \quad A(t) = a(t)k(t)^{-1}Ak(t),$$

respectively, for $t \in (\varepsilon_1, \varepsilon_2)$.

Theorem 16 provides a useful tool for addressing long-time existence and regularity questions, since it shows that the Laplacian coflow and the bracket flow have the same maximal interval of solution. Hence, the bracket flow (4.9) is explicitly given in the following proposition:

Proposition 17. *Let $\mathcal{L} \simeq \mathfrak{gl}(\mathbb{R}^6)$ be the family of 7-dimensional almost Abelian Lie algebras. The subfamily $\mathcal{L}_{coclosed} \simeq \mathfrak{sp}(\mathbb{R}^6) \subset \mathcal{L}$ of coclosed G_2-structures is invariant under the bracket flow (4.9), which becomes equivalent to the following ODE for a one-parameter family of matrices $A = A(t) \in \mathfrak{sp}(\mathbb{R}^6)$:*

$$\frac{d}{dt}A = -\Big(\frac{1}{2}\operatorname{tr}(S_A)^2 + \frac{1}{4}(\operatorname{tr} JA)^2\Big)A + \frac{1}{2}[A, [A, A^t]] + \frac{1}{2}[A, S_A \circ_6 S_A]. \tag{4.10}$$

Proof. Notice that the velocity $\dot{A}(t) = q_{A(t)}A + [A, Q^{\mathfrak{h}}_{A(t)}]$ lies in $\mathfrak{sp}(\mathbb{R}^6)$, since $S_A \circ_6 S_A \in \mathfrak{sp}(\mathbb{R}^6)$ by Lemma 14, hence, the family $\mathcal{L}_{coclosed} \subset \mathcal{L}$ is invariant under the bracket flow. Finally, replacing (4.5) into (4.9), we obtain (4.10). □

Proposition 18. *If $A(t)$ is a solution of* (4.10) *associated to the Laplacian coflow, then its norm evolves by*

$$\frac{d}{dt}|A|^2 = -\Big(|S_A|^2 + \frac{1}{2}(\operatorname{tr} JA)^2\Big)|A|^2 - |[A, A^t]|^2 - \langle S_A \circ_6 S_A, [A, A^t]\rangle. \tag{4.11}$$

Proof. From equation (4.10), we have

$$\begin{aligned}
&\frac{d}{dt}|A|^2 = 2\langle \dot{A}, A\rangle = 2\operatorname{tr}(\dot{A}A^t) \\
&= -\Big(\operatorname{tr}(S_A)^2 + \frac{1}{2}(\operatorname{tr} JA)^2\Big)|A|^2 + \operatorname{tr}([A, [A, A^t]]A^t) + \operatorname{tr}([A, S_A \circ_6 S_A], A^t) \\
&= -\Big(|S_A|^2 + \frac{1}{2}(\operatorname{tr} JA)^2\Big)|A|^2 - |[A, A^t]| - \langle S_A \circ_6 S_A, [A, A^t]\rangle.
\end{aligned}$$

□

In order to prove long-time existence solution for (4.10) we need the following identity.

Lemma 19. *For the symmetric part S_A of the matrix $A \in \mathfrak{sp}(\mathbb{R}^6)$, we have*

$$|S_A \circ_6 S_A|^2 = 4(|S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2).$$

Proof. This identity follows by direct computations, using the contractions (A.7) and (A.8). □

Theorem 20. *The Laplacian coflow solution $(\mathfrak{g}_A, \varphi(t))$ starting at any coclosed (non-flat)* G_2*-structure is defined for all $t \in (\varepsilon_1, \infty)$.*

Proof. Let $\varphi(t)$ a solution of the Laplacian coflow defined for all $t \in (\varepsilon_1, \varepsilon_2)$, according to Theorem 16, we get that the solution $A(t) \in \mathfrak{sp}(\mathbb{R}^6)$

of (4.10) is defined for all $t \in (\varepsilon_1, \varepsilon_2)$. Now, using the Cauchy-Schwarz and Peter-Paul inequalities (i.e. $ab \leq \frac{a^2}{4} + b^2$ for $a, b \geq 0$), we have

$$\begin{aligned}
-\langle S_A \circ_6 S_A, [A, A^t]\rangle \leq & |S_A \circ_6 S_A||[A, A^t]| \\
\leq & \frac{|S_A \circ_6 S_A|^2}{4} + |[A, A^t]|^2 \\
= & |S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 + |[A, A^t]|^2.
\end{aligned} \tag{4.12}$$

Replacing the last inequality into equation (4.11), we have

$$\begin{aligned}
\frac{d}{dt}|A|^2 \leq & -\Big(|S_A|^2 + \frac{1}{2}(\operatorname{tr} JA)^2\Big)|A|^2 - |[A, A^t]|^2 + |S_A|^2|S_A|^2 \\
& - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 + |[A, A^t]|^2 \\
= & - |S_A|^2|S_A|^2 - \frac{1}{4}|S_A|^2|A - A^t|^2 - \frac{1}{2}(\operatorname{tr} JA)^2|A|^2 \\
& + |S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 \\
= & - \frac{1}{4}|S_A|^2|A - A^t|^2 - \frac{1}{2}(\operatorname{tr} JA)^2|A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 \leq 0.
\end{aligned}$$

Thus, $|A|^2$ is non-increasing and non-negative, therefore $A(t)$ is an immortal solution, i.e. it is defined for all $t \in (\varepsilon_1, \infty)$. In particular, $|A|^2$ is strictly decreasing unless $(\mathfrak{g}_{A(t)}, \varphi)$ is torsion-free, that is

$$|\dot{A}|^2 = 0 \quad \Leftrightarrow \quad A^t = -A \quad \text{and} \quad \operatorname{tr} JA = 0,$$

and thus $A(t) \equiv A_0 \in \mathfrak{sl}(\mathbb{C}^3) \cap \mathfrak{sp}(\mathbb{R}^6) = \mathfrak{su}(3)$ the bracket flow solution is constant. □

Remark 21. In [3] Bagaglini and Fino address also the Laplacian coflow on almost Abelian Lie algebras, there the approach is different from ours, the authors find explicit solutions of the Laplacian coflow when $A \in \mathfrak{sp}(\mathbb{R}^6)$ is normal. Notice that the above theorem holds for any $A \in \mathfrak{sp}(\mathbb{R}^6)$.

Example 22. Consider the almost Abelian Lie algebra $\mathfrak{g}_A$ with the matrix A defined by

$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & -B^t \end{array}\right] \quad \text{with} \quad B = \begin{bmatrix} 0 & x & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad x, y \in \mathbb{R}. \tag{4.13}$$

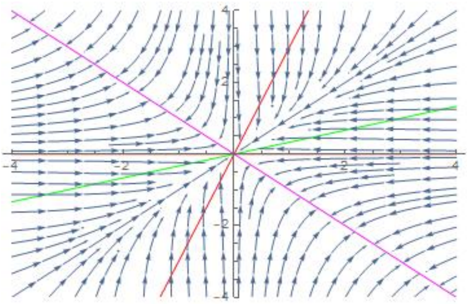

Figure 4.1: x-nullcline, y-nullcline, equilibrium points

The later 2-parameter family illustrates an example where the bracket flow $A(t)$ is stable: For $A(t)$ given by (4.13) we have that $\operatorname{tr}(S_A)^2 = (x+y)^2$, $\operatorname{tr}(JA) = 0$, $[A, S_A \circ_6 S_A] = 0$ and the non-vanishing terms of $[A,[A,A^t]]$ are

$$\begin{aligned} [A,[A,A^t]]_{12} &= -\,[A,[A,A^t]]_{54} = 2x(y^2 - x^2) \\ [A,[A,A^t]]_{21} &= -\,[A,[A,A^t]]_{45} = 2y(x^2 - y^2). \end{aligned}$$

Replacing the above into (4.13), we obtain that the bracket flow is equivalent with the following nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x})$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$; $\mathbf{x} \mapsto f = (f_1(\mathbf{x}), f_2(\mathbf{x}))$ given by

$$\dot{x} = -\frac{x}{2}(3x - y)(x+y) \quad \text{and} \quad \dot{y} = -\frac{y}{2}(3y - x)(x+y). \tag{4.14}$$

A point $\mathbf{x} \in \mathbb{R}^2$ is an *equilibrium point* if $f(\mathbf{x}) = 0$ which is given by the surface $S = \{(x,y) \in \mathbb{R}^2 : x = -y\}$. The x-nullclines (i.e, $\mathbf{x} \in \mathbb{R}^2$ where $f_1(\mathbf{x}) = 0$) are the lines $x = 0$, $x = -y$ and $y = 3x$ and the y-nullclines (i.e, $\mathbf{x} \in \mathbb{R}^2$ where $f_2(x) = 0$) are the lines $y = 0$, $x = -y$ and $y = \frac{1}{3}x$. The intersection of the x-nullclines and y-nullclines yield the equilibrium points. On the other hand, the lines $y = 0$ and $x = 0$ are invariants for the system (4.14). If we set $y = 0$ then we obtain $\dot{x} = -6x^3$. Therefore, $\dot{x}$ is positive if $x > 0$ and negative if $x < 0$ which clearly shows the stability along the line $y = 0$.

To determine the trajectories, if $\mathbf{x}_0$ is not a equilibrium point then, at least one of $f_1(\mathbf{x}_0)$ or $f_2(\mathbf{x}_0)$ is not zero. Let us suppose that $f_1(\mathbf{x}_0) \neq 0$. Then, there is an open neighborhood of $\mathbf{x}_0$, such that $f_1(\mathbf{x}_0) \neq 0$, so the orbit through $\mathbf{x}_0$ can be defined as a solution of the nonautonomous scalar equation

$$\frac{dy}{dx} = -\frac{y(x-3y)}{x(3x-y)}.$$

This differential equation is homogeneous. Setting $y = xv(x)$, we obtain

$$v + x\frac{dv}{dx} = -\frac{v(1-3v)}{3-v}.$$

That is,

$$x\frac{dv}{dx} = 4v\left(\frac{v-1}{3-v}\right).$$

The resulting ODE is separable, with solution $x^{-4}v^{-3}(v-1)^2 = c$. Reverting back to the original variables, the trajectories are level curves of

$$H(x(t), y(t)) = \frac{(y(t)-x(t))^2}{y(t)^3x(t)^3}.$$

On the other hand, let

$$V(\mathbf{x}) = x^2 + y^2 + 2xy,$$

be a Lyapunov function. In fact, $V(\mathbf{x}) = 0$ when $\mathbf{x}$ is an equilibrium point for this system and $V(\mathbf{x}) = (x+y)^2 \geq 0$ if $\mathbf{x}$ is not an equilibrium point. Computing $\dot{V}(\mathbf{x})$, we find

$$\dot{V}(\mathbf{x}) = -2(x+y)^2(6x^2 - 4xy + 6y^2),$$

where $\dot{V}(x) = 0$ if $x = -y$ and $\dot{V}(x) \leq 0$ otherwise. For any curve $\gamma(r,\theta) = (r\cos\theta, r\sin\theta)$ with $r > 0$ and $0 \leq \theta \leq 2\pi$, we obtain

$$\dot{V}(\gamma(r,\theta)) = -2r^2(r\sin\theta + r\cos\theta)^2(6 - 2\sin(2\theta)) \leq 0,$$

since $|\sin(2\theta)| \leq 1$ then we have $6 - 2\sin(\theta) > 0$. Therefore, the system is stable if $\mathbf{x}_0$ is a equilibrium point.

Proposition 23. *Let* $(\mathfrak{g}_A, \varphi(t))$ *be a Laplacian coflow solution on an almost Abelian Lie algebra, starting at any coclosed (non-flat)* G_2*-structure. Then, the scalar curvature* $R(t)$ *of* $(\mathfrak{g}_A, \varphi(t))$ *is strictly increasing and satisfies the inequality*

$$\frac{1}{-\frac{|t|}{2}+\frac{1}{R(0)}} \leq R(t) \leq 0 \quad \textit{for} \quad \textit{any} \quad t \in (\varepsilon_1, \infty).$$

In particular, $|T|^2$ *is strictly decreasing. and it converges to zero when* $t \to \infty$ *as* $|A(t)|^2 \to 0$.

Proof. From (4.4), we have $R = -\operatorname{tr} S_A^2 = -\frac{1}{4}\operatorname{tr}(A+A^t)^2$. Thus, using the bracket flow equation (4.10) we have

$$\begin{aligned}
&\frac{d}{dt}\operatorname{tr}(A+A^t)^2 = 2\operatorname{tr}\Big((A+A^t)\frac{d}{dt}(A+A^t)\Big)\\
&= -\Big(\operatorname{tr} S_A^2 + \frac{1}{2}(\operatorname{tr} JA)^2\Big)\operatorname{tr}(A+A^t)^2 + \operatorname{tr}\Big((A+A^t)[A-A^t,[A,A^t]]\Big)\\
&\quad + \operatorname{tr}\Big((A+A^t)[A-A^t, S_A \circ_6 S_A]\Big)\\
&= -4\Big(\operatorname{tr} S_A^2 + \frac{1}{2}(\operatorname{tr} JA)^2\Big)\operatorname{tr} S_A^2 + \operatorname{tr}\Big([A+A^t, A-A^t][A,A^t]\Big)\\
&\quad + \operatorname{tr}\Big([A+A^t, A-A^t]S_A \circ_6 S_A\Big)\\
&= -2\Big(|S_A|^2 + (\operatorname{tr} JA)^2\Big)|S_A|^2 - 2|[A,A^t]|^2 - 2\langle [A,A^t]S_A \circ S_A\rangle.
\end{aligned}$$

Using the inequality (4.12), we obtain

$$\begin{aligned}
\frac{d}{dt}\operatorname{tr}(A+A^t)^2 &\leq -2\Big(2|S_A|^2 + (\operatorname{tr} JA)^2\Big)|S_A|^2 + 2|S_A|^4\\
&\quad - 4|S_A^2|^2 - 2(\langle JS_A, S_A\rangle)^2\\
&\leq -2(\operatorname{tr} S_A^2)^2 = -\frac{1}{8}(\operatorname{tr}(A+A^t)^2)^2.
\end{aligned}$$

For any $t_1, t_2 \in (\varepsilon_1, \infty)$ satisfying $t_1 \leq t_2$, the last inequality implies

$$\frac{1}{R(t_2)} - \frac{1}{R(t_1)} \geq \frac{t_2 - t_1}{2}.$$

If $t_1 = 0$ then we get

$$\frac{1}{-\frac{t_2}{2}+\frac{1}{R(0)}} \leq R(t_2) < 0 \quad \text{any} \quad t_2 \in [0, \infty).$$

If $t_2 = 0$ then we obtain

$$\frac{1}{\frac{t_1}{2} + \frac{1}{R(0)}} \leq R(t_1) < 0 \quad \text{any} \quad t_1 \in (\varepsilon_1, 0].$$

Finally, by (2.10) and (4.2), the scalar curvature of a coclosed G_2-structure is

$$R_A = -|T|^2 + (\mathrm{tr}(JA))^2.$$

Hence, using the Cauchy-Schwarz inequality, we have

$$|T|^2 \leq -R(t) + |J|^2|A(t)|^2 = -R(t) + 6|A(t)|^2 \leq \frac{1}{\frac{|t|}{2} - \frac{1}{R(0)}} + 6|A(t)|^2.$$

Therefore, $|T|^2$ is strictly decreasing, since $|A(t)|^2$ is strictly decreasing as well and $|T|^2$ goes to zero as $|A(t)| \to 0$. □

4.2 Algebraic solitons

In this section, we characterize the invariant G_2-structures on almost Abelian Lie algebras which are semi-algebraic solitons of the Laplacian coflow, in terms of the Lie bracket induced by $A \in \mathfrak{sp}(\mathbb{R}^6)$.

The solution (3.2) on the almost Abelian Lie group G_A *is self-similar relative to equivariant equivalence* if $\lambda(t) \in \mathbb{R}^*$ and $f(t) \in \mathrm{Aut}(G_A)$ (see [22, Equation (16)]). Then, the corresponding solution (3.2) on $\mathfrak{g}_A$ is

$$\psi(t) = \lambda(t)h(t)^*\psi \in \Lambda^4(\mathfrak{g}_A)^* \quad \text{with} \quad \lambda(t) \in \mathbb{R}^* \quad \text{and} \quad h(t) \in \mathrm{Aut}(\mathfrak{g}_A), \tag{4.15}$$

with $df(t)_1 = h(t)$ and then, the soliton equation (3.3) becomes $\Delta_\psi\psi = \lambda\psi + \mathcal{L}_{X_D}\psi \in \Lambda^4(\mathfrak{g}_A)^*$ with $\lambda \in \mathbb{R}$ and $X_D := \frac{d}{dt}|_{t=0}h(t) =: -D \in \mathrm{Der}(\mathfrak{g}_A)$. Using the representation (4.1), we have

$$\begin{aligned}\theta(Q_A)\psi =& \Delta_\psi\psi = \lambda\psi + \mathcal{L}_{X_D}\psi \\ =& \theta\left(-\frac{\lambda}{4}I_7\right)\psi + \frac{d}{dt}|_{t=0}h(t)^*\psi \\ =& \theta\left(-\frac{\lambda}{4}I_7 + D\right)\psi.\end{aligned}$$

By Proposition 13, the matrix Q_A is symmetric, hence, setting $\lambda = -4c$ we say that ψ is a *semi-algebraic* soliton if

$$Q_A = cI_7 + \frac{1}{2}(D + D^t),$$

and ψ is an *algebraic soliton* if $D^t \in \mathrm{Der}(\mathfrak{g}_A)$. Moreover, the self-similar solution (4.15) is given

$$\lambda(t) = (1-2ct)^2 \quad \text{and} \quad h(t) = e^{-s(t)D} \quad \text{where} \quad s(t) = -\frac{1}{2c}\log|2ct-1|,$$

(for $c = 0$ set $s(t) = t$). And the corresponding bracket solution of a semi-algebraic soliton is induced by

$$A(t) = (1-2ct)^{-1/2} e^{s(t)E} A e^{-s(t)E} \quad \text{where} \quad E = \frac{1}{2}(D - D^t), \tag{4.16}$$

(e.g. [19, Remark 3.4] for the homogeneous Ricci soliton case). The next theorem shows the (semi-) algebraic soliton equation in terms $A \in \mathfrak{sp}(\mathbb{R}^6)$.

Theorem 24. *Let $(\mathfrak{g}_A, \varphi)$ be an almost Abelian Lie algebra with coclosed G_2-structure:*

(i) ψ is an algebraic soliton for the Laplacian coflow if and only if

$$[[A, A^t] + S_A \circ_6 S_A, A] = \frac{|[A, A^t]|^2 + \langle S_A \circ_6 S_A, [A, A^t]\rangle}{|A|^2} A. \tag{4.17}$$

In this case, $D = Q_A - cI_7 \in \mathrm{Der}(\mathfrak{g}_A)$ for

$$c = -\frac{1}{2}\Big(\operatorname{tr} S_A^2 + \frac{1}{2}(\operatorname{tr} JA)^2 + \frac{|[A, A^t]|^2}{|A|^2} + \frac{\langle S_A \circ_6 S_A, [A, A^t]\rangle}{|A|^2}\Big).$$

(ii) ψ is a semi-algebraic soliton if and only if

$$[A, A^t] + S_A \circ_6 S_A = -\Big(\operatorname{tr} S_A^2 - \frac{1}{2}(\operatorname{tr} JA)^2 + 2d\Big) I_6 + D_1 + D_1^t, \tag{4.18}$$

for some $D_1 \in \mathfrak{gl}(\mathbb{R}^6)$ such that $[D_1, A] = dA$, where

$$d = \frac{|[A, A^t]|^2 + \langle S_A \circ_6 S_A, [A, A^t]\rangle}{2|A|^2}.$$

In this case $Q_A = cI_7 + \frac{1}{2}(D + D^t)$ for

$$c = -\frac{1}{2}\Big(\operatorname{tr} S_A^2 + \frac{1}{2}(\operatorname{tr} JA)^2 + \frac{|[A, A^t]|^2}{|A|^2} + \frac{\langle S_A \circ_6 S_A, [A, A^t]\rangle}{|A|^2}\Big). \tag{4.19}$$

Proof. (i) Suppose that $(\mathfrak{g}_A, \varphi)$ is an algebraic soliton i.e. $Q_A = cI + D$ for $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g}_A)$. Then,

$$De_7 = de_7 \quad \text{for} \quad \text{some} \quad d \in \mathbb{R} \quad \text{and} \quad [Q_A^{\mathfrak{h}}, A] = [D|_{\mathfrak{h}}, A] = dA.$$

Thus, by Proposition 13 we get

$$[[A, A^t], A] + [S_A \circ_6 S_A, A] = 2dA.$$

Taking the inner product between A and the above equation we obtain

$$d = \frac{|[A, A^t]|^2 + \langle S_A \circ_6 S_A, [A, A^t]\rangle}{2|A|^2}.$$

The converse follows by taking $D = Q_A - cI \in \mathrm{Der}(\mathfrak{g}_A)$ and

$$c = q - d = -\frac{1}{2}\Big(\operatorname{tr} S_A^2 + \frac{1}{2}(\operatorname{tr} JA)^2 + \frac{|[A, A^t]|^2}{|A|^2} + \frac{\langle S_A \circ_6 S_A, [A, A^t]\rangle}{|A|^2}\Big).$$

(ii) Suppose that $(\mathfrak{g}_A, \varphi)$ is a semi algebraic soliton, i.e. $Q_A = cI_7 + \frac{1}{2}(D + D^t)$ for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g}_A)$. It implies the equations

$$Q_A^{\mathfrak{h}} = cI_6 + \frac{1}{2}(D_1 + D_1^t) \quad \text{and} \quad q = c + d$$

where

$$De_7 = de_7 \quad \text{for} \quad d \in \mathbb{R} \quad \text{and} \quad [D_1, A] = dA \quad \text{where} \quad D_1 = D|_{\mathfrak{h}}.$$

Since $\langle [D_1, A], A\rangle = \langle A, [D_1^t, A]\rangle$, by Proposition 13 we obtain (4.18). The converse follows immediately, and the formulae for c and d are obtained as in (i).

□

Using the condition (4.17) we describe a class of algebraic solitons.

Corollary 25. *If $A \in \mathfrak{sp}(\mathbb{R}^6)$ is skew-symmetric then $(\mathfrak{g}_A, \varphi)$ is an algebraic soliton.*

Using Lemma 19, we can prove the absence of shrinking (semi-) algebraic solitons for the Laplacian coflow on almost Abelian Lie algebras.

Proposition 26. *If* $(\mathfrak{g}_A, \varphi)$ *is a (semi-) algebraic soliton for the Laplacian coflow then it is expanding, and it is steady if it is torsion-free.*

Proof. Using the inequality (4.12) in the equation (4.19), we have

$$\begin{aligned}
2c \leq & -\left(\operatorname{tr} S_A^2 + \frac{1}{2}(\operatorname{tr} JA)^2 + \frac{|[A, A^t]|^2}{|A|^2}\right) \\
& + \frac{1}{|A|^2}\left(|S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 + |[A, A^t]|^2\right) \\
\leq & -\frac{1}{|A|^2}\Big(|S_A|^2(|A|^2 - |S_A|^2) + \frac{1}{2}(\operatorname{tr} JA)^2|A|^2 + 2|S_A^2|^2 + \langle JS_A, S_A\rangle^2\Big) \\
\leq & 0.
\end{aligned}$$

If $c = 0$ then

$$\operatorname{tr} JA = 0 \quad \text{and} \quad S_A^2 = 0.$$

In particular $S_A = 0$, and thus A is skew-symmetric. And since $A \in \mathfrak{sp}(\mathbb{R}^6)$ it implies that $[J, A] = 0$. Therefore, by equation (4.2) we get that the full torsion tensor T vanishes. $\square$

Remark 27. We remark that the previous proposition was proved in [3, Corollary 4.4] for the context of algebraic solitons and assuming that A is normal.

We conclude this section with an example of a semi-algebraic soliton which is not an algebraic one.

Example 28. Let $(\mathfrak{g}_A, \varphi)$ be an almost Abelian Lie algebra with G_2-structure $\varphi = \omega \wedge e^7 + \rho^+$, where

$$\omega = e^{14} + e^{25} + e^{36} \quad \text{and} \quad \rho^+ = e^{123} - e^{156} + e^{246} - e^{345},$$

and the Lie bracket is determined by the 3-step nilpotent matrix

$$A = \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \in \mathfrak{sp}(\mathbb{R}^6), \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that the matrix

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & d \end{pmatrix}, \quad D_1 = \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & & & \\ 0 & 2 & 0 & & & \\ -\sqrt{2} & 0 & 4 & & & \\ \hline & & & 3 & 0 & \sqrt{2} \\ & & & 0 & 3 & 0 \\ & & & 0 & 0 & 1 \end{array}\right), \quad d = 1,$$

satisfies the relation $[D_1, A] = A$, it means that $D \in \mathrm{Der}(\mathfrak{g}_A)$. Now, for each term of (4.5), we obtain $\operatorname{tr} S_A^2 = 3$, $\operatorname{tr} JA = 0$, $|[A, A^t]|^2 = 12$, $|A|^2 = 6$ and $\langle S_A \circ_6 S_A, [A, A^t]\rangle = 0$ where

$$[A, A^t] = \left(\begin{array}{c|c} P & \\ \hline & -P \end{array}\right), \quad S_A \circ_6 S_A = \left(\begin{array}{c|c} R & \\ \hline & -R \end{array}\right)$$

with

$$P = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 2 \end{pmatrix}.$$

Since the matrices A and D satisfy the equation (4.18), we have that $(\mathfrak{g}_A, \varphi)$ is a semi-algebraic soliton with

$$Q_A = -\frac{5}{2} I + \frac{1}{2}(D + D^t).$$

Notice that $[D_1^t, A] \neq A$, so $D^t \notin \mathrm{Der}(\mathfrak{g}_A)$ thus $(\mathfrak{g}_A, \varphi)$ is not an algebraic soliton. According to (4.16), the associated bracket flow solution is

$$A(t) = (1+5t)^{-1/2} e^{s(t)E} A e^{-s(t)E} = (1+5t)^{-1/2} \left(\cos \frac{s(t)}{\sqrt{2}} A + \sin \frac{s(t)}{\sqrt{2}} A^{\perp} \right),$$

where

$$E = \frac{1}{2}(D - D^t) = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c|c} E_1 & 0 & \\ \hline 0 & E_1 & \\ \hline & & 0 \end{array}\right), \quad E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and

$$A^{\perp} = \left(\begin{array}{c|c} 0 & B' \\ \hline C' & 0 \end{array}\right) \quad B' = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

As in [21, Example 5.28], we obtain that $A(t)/|A(t)|$ runs on a circle and $A(t)$ converges to zero rounding in a cone.

A Contraction of G_2 and $\mathrm{SU}(3)$-identities

Let φ be a G_2-structure with Hodge dual 4-form ψ and induced $\mathrm{SU}(3)$-structure $(\omega, \rho^+ + i\rho^-) \in \Lambda^2(\mathbb{R}^6)^* \oplus \Lambda^3(\mathbb{C}^3)^*$. From [14, §A.3] and [4, §2.2], we gather the following contraction identities for G_2 and $\mathrm{SU}(3)$-structures, respectively.

Contractions of φ with φ:

$$\varphi_{abj}\varphi^{ab}_{\ \ k} = 6g_{jk}, \tag{A.1}$$

$$\varphi_{apq}\varphi^{a}_{\ jk} = g_{pj}g_{qk} - g_{pk}g_{qj} + \psi_{pqjk}. \tag{A.2}$$

Contractions of φ with ψ:

$$\varphi_{ijq}\psi^{ij}_{\ \ kl} = 4\varphi_{qkl}, \tag{A.3}$$

$$\begin{aligned}\varphi_{ipq}\psi^{i}_{\ jkl} &= g_{pj}\varphi_{qkl} - g_{jq}\varphi_{pkl} + g_{pk}\varphi_{jql} \\ &\quad - g_{kq}\varphi_{jpl} + g_{pl}\varphi_{jkq} - g_{lq}\varphi_{jkp}.\end{aligned} \tag{A.4}$$

Contractions of ψ with ψ:

$$\psi_{abcd}\psi^{ab}_{\ \ mn} = 4g_{cm}g_{dn} - 4g_{cn}g_{dm} + 2\psi_{abmn}, \tag{A.5}$$

$$\psi_{abcd}\psi_{m}^{\ bcd} = 24g_{am}, \tag{A.6}$$

Contractions of ω with ω and $\rho^{\pm}$ with ω:

$$\omega_{ip}\omega^{p}_{\ j} = -\delta_{ij}, \quad \rho^{+}_{iab}\omega^{ab} = 0, \quad \rho^{+}_{ijp}\omega^{p}_{\ k} = \rho^{-}_{ijk}, \quad \rho^{-}_{ijp}\omega^{p}_{\ k} = -\rho^{+}_{ijk}. \tag{A.7}$$

Contraction of $\rho^{\pm}$ with $\rho^{\pm}$:

$$\rho^{+}_{ijp}\rho^{+\ p}_{kl} = -\omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk} + \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} = \rho^{-}_{ijp}\rho^{-\ p}_{kl}. \tag{A.8}$$

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