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Parallelisms on the 7-sphere

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Abstract. We survey Kirchhoff's classical construction of parallelisms on spheres, induced by almost complex structures on their equator. Motivated by the Hopf problem, we focus on the 7-dimensional context, highlighting the construction's proximity to notions of Lie group structure and integrability of almost-complex structures on $\mathbb{S}^6$. We explain how integrability of such almost complex structures and parallelisms are mediated by a particular notion of torsion, a fact which has interesting algebraic and geometric repercussions.

Keywords: Almost-complex structures, Hopf conjecture, 7-sphere, parallelisms, Kirchhoff frame.

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1 Introduction

The existence of special algebraic and geometric structures on spheres is a long-standing matter of mathematical inquiry, arguably dating back to Cartan's seminal proof that high-dimensional spheres cannot be Lie groups. In particular, the existence of global frame fields, or *parallelisms*, on odd-dimensional spheres has been shown by Kirchhoff in 1947 to bear relation to almost complex structures on their even-dimensional equatorial sphere. Since then, a fruitful classification programme spanned over almost two decades, to which Borel-Serre, Kervaire and Bott–Milnor (independently), and Adams added crucial contributions:

1. [6] $\mathbb{S}^{n+1}$ is a Lie group $\implies n = 0, 2$.
2. [12] $\mathbb{S}^n$ is almost complex $\implies \mathbb{S}^{n+1}$ is parallelisable.
3. [3] $\mathbb{S}^n$ is almost complex $\implies n = 0, 2, 6$.
4. [11, 4] $\mathbb{S}^{n+1}$ is parallelisable $\implies n = 0, 2, 6$.
5. [1] $\mathbb{S}^{n+1}$ is an H-space $\iff n = 0, 2, 6$ (Adams)

NB.: Of course (2)&(4) $\implies$ (3), but the historical timeline worked out differently.

This successful line of investigation culminates at the specificity of the problem of parallelisation on the spheres $\mathbb{S}^3$ and $\mathbb{S}^7$, which correlates to the study of almost complex structures (ACSs) respectively on $\mathbb{S}^2$ and $\mathbb{S}^6$. Moreover, it acquires additional texture from the perspective of differential topology, once we pose the integrability question about those structures, even if only to low-order obstructions. For a parallelism, it takes the form of global constancy of its structure functions, ie. of *constant torsion*, whereas for an almost complex structure it amounts to the vanishing of its Nijenhuis tensor. The situation on $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ is completely understood, and a smooth constant parallelism can be explicitly gathered from any three independent left-invariant vector fields given by its Lie group structure, as well as being directly associated to the canonical (integrable) complex structure on $\mathbb{S}^2$. Meanwhile, a similar narrative is far from clear on $\mathbb{S}^7$, where indeed it stumbles upon the famous Hopf problem: *whether a(n integrable) complex structure exists at all on* $\mathbb{S}^6$. If such a structure could exist, then it makes sense to ponder what consequences that would bear upon the torsion of the corresponding parallelism on $\mathbb{S}^7$, since after all the latter *does not admit* a Lie group structure.

For the history of the Hopf problem, we refer the reader to the excellent survey [2]. Our central motivation here is the fact that the construction of Kirchhoff frames is an example of simple mathematics yielding thought-provoking conclusions, intertwining geometry, algebra and topology. In particular, we feel that a number of interesting connections in this story have been somewhat estranged by maths curricula over the past decades, and it might be a good moment to refresh the topic under a modern light. For example, although the relation between the associator and the Nijenhuis tensor in the case of the classical almost complex structure induced by the octonions is probably well-known to experts, it is not well-documented, cf. [15, Remark 2.3]. We should also mention that a contemporary perspective on $\{e\}$-structures, seen as homogeneous sections defining a (trivial) reduction of the frame bundle, as well as their torsion and their infinitesimal deformations, can be found in [16, 8].

The material is organised as follows. In §2 we review the relations between parallelisms and algebraic structures on spheres, as multiplicative spaces. In particular, we explain how the non-associativity of the octonions is in a precise sense 'to blame' for the non-constancy of the canonical parallelism on $\mathbb{S}^7$ and for the non-integrability of the standard ACS on $\mathbb{S}^6$. In §3, we review Kirchhoff's construction of global frames on a sphere from an ACS on the equator, with a few additions such as a class of spherical metrics on $\mathbb{S}^7$ which are particularly compatible with Kirchhoff frames, and a computation of their torsion therewith. Finally, in §4 we address the Hopf problem, explaining how the hypothetical integrability of an ACS on $\mathbb{S}^6$ could lead to a contradiction, by way of an analytical argument yielding an integrable parallelism on $\mathbb{S}^7$.

2 Parallelisms on spheres and algebraic structures

2.1 When does a parallelism come from a Lie group structure?

Each smooth global frame $\sigma = \{X_1, \cdots, X_n\}$ on a manifold M induces a flat (zero curvature) connection Γ, corresponding to the covariant derivative defined by

$$\nabla_Z \left(\sum f^i X_i\right) := \sum Z(f^i) X_i, \quad \text{for} \quad Z \in \mathscr{X}(M).$$

The global structure equations of Γ in the frame σ are:

$$d\omega^i = \tfrac{1}{2} T^i_{\ jk}\, \omega^j \wedge \omega^k, \tag{2.1}$$

where $\{\omega^1, \cdots, \omega^n\}$ is the dual coframe of $\{X_1, \cdots, X_n\}$. The torsion tensor of Γ is given by

$$T(X_j, X_k) = \sum_{i=1}^{n} T^i_{\ jk}\, X_i = -[X_j, X_k]. \tag{2.2}$$

Furthermore the torsion tensor of Γ is parallel if and only if the *structure functions* $T^i_{\ jk} : M \to \mathbb{R}$ are constant. More generally, a tensor field on M is parallel with respect to Γ if and only if it has constant components with respect to the frame field $\{X_1, \cdots, X_n\}$. Equation (2.2) resembles the way in which one defines the structure constants of a Lie algebra, i.e., (2.1) looks like the Maurer-Cartan equation; this is not a coincidence, since Lie groups are always parallelisable. Moreover, the following converse states which parallelisms come from a Lie group structure on M:

Theorem 2.1 ([9], Theorem 5)**.** *Let M be a simply connected manifold admitting a complete flat linear connection with torsion invariant under parallel translation. Then M admits a Lie group structure such that left-translations induce the original connection.*

The above theorem was proved by Chern [7, Section 5, page 128], in terms of an $\{e\}$-structure on M. Generalisations of this result to not necessarily simply-connected manifolds have been proved many times in the literature, e.g., Wolf [23, Proposition 2.5] within the context of absolute parallelisms. All of those are rooted, one way or another, on Cartan's local equivalence method, see Sternberg [20, Theorem 2.4, Chapter V].

Proposition 2.2 ([23], Proposition 2.5)**.** *Let σ be a smooth parallelism on a connected manifold M. The following statements are equivalent:*

(i) *σ has complete associated connection and parallel torsion;*

(ii) *M has the structure of the coset space $G/D = \{Dg : g \in G\}$, for a connected Lie group G and a discrete subgroup $D \subset G$, such that σ is induced by left-translations of G.*

In general, a linear connection on a compact manifold is not necessarily complete. However, as the geodesics of the associated connection Γ consist of the integral curves of the vector fields $\{X_1, \cdots, X_n\}$, compactness of M implies that Γ is complete. In a local coordinate patch $(U, x_1, \cdots, x_n)$, we can write the vector fields $X_j = \sum X_j^l \frac{\partial}{\partial x^l}$ and the dual forms $\omega^j = \sum \omega_l^j dx^l$, where (ω_l^j) is the inverse matrix of $\left(X_j^l\right)$) in terms of the local basis of vectors fields and differential forms. The structure functions may then be computed by the formula:

$$T^i_{\ jk} = \sum_{r,s=1}^{n} X_j^r X_k^s \left(\frac{\partial \omega_s^i}{\partial x^r} - \frac{\partial \omega_r^i}{\partial x^s} \right). \tag{2.3}$$

2.2 The octonions and the classical parallelism of the 7-sphere

Let us briefly discuss the parallelism of $\mathbb{S}^7$ induced by the octonions, in the light of Hicks' Theorem 2.1. The next proposition explains how orthogonal multiplications define vector fields on spheres.

Proposition 2.3 ([5], Proposition 7.3.1)**.** *Suppose we have a map ν : $\mathbb{R}^{k+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, linear in the first factor and continuous in the second factor, satisfying:*

i) $\nu(v,z)=0$ *implies* $z=0$ *or* $v=0$,

ii) *there exists* $e\in\mathbb{R}^{k+1}$ *such that* $\nu(e,z)=z$ *for all* $z\in\mathbb{R}^{n+1}$,

then $\mathbb{S}^n$ *admits* k *independent vector fields.*

The proof can be found in [5] and, though stated for bilinear maps ν, it only uses linearity in the first factor. As a corollary of the above Proposition, the multiplications in the complex, quaternion, and octonion numbers induce parallelisms on $\mathbb{S}^1,\mathbb{S}^3$ and $\mathbb{S}^7$, respectively. For instance, fix the canonical basis of $\mathbb{O}$ given by the identity 1 and seven imaginary octonions e_i, $i=1,\cdots,7$, satisfying the multiplication rule

$$e_ie_j=-\delta_{ij}+a_{ijk}e_k,$$

where the structure constants a_{ijk} are totally skew-symmetric. Using Proposition 2.3, we construct seven linearly independent vector fields X_i on the sphere $\mathbb{S}^7\subset\mathbb{O}$ of unit octonions as follows:

$$X_i(x)=e_ix \quad \text{for} \quad x\in\mathbb{S}^7,\ i=1,\cdots,7.$$

Let us compute the structure functions of this global frame. Note that multiplication in this particular case is linear in both factors, therefore the Lie brackets $[X_i,X_j]$ can be computed by the commutator of the corresponding linear maps.

$$\begin{aligned}[X_i,X_j](x)&=e_i(e_jx)-e_j(e_ix)\\&=2a_{ijk}e_kx-2[e_i,e_j,x]\\&=2\left(a_{ijk}-\langle[e_i,e_j,x],e_kx\rangle\right)X_k(x),\end{aligned}\tag{2.4}$$

where $[a,b,c]:=(ab)c-a(bc)$ is the *associator*, $\langle a,b\rangle:=\frac{1}{2}\left(a\bar b+b\bar a\right)$ is the standard inner product, and conjugation is defined by $\bar 1=1$, $\bar e_i=-e_i$ and $\overline{ab}=\bar b\bar a$.

Remark 2.4. While the non-commutativity of the octonions causes the non-vanishing of the torsion, *their non-associativity causes the non-constancy of the structure functions of the classical parallelism of* $\mathbb{S}^7$. Compare to Remark 2.7, below.

Remark 2.5. We used the alternativity of the octonionic product to prove the second equality in (2.4). Compare to Remark 2.8, below.

Remark 2.6. Note the structure functions coincide with the structure constants of the algebra at the North and South poles, i.e., at 1 and -1 in $\mathbb{O}$. Compare to Remark 3.8, below.

2.3 The octonions and the induced almost complex structure on the 6-sphere

Let us briefly recall how the multiplication in the octonions induces an almost complex structure on $\mathbb{S}^6$. As we will see in §3, Kirchhoff's construction is modeled on this, in fact its proof reverses this process, by reconstructing the 'multiplication' of $\mathbb{R}^8$ from the almost complex structure, see also Remark 3.10 and Section 4.2.

Let $\Im\mathbb{O} \subset \mathbb{O}$ denotes the hyperplane of imaginary octonions orthogonal to $1 \in \mathbb{O}$, and let $\mathbb{S}^6 \subset \Im\mathbb{O}$ be the sphere of unit imaginary octonions. Right-multiplication by $y \in \mathbb{S}^6$ induces an orthogonal linear map $R_y : \mathbb{O} \to \mathbb{O}$ such that $(R_y)^2 = -1$. Moreover, R_y maps $1 \mapsto y$ and $y \mapsto -1$, hence it preserves the 2-plane spanned by 1 and y, as well as its orthogonal 6-plane, which can be identified with $T_y\mathbb{S}^6 \subset \mathbb{O}$. It follows that R_y induces an almost complex structure on $\mathbb{S}^6$. Now we are going to show that its Nijenhuis tensor can be expressed in terms of the associator of the octonions $\mathbb{O}$.

The Nijenhuis tensor can be computed by:

$$\begin{aligned} N(X,Y) = {} & d(JY)(JX) - d(JX)(JY) - dY(X) + dX(Y) \\ & - J\left(d(JY)(X) - dX(JY)\right) - J\left(dY(JX) - d(JX)(Y)\right). \end{aligned}$$

To see this, note that in Euclidean space we can compute the Lie bracket of vector fields $X, Y : \mathbb{S}^6 \to \mathbb{R}^7$ by

$$[X,Y] = dY(X) - dX(Y),$$

where dX and dY denote respectively the differentials of X and Y, as maps. By definition $J_aY_a = Y_a \cdot a$, where $a \in \mathbb{S}^6$ and Y is a vector field on

$\mathbb{S}^6$. Differentiating, we get

$$d(JY)(JX) = J\left(dY(JX)\right) + Y \cdot JX,$$
$$J\left(d(JY)(X)\right) = (-1)dY(X) + J(Y \cdot X).$$

Then

$$N(X,Y) = Y \cdot JX - X \cdot JY - J(Y \cdot X) + J(X \cdot Y).$$

For $b, c \in T_a\mathbb{S}^6$ we get:

$$N_a(b,c) = c \cdot (b \cdot a) - b \cdot (c \cdot a) - (c \cdot b) \cdot a + (b \cdot c) \cdot a = 2[a,b,c]. \quad (2.5)$$

Remark 2.7. Therefore *the non-associativity of the octonions is responsible for the non-integrability of the almost complex structure* J. Compare to Remark 2.4.

Remark 2.8. To establish the last equality in (2.5) we used the fact that the algebra of octonions is alternative. Compare to Remark 2.5.

3 Kirchhoff's theorem: old and new

3.1 Statement and proof

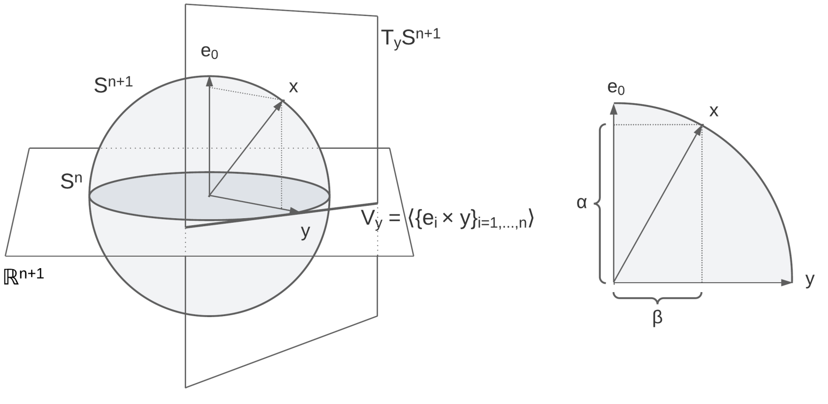
Now we state Kirchhoff's theorem and then survey its proof, which is fairly straightforward but maybe not so widely remembered nowadays, cf. [12], [13, Theorem V], see also Kobayashi-Nomizu [14, Chapter IX, Example 2.6].

Theorem 3.1 ([12], Theorem 4)**.** *If the sphere $\mathbb{S}^n$ admits an almost complex structure, then $\mathbb{S}^{n+1}$ is parallelisable.*

We need to exhibit a field σ of linear frames on $\mathbb{S}^{n+1}$. Let J be an almost complex structure on the equatorial sphere $\mathbb{S}^n$. Fix a subspace $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ in the ambient vector space and a unit vector $e := e_0 \in \mathbb{R}^{n+2}$ perpendicular to $\mathbb{R}^{n+1}$, in the standard Euclidean inner-product. Denote by $\mathbb{S}^n$ and $\mathbb{S}^{n+1}$ the unit spheres in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+2}$ respectively.

Lemma 3.2. *At each $y \in \mathbb{S}^n$, the almost complex structure J_y can be extended to a map $\widetilde{J}_y : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ such that $\widetilde{J}_y^2 = -\mathrm{I}$. Moreover, if J_y is compatible with the inner product on $\mathbb{S}^n$, then $\widetilde{J}_y \in \mathrm{SO}(n+2)$.*

Proof. Given $y \in \mathbb{S}^n$ denote by V_y the n-dimensional vector subspace of $\mathbb{R}^{n+2}$ parallel to the tangent space $T_y(\mathbb{S}^n)$ in $\mathbb{R}^{n+2}$ and J_y the linear endomorphism of V_y corresponding to the linear endomorphism of $T_y(\mathbb{S}^n)$ given by J. Define a linear transformation $\widetilde{J}_y : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ by $\widetilde{J}_y(e) = y$, $\widetilde{J}_y(y) = -e$ and $\widetilde{J}_y(z) = J_y(z)$ for $z \in V_y$. It follows from $J^2 = -\mathrm{I}$ that $\widetilde{J}_y^2 = -\mathrm{I}$. ☐



Let $x \in \mathbb{R}^{n+2}$, then it can be written uniquely as follows:

$$x = \alpha e + \beta y, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \geq 0, \quad \text{and} \quad y \in \mathbb{S}^n. \tag{3.1}$$

We will refer to y as the *equatorial projection of x*. Define the linear transformation:

$$\widetilde{\sigma}_x : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}, \quad \widetilde{\sigma}_x := \alpha I + \beta \widetilde{J}_y. \tag{3.2}$$

Lemma 3.3. *The map $\widetilde{\sigma}_x$ is an automorphism of $\mathbb{R}^{n+2}$. Moreover, if $x \in \mathbb{S}^{n+1}$ and J_y is compatible with the inner product on $\mathbb{S}^n$, then $\widetilde{\sigma}_x \in \mathrm{SO}(n+2)$.*

Proof. It is straightforward to check that $\frac{1}{\alpha^2+\beta^2}\widetilde{\sigma}_x^t = \frac{1}{\alpha^2+\beta^2}(\alpha I - \beta \widetilde{J}_y) = \widetilde{\sigma}_x^{-1}$, which yields the claim when $\alpha^2 + \beta^2 = 1$. □

When $x \in \mathbb{S}^{n+1}$, note also that $\widetilde{\sigma}_x(e) = x$, and denote each restriction of $\widetilde{\sigma}_x$ to $\mathbb{R}^{n+1}$ by

$$\sigma_x := \widetilde{\sigma}_x|_{\mathbb{R}^{n+1}}, \quad x \in \mathbb{S}^{n+1}.$$

Lemma 3.4. *When $x \in \mathbb{S}^{n+1}$, each linear frame $\sigma_x : \mathbb{R}^{n+1} \to T_x(\mathbb{S}^{n+1})$ is an isomorphism.*

Proof. Recall that $\mathbb{R}^{n+1}$ is spanned by y (the equatorial projection of x) and V_y. Clearly $\sigma_{\pm e} = \pm \mathrm{I} : \mathbb{R}^{n+1} \to T_{\pm e}\mathbb{S}^{n+1}$, respectively. For a point $x \in \mathbb{S}^{n+1}_\times := \mathbb{S}^{n+1} \setminus \{\pm e\}$ and a vector $v \in \mathbb{R}^{n+1} = \mathbb{R}y \oplus V_y$, using the standard inner-product of $\mathbb{R}^{n+2}$ and recalling that $\langle y, e\rangle = 0$, we obtain

$$\sigma_x(v) = \alpha v - \beta\langle v, y\rangle e + \beta \tilde{J}_x(v^\perp),$$

where $v^\perp \in V_y$. Note that $\sigma_x(y) = \alpha y - \beta e$ and $\sigma_x(v) = \alpha v + \beta \tilde{J}_x(v)$ when $v \in V_y$. Observe that as $v^\perp \in V_y$ then $\tilde{J}_x(v^\perp) \in V_y$, so $\tilde{J}_x(v^\perp)$ is orthogonal to both y and e, hence to $x = \alpha e + \beta y$.

It is then straightforward to verify that σ_x indeed maps into $T_x\mathbb{S}^{n+1}$:

$$\begin{aligned}
\langle \sigma_x(v), x\rangle &= \langle \alpha v - \beta\langle v, y\rangle e + \beta \tilde{J}_x(v^\perp), x\rangle \\
&= \alpha\langle v, x\rangle - \beta\langle v, y\rangle\langle e, x\rangle + \beta\langle \tilde{J}_x(v^\perp), x\rangle \\
&= \alpha\beta\langle v, y\rangle - \alpha\beta\langle v, y\rangle \\
&= 0.
\end{aligned}$$

Finally, σ_x is a linear map and, as $\tilde{J}_x^2 = -\mathrm{I}_{n+1}$, σ_x is an isomorphism

$$\begin{aligned}
\tilde{J}_x(\sigma_x(v)) &= \alpha \tilde{J}_x(v) + \beta \tilde{J}_x^2(v) \\
&= \alpha \tilde{J}_x(v) - \beta v
\end{aligned}$$

so $v = \alpha\sigma_x(v) + \beta \tilde{J}_x(\sigma_x(v))$. and σ_x is an isomorphism from $\mathbb{R}^{n+1}$ to $T_x(\mathbb{S}^{n+1})$. □

Remark 3.5. Kirchhoff's theorem does not assume any property of the almost complex structure J.

Remark 3.6. If moreover J is an almost Hermitian structure, i.e., compatible with some Riemannian metric g on $\mathbb{S}^n$ (this is always possible), then the theorem above can be found in [19, Theorem 41.19]. Steenrod noted that in this case the global frame $\widetilde{\sigma}$ is in fact orthogonal, that is, $\widetilde{\sigma}_x\left(\widetilde{\sigma}_x\right)^t = I$, for $x \in \mathbb{S}^{n+1}$. More generally, we have $\widetilde{\sigma}_x\left(\widetilde{\sigma}_x\right)^t = \|x\|^2 I$, for $x \in \mathbb{R}^{n+2}$. This will be used in Section 4.2. In what follows, $\|x\|$ will denote the Euclidean norm of x.

Remark 3.7. The vector fields $\{X_i(x) := \sigma_x(e_i)\}_{i=1,\cdots,n+1}$ defining the parallelism in Theorem 3.1 are given explicitly by

$$X_i(x) = x_0 e_i - x_i e_0 + \beta(x) J_y\left(e_i - \langle y, e_i\rangle y\right),$$

where $\{e_i\}_{i=0,\cdots,n+1}$ is the canonical basis of $\mathbb{R}^{n+2}$.

Remark 3.8. The linear frame σ is smooth at all points of $\mathbb{S}^{n+1}$ except at e and $-e$, where it is merely Lipschitz continuous. The following approximation theorem, originally proved in somewhat outdated language by Steenrod [19, Theorem 6.7], guarantees that every continuous section of a smooth fibre bundle can be approximated arbitrarily closely (in a suitable topology) by a smooth section:

Theorem 3.9. *[18, Chap. III, Prop. 5.11 & 5.12] Let $E \to X$ be a smooth vector bundle over a smooth manifold X. Denote by $\mathscr{F}$ the $C^0(X)$-module of (continuous) sections $X \to E$, and let $\|.\| : \mathscr{F} \to C^0(X)$ be any Euclidean norm. For any strictly positive function $\epsilon \in C^0(X)$ and continuous section $\sigma : X \to E$, there is then a smooth section $\hat{\sigma} : X \to E$ such that $\|\hat{\sigma} - \sigma\| < \epsilon$. In particular, if there is a continuous nowhere-vanishing section $\sigma : X \to E$, there is also a smooth nowhere-vanishing section $\hat{\sigma} : X \to E$.*

Therefore, unless otherwise mentioned, we may assume σ is a smooth linear frame that coincides on $\mathbb{S}^{n+1}_\times := \mathbb{S}^{n+1} \setminus \{e, -e\}$ with the one constructed in the above theorem, except at a small neighbourhood of the poles; we will refer to it in this case as the *smooth Kirchhoff frame*.

Remark 3.10. We can use Proposition 2.3 to conclude that $\mathbb{S}^{n+1}$ is parallelisable, by Kirchhoff's theorem, since the map $\nu : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $\nu(v, z) := \widetilde{\sigma}_z v$ satisfies the hypotheses of Proposition 2.3.

Example 3.11. Let us illustrate the construction in the familiar $n = 2$ case, when the standard vector cross-product on $\mathbb{R}^3$ defined by quaternion multiplication induces an (almost) complex structure J on $\mathbb{S}^2$. At the poles we have $\sigma_{\pm e_0} = \pm I$; at any other $x \neq \pm e_0$, we have $\beta = 1 - x_0^2 \neq 0$ and the equatorial projection is $y = \frac{1}{\beta}(x - x_0 e_0) \in \mathbb{S}^2$, so

$$y = \frac{1}{1 - x_0^2}\begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad [J_y] = \frac{1}{1 - x_0^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & x_2 \\ 0 & x_3 & 0 & -x_1 \\ 0 & -x_2 & x_1 & 0 \end{pmatrix}.$$

The induced Kirchhoff frame on $\mathbb{S}^3$ is obtained explicitly as follows:

$$\begin{aligned}
X_1(x) &= x_0 e_1 - x_1 e_0 + (1 - x_0^2) J_y(e_1) \\
&= \begin{pmatrix} -x_1 \\ x_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & x_2 \\ 0 & x_3 & 0 & -x_1 \\ 0 & -x_2 & x_1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_0 \\ x_3 \\ -x_2 \end{pmatrix} \\
X_2(x) &= x_0 e_2 - x_2 e_0 + (1 - x_0^2) J_y(e_2) \\
&= \begin{pmatrix} -x_2 \\ 0 \\ x_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & x_2 \\ 0 & x_3 & 0 & -x_1 \\ 0 & -x_2 & x_1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -x_2 \\ -x_3 \\ x_0 \\ x_1 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
X_3(x) &= x_0e_3 - x_3e_0 + (1-x_0^2)J_y(e_3)\\
&= \begin{pmatrix} -x_3\\ 0\\ 0\\ x_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -x_3 & x_2\\ 0 & x_3 & 0 & -x_1\\ 0 & -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} 0\\ 0\\ 0\\ 1 \end{pmatrix} = \begin{pmatrix} -x_3\\ x_2\\ -x_1\\ x_0 \end{pmatrix}
\end{aligned}
$$

ie.,

$$
\sigma_x = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3\\ x_1 & x_0 & -x_3 & x_2\\ x_2 & x_3 & x_0 & -x_1\\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \in \mathrm{SO}(4).
$$

These ideas lead to the following question, regarding the Hopf problem. Assume, for a moment, that $\mathbb{S}^6$ admits an *integrable* almost complex structure J. By Kirchhoff's Theorem 3.1, this induces a parallelism on the sphere $\mathbb{S}^7$, and we have the explicit form of the corresponding global frame in terms of J.

Question 1. To what extent does the integrability condition on J on $\mathbb{S}^6$ amount to the constancy of the structure functions of the smooth Kirchhoff frame defining the parallelism on $\mathbb{S}^7$?

We will return to this Question in §4.1. Notice finally that, even if in general the structure functions of the Kirchhoff frame are not constant, we can certainly aim at expressing the full torsion tensor for this $\{e\}$-structure on the punctured sphere $\mathbb{S}^7_\times$ in terms of the almost complex structure J on $\mathbb{S}^6$.

3.2 Example: Kirchhoff frames on $\mathbb{S}^7$

We will now specialise the language of Kirchhoff's proof to the context of the unit sphere $\mathbb{S}^7 \subset \mathbb{R}^8$ and its (equatorial) sphere $\mathbb{S}^6 := \{x \in \mathbb{S}^7 \mid x_8 = 0\}$ and fill in some of the details of the previous section. We still denote the North pole by the vector $e := (1, 0, \dots, 0) \in \mathbb{S}^7$.

We assume that $\mathbb{S}^6$ is equipped with an almost complex structure J, that is for any $y \in \mathbb{S}^6$, $J_y : T_y\mathbb{S}^6 \to T_y\mathbb{S}^6$ with $J_y^2 = -\mathrm{I}$. Given $x \in \mathbb{S}^7$, we write

$$x = \cos\phi\, e + \sin\phi\, y,$$

for the zenith angle $\phi \in [0, \pi]$ and the equatorial projection $y \in \mathbb{S}^6$. We denote by V_y the tangent space $T_y\mathbb{S}^6$ of $\mathbb{S}^6$ at the point y, so that

$$T_y\mathbb{S}^7 = \mathbb{R}y \oplus V_y.$$

At each point $x \in \mathbb{S}^7 \subset \mathbb{R}^8$, we extend the almost complex structure J on $\mathbb{S}^6$ to $\tilde{J} : \mathbb{R}^8 \to \mathbb{R}^8$ by

$$\tilde{J}_x(e) = y, \quad \tilde{J}_x(y) = -e, \quad \tilde{J}_x(z) = J_y(z), \forall z \in V_y.$$

Note that $\tilde{J}_x^2 = -\mathrm{I}_8$. This definition could easily be extended into an almost complex structure on $\mathbb{R}^8$.

Setting $\alpha = \cos\phi$ and $\beta = \sin\phi$, the Kirchhoff frame is defined at each point $x \in \mathbb{S}^7$ by

$$\begin{aligned} \sigma_x : \mathbb{R}^7 &\to T_x\mathbb{S}^7 \\ v &\mapsto \sigma_x(v) = \alpha v + \beta\tilde{J}_x(v). \end{aligned}$$

The Kirchhoff frame assumes a particularly simple form in spherical coordinates. At any given $y \in \mathbb{S}^6$, let $\{f_i\}_{i=1,\dots,6}$ be a basis of V_y, so that $\{y, f_i\}_{i=1,\dots,6}$ spans $\mathbb{R}^7 \in \mathbb{R}^8$. Now, introducing the zenith coordinate vector $\partial_\phi = -\beta e + \alpha y$, we obtain a basis of $T_x\mathbb{S}^7_\times$ as $\{\partial_\phi, f_i\}_{i=1,\dots,6}$, in which the Kirchhoff frame field is expressed as the (7×7)-matrix:

$$\begin{pmatrix} -1 & 0\dots0 \\ 0 & \\ \vdots & J_y \\ 0 & \end{pmatrix} \tag{3.3}$$

Remark 3.12. While the matrix expression (3.3) of the Kirchhoff frame field is simple enough, it is written with respect to a moving frame – which will change from point to point. An alternative choice is to consider now the canonical basis (up to order) $E = \{e_0 = e, e_1, \dots, e_7\}$ of $\mathbb{R}^8$. For a point $x = \alpha e + \beta y \in \mathbb{S}^7$, with equatorial projection $y = (0, y_1, \dots, y_7) \in \mathbb{S}^6 \subset \mathbb{R}^7$, we know that $\tilde{J}_x(e_0) = \tilde{J}_x(e) = y$ and

$$\begin{aligned}\tilde{J}_x(e_i) &= \tilde{J}_x\Big(\langle e_i, y\rangle y + (e_i - \langle e_i, y\rangle y)\Big)\\ &= y_i \tilde{J}_x(y) + \tilde{J}_x\Big((e_i - \langle e_i, y\rangle y)\Big)\\ &= y_i \tilde{J}_x(y) + J_y\Big((e_i - \langle e_i, y\rangle y)\Big), \quad \text{for} \quad i = 1, \dots, 7.\end{aligned}$$

since $e_i - \langle e_i, y\rangle y \in V_y$. Therefore, in the basis E, the operator $\tilde{J}_x$ is given by the (8×8)-matrix:

$$[\tilde{J}_x]_E = \begin{pmatrix} 0 & -y_1 \cdots - y_7 \\ y_1 & \\ \vdots & [J_y \circ \gamma_y] \\ y_7 & \end{pmatrix}$$

where $[J_y \circ \gamma_y]$ is the matrix of the operator $J_y \circ \gamma_y : \mathbb{R}^7 \to \mathbb{R}^7$ in the basis $\{e_1, \dots, e_7\}$, and

$$\gamma_y = \begin{pmatrix} 1 - y_1^2 & -y_1 y_2 & \dots & -y_1 y_7 \\ -y_1 y_2 & 1 - y_2^2 & \dots & -y_2 y_7 \\ \vdots & \vdots & \ddots & \vdots \\ -y_1 y_7 & -y_2 y_7 & \dots & 1 - y_7^2 \end{pmatrix} : v \mapsto v - \langle v, y\rangle y \in V_y \subset \mathbb{R}^7$$

is an orthogonal projection. Then the Kirchhoff frame field reads $[\sigma_x]_E = \alpha\, \mathrm{I}_8 + \beta [\tilde{J}_x]_E$.

3.3 A metric on $\mathbb{S}^7$ adapted to generalised Kirchhoff parallelisms

All the classical discussion so far considers $\mathbb{S}^6 \subset \mathbb{S}^7$ as *round spheres* in the Euclidean metric of $\mathbb{R}^8$. Let us now try and address the same construction from the more general perspective of *spherical* metrics $\tilde{g}$ induced on a smooth 7-sphere by an arbitrary Hermitian metric g on its equatorial 6-sphere, compatible with a given almost complex structure J.

Suppose, without loss of generality up to homotopy, that the embedding of $(\mathbb{S}^6, g)$ in $\mathbb{R}^7$ is star-shaped with respect to the origin. Then g has a natural *polar extension* metric over $\mathbb{R}^8$, obtained in the following way. Let

$$\tilde{\mathbb{S}}^7 := \{x = \cos\phi\, e + \sin\phi\, y \mid y \in \mathbb{S}^6,\ \phi \in [0,\pi]\}.$$

be the new unit 7-sphere, which is homotopic in $\mathbb{R}^8$ to the standard Euclidean $\mathbb{S}^7$. Then, for any point $p \in \mathbb{R}^8$, we define $r = |p|_{\tilde{g}}$ by $p =: rx$, where $x \in \tilde{\mathbb{S}}^7$ is unique by assumption, and let

$$\tilde{g} := dr^2 + r^2 d\phi^2 + r^2(\sin^2\phi) g.$$

Notice in particular that $\tilde{g}(e,y) = 0$, for any $y \in \mathbb{S}^6$. We can then define the group of isometries of $(\mathbb{R}^8, \tilde{g})$:

$$\widetilde{\mathrm{SO}}(8) = \{A \in \mathrm{End}(T\mathbb{R}^8) : A^t \tilde{g} A = \tilde{g}\}.$$

Please observe that this notation is merely suggestive, since the total space, in this case, is not in itself a matrix Lie group. The compatibility condition between $\tilde{g}$ and an almost complex structure $\tilde{J}$ on $\mathbb{R}^8$ is

$$\tilde{g}(u,v) = \tilde{g}(\tilde{J}u, \tilde{J}v) = u^t \tilde{J}^t \tilde{g} \tilde{J} v,$$

which is equivalent to $\tilde{J}^t \tilde{g} \tilde{J} = \tilde{g}$.

Lemma 3.13. *If g is compatible with an almost complex structure J on $\mathbb{S}^6$, then its polar extension $\tilde{g}$ is compatible with $\tilde{J}$ on $\mathbb{R}^8$, i.e.,*

$$\tilde{g}(\tilde{J}u, \tilde{J}v) = \tilde{g}(u,v), \quad \forall u,v \in T_p\mathbb{R}^8,\ p \in \mathbb{R}^8.$$

Proof. Let $x = \frac{p}{|p|_{\tilde{g}}} = \cos\phi e + \sin\phi y \in \tilde{\mathbb{S}}^7$, with equatorial projection $y \in \mathbb{S}^6$. If both $u, v \in T_y\mathbb{S}^6$, then the conclusion is immediate, because $\tilde{g}|_{\mathbb{S}^6} = g$. If $u \in (T_y\mathbb{S}^6)^{\tilde{\perp}}$, then it is contained in the 2-plane $\Pi := \mathrm{span}\{x, \partial_\phi\} = \mathrm{span}\{e, y\}$, and by definition $\tilde{J} : \Pi \to \Pi$ is a rotation by 90 degrees. The conclusion follows. $\square$

Adopting slightly more convenient notation, we redefine the (basepoint-dependent) transpose by $A^{\tilde{\mathrm{t}}} = \tilde{g}^{-1}A^{\mathrm{t}}\tilde{g}$, and the corresponding orthonormal frame bundle over $\tilde{\mathbb{S}}^7$ is characterised by pointwise projection onto the first column:

$$\widetilde{\mathrm{SO}}(8) = \{A \in \mathrm{End}(T\mathbb{R}^8) : A^{\tilde{\mathrm{t}}} = A^{-1}\} \xrightarrow{\pi} (\tilde{\mathbb{S}}^7, \tilde{g}).$$

The associated bundle of Lie algebras is $\widetilde{\mathfrak{so}}(8) := \{B \in \mathrm{End}(T\mathbb{R}^8|_{\tilde{\mathbb{S}}^7}) : B^{\tilde{\mathrm{t}}} = -B\} = \tilde{g}^{-1}\mathfrak{so}(8)$.

Example 3.14. The Kirchhoff almost complex structure $\tilde{J}$ on $\mathbb{R}^8$ is compatible with $\tilde{g}$ so

$$\tilde{J}^t\tilde{g}\tilde{J} = \tilde{g}$$

therefore $\tilde{g}\tilde{J} + \tilde{J}^t\tilde{g} = 0$, so that $\tilde{J}$ is in $\mathfrak{so}(8,\tilde{g})$. Then the Kirchhoff frame constructed from J is $\sigma = \alpha I + \beta\tilde{J}$, with $\alpha^2 + \beta^2 = 1$ takes values in $\widetilde{\mathrm{SO}}(8)$:

$$\sigma^t\tilde{g}\sigma = \left(\alpha I + \beta\tilde{J}^t\right)\tilde{g}\left(\alpha I + \beta\tilde{J}\right) = \alpha^2 g + \alpha\beta\left(\tilde{J}^t g + g\tilde{J}\right) + \beta^2\tilde{J}^t\tilde{g}\tilde{J} = \tilde{g}.$$

3.4 Torsion of a parallelism

Our procedure to compute the torsion of parallelisms on $\tilde{\mathbb{S}}^7$ will follow closely that which was carried out for the Euclidean 3-sphere in [16, §3.1], and we refer the reader there for additional details. In particular, we avoid repeating here the abstract definition of the torsion of a geometric structure as the vertical differential of a homogeneous section.

The tangent space to the orthonormal frame bundle at $z \in \widetilde{\mathrm{SO}}(8)$ is given by:

$$T_z\widetilde{\mathrm{SO}}(8) = \left\{\frac{1}{2}(M - zM^{\tilde{\mathrm{t}}}z) : M \in \mathrm{GL}(8,\mathbb{R})\right\} = z\widetilde{\mathfrak{so}}(8). \tag{3.4}$$

The vertical distribution at the identity of $\widetilde{\mathrm{SO}}(8)$ is

$$\mathcal{V}_{\mathrm{I}} := \ker p_* = \left\{M \in \widetilde{\mathfrak{so}}(8) : M = \begin{pmatrix} 0 & 0 \\ 0 & \hat{M} \end{pmatrix}, \hat{M} \in \widetilde{\mathfrak{so}}(7)\right\}$$

and, at $z \in \widetilde{\mathrm{SO}}(8)$, $\mathcal{V}_z = z\mathcal{V}_{\mathrm{I}}$. Here we define

$$\widetilde{\mathfrak{so}}(7) := \{C \in \mathrm{GL}(7, \mathbb{R}) : C^{\tilde{\mathrm{t}}} = -C\},$$

under the identification $\mathbb{R}^7 = \{p \in \mathbb{R}^8 : \langle p, e\rangle = 0\} = \{p \in \mathbb{R}^8 : \phi = \frac{\pi}{2}\}$.

A $\tilde{g}$-orthonormal parallelism is a section

$$\begin{aligned} \sigma : \tilde{\mathbb{S}}^7 &\to \widetilde{\mathrm{SO}}(8) \\ x &\mapsto \sigma(x) = (x, \sigma_1(x), \dots, \sigma_7(x)) \end{aligned}$$

of the orthonormal frame bundle $\widetilde{\mathrm{SO}}(8) \xrightarrow{\pi} (\tilde{\mathbb{S}}^7, \tilde{g})$. At each $x \in \tilde{\mathbb{S}}^7$, its differential is given by

$$\begin{aligned} d\sigma_x : T_x\tilde{\mathbb{S}}^7 &\to T_{\sigma(x)}\widetilde{\mathrm{SO}}(8) \\ X &\mapsto d\sigma_x(X) =: \mathrm{Proj}_{T_{\sigma(x)}\widetilde{\mathrm{SO}}(8)}(M) \end{aligned}$$

with $M = (X, d\sigma_1(X), \dots, d\sigma_7(X))$. Therefore

$$\begin{aligned} d\sigma_x(X) &= \tfrac{1}{2}\Big(M - \sigma(x)M^{\tilde{\mathrm{t}}}\sigma(x)\Big) = \sigma(x).\tfrac{1}{2}\Big(\sigma(x)^{\tilde{\mathrm{t}}}M - M^{\tilde{\mathrm{t}}}\sigma(x)\Big) \\ &= \sigma(x)\begin{pmatrix} 0 & -v^t \\ v & \hat{M} \end{pmatrix}, \end{aligned}$$

with $\hat{M} \in \widetilde{\mathfrak{so}}(7)$ and $v \in \mathbb{R}^7$, so the torsion of the parallelism is the vertical component

$$T(X_x) = d^{\mathcal{V}}\sigma_x(X) = \sigma(x)\begin{pmatrix} 0 & 0 \\ 0 & \hat{M} \end{pmatrix}.$$

Denoting $\tilde{g}_x$ by $\langle \cdot, \cdot \rangle$, we have

$$\sigma(x)^{\tilde{\mathrm{t}}}M = \begin{pmatrix} \langle x, X\rangle & \langle x, d\sigma_1(X)\rangle & \dots & \langle x, d\sigma_7(X)\rangle \\ \langle \sigma_1, X\rangle & \langle \sigma_1, d\sigma_1(X)\rangle & \dots & \langle \sigma_1, d\sigma_7(X)\rangle \\ \vdots & \vdots & & \vdots \\ \langle \sigma_7, X\rangle & \langle \sigma_7, d\sigma_1(X)\rangle & \dots & \langle \sigma_7, d\sigma_7(X)\rangle \end{pmatrix},$$

and, since $d\sigma_i(X) = X^{\mathcal{H}} + (\tilde{\nabla}_X\sigma_i)^{\mathcal{V}}$, several entries simplify as follows:

$$\begin{aligned} \langle x, X\rangle = \langle \sigma_i, d\sigma_i(X)\rangle = 0, &\quad \langle x, d\sigma_i(X)\rangle = -\langle \sigma_i, X\rangle \\ \langle \sigma_i, d\sigma_j(X)\rangle = \langle \sigma_i, \tilde{\nabla}_X\sigma_j\rangle, &\quad \text{for} \quad i, j = 1, ..., 7. \end{aligned}$$

We obtain, cf. (3.4):

$$d\sigma_x(X) = \sigma(x)\begin{pmatrix} 0 & -\langle\sigma_1, X\rangle & -\langle\sigma_2, X\rangle & -\langle\sigma_3, X\rangle \\ \langle\sigma_1, X\rangle & 0 & \langle\sigma_1, \tilde{\nabla}_X\sigma_2\rangle & \langle\sigma_1, \tilde{\nabla}_X\sigma_3\rangle \\ \langle\sigma_2, X\rangle & \langle\sigma_2, \tilde{\nabla}_X\sigma_1\rangle & 0 & \langle\sigma_2, \tilde{\nabla}_X\sigma_3\rangle \\ \langle\sigma_3, X\rangle & \langle\sigma_3, \tilde{\nabla}_X\sigma_1\rangle & \langle\sigma_3, \tilde{\nabla}_X\sigma_2\rangle & 0 \end{pmatrix}$$

$$\in T_{\sigma(x)}\widetilde{\mathrm{SO}}(8)$$

$$\Rightarrow \quad d^{\mathcal{V}}\sigma_x(X) =$$

$$\sigma(x)\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \langle\sigma_1, \tilde{\nabla}_X\sigma_2\rangle & \dots & \langle\sigma_1, \tilde{\nabla}_X\sigma_7\rangle \\ 0 & \langle\sigma_2, \tilde{\nabla}_X\sigma_1\rangle & 0 & & \vdots \\ 0 & \vdots & & \ddots & \langle\sigma_6, \tilde{\nabla}_X\sigma_7\rangle \\ 0 & \langle\sigma_7, \tilde{\nabla}_X\sigma_1\rangle & \dots & \langle\sigma_7, \tilde{\nabla}_X\sigma_6\rangle & 0 \end{pmatrix}$$

In words, the typical (off-diagonal) elements of $d^{\mathcal{V}}\sigma$ are the 1-forms:

$$(d^{\mathcal{V}}\sigma)_{ij} = \tilde{g}(\tilde{\nabla}\sigma_j, \sigma_i), \quad 1 \leq i \neq j \leq 7.$$

Indeed, those are the Christoffel symbols of $\tilde{\nabla}$ in the frame field σ, if we care to raise an index etc, which are precisely the 'structure functions' introduced in §2.1. By Koszul's formula, we find:

$$T_{ij}^k := (d_i^{\mathcal{V}}\sigma)_j^k = \tilde{g}(\tilde{\nabla}_i\sigma_j, \sigma_k) = \sum \mathrm{sign}(ijk)\tilde{g}([\sigma_i, \sigma_j], \sigma_k), \quad 1 \leq i, j \leq 7. \tag{3.5}$$

In §4.1, we will specialise the above discussion to the particular case of Kirchhoff frames.

Remark 3.15. An alternative approach to computing the torsion of a parallelism is to consider it a section of the SO(7)-principal bundle of orthonormal frames of $(\mathbb{S}^7, \tilde{g})$, ie. the Euclidean unit 7-sphere endowed with the unusual Riemannian metric from $(\mathbb{R}^8, \tilde{g})$. While its total space can no longer be seen as a matrix Lie group, its fibre of frames on the point $x \in \mathbb{S}^7$ is equal to $\widetilde{\mathrm{SO}}(7) = \{A \in \mathrm{GL}(7, \mathbb{R}) : A^{-1} = \tilde{g}_x^{-1}A^{\mathrm{t}}\tilde{g}_x\}$. As we work with the vertical part of the differential of σ, the computations are the same.

4 Reflections on the Hopf problem

4.1 Integrability on $\mathbb{S}^6$ and torsion of smooth Kirchhoff parallelisms

Let us now address the issue raised in Question 1 and reason, for argument's sake, under the outrageous assumption that J is actually an *integrable* complex structure on $\mathbb{S}^6$. Around a given point $y \in \mathbb{S}^6$, one may choose a local system of coordinates $(U, \{\theta_1, \dots, \theta_6\})$ such that

$$J\partial_{\theta_j} = \partial_{\theta_{j+1}}, \quad \text{for} \quad j = 1, 3, 5,$$

where we write ∂_{θ_j} for $\frac{\partial}{\partial\theta_j}$. Then, in a neighbourhood of $x = \alpha e + \beta y \in \tilde{\mathbb{S}}^7$, we have a local system of coordinates $(\phi, \theta_1, \dots, \theta_6)$. The Kirchhoff frame field is then given by the image of the local frame $\{\partial_\phi, \partial_{\theta_1}, \dots, \partial_{\theta_6}\}$ by the matrix (3.3):

$$\begin{aligned}
\sigma_0 &= \sigma_{(\phi,\theta_1,\dots,\theta_6)}(y) = \partial_\phi \\
\sigma_j &= \sigma_{(\phi,\theta_1,\dots,\theta_6)}(\partial_{\theta_j}) = J_y(\partial_{\theta_j}), \quad \text{for} \quad j = 1, \dots, 6.
\end{aligned}$$

Proposition 4.1. *The existence of a complex structure on $\mathbb{S}^6$ would induce a smooth* torsion-free *parallelism on the punctured sphere $\tilde{\mathbb{S}}^7_\times$.*

Proof. It becomes easy to compute the brackets of the vector fields of the Kirchhoff frame field at any point where it is smooth, ie. away from the poles, since our assumption that J is integrable implies that, for any $j = 1, \dots, 6$, $J_y(\partial_{\theta_j}) = \pm\partial_{\theta_k}$, for some $1 \le k \le 6$:

$$\begin{aligned}
[\sigma_0, \sigma_j] &= [\partial_\phi, J_y(\partial_{\theta_j})] = \pm[\partial_\phi, \partial_{\theta_k}] \\
&= 0, \\
[\sigma_{j_1}, \sigma_{j_2}] &= [J_y(\partial_{\theta_{j_1}}), J_y(\partial_{\theta_{j_2}})] = \pm[\partial_{\theta_{k_1}}, \partial_{\theta_{k_2}}] \\
&= 0, \quad \text{for} \quad 1 \le j_1, j_2 \le 6.
\end{aligned}$$

Here we are simply using the fact that local coordinate vector fields commute. Replacing the above vanishings in Koszul's formula (3.5), we find

$$T_{ij}^k = 0, \quad \forall 1 \le i, j \le 7,$$

i.e. the generalised Kirchhoff parallelism induced on $\tilde{\mathbb{S}}^7$ by an integrable almost complex structure J on $\mathbb{S}^6$ is *torsion-free* wherever smooth. $\square$

In the light of Remarks 3.5–3.10, applied to $\mathbb{S}^6 \subset \tilde{\mathbb{S}}^7$, Proposition 4.1 motivates the following observations:

Remark 4.2. If the associated connection of σ can be shown to be complete, this would trigger from Wolf's Proposition 2.2 the conclusion that the 7-sphere *punctured at its poles*, away from which the Kirchhoff parallelism is indeed smooth, is a 'Lie group' (possibly quotiented by a discrete subgroup). This rather unimpressive conclusion opens however the following analytical programme.

Remember that we only need that the torsion functions of a given parallelism be *constant*, not necessarily vanishing, in order to guarantee a Lie group structure on a simply-connected manifold. If this could be achieved from the Kirchhoff parallelism on $\tilde{\mathbb{S}}^7$, under the initial assumption of an integrable J on $\mathbb{S}^6$, then by contradiction Hopf's problem would be answered in the negative. Thus, first, one might try to extend this torsion-free parallelism, along with its complete connection, over the poles with *nonconstant torsion concentrated* around the polar caps. Then, one may hope to deform it homotopically, eg. by a suitable geometric flow, towards a smooth limit with constant torsion everywhere. As a teaser, let us mention that Shi-type estimates for a general flow of $\{e\}$-structures have a particularly amenable form, cf. [8, Remark 1.35]. In any event, this would still be, of course, a very challenging problem.

Remark 4.3. Due to the nature of H-spaces, as we will discuss below, it suffices to achieve through this process a continuous associative multiplication on $\mathbb{S}^7$, in order to obtain a contradiction.

4.2 H-space structures on spheres

We can rephrase Kirchhoff's theorem as follows: if $\mathbb{S}^n$ admits an almost complex structure J, then $\mathbb{S}^{n+1}$ is a an H-space, ie. it admits a continuous multiplication with a two-sided identity element. This is trivial at glance because a parallelisable sphere is well-known to be an H-space [1]. The point is that the induced multiplication on $\mathbb{S}^{n+1}$ is written explicitly in terms of J.

In what follows we keep the notation of Theorem 3.1. Let J be an almost complex structure on $\mathbb{S}^6$, and let $\widetilde{\sigma}$ be the global frame given by Kirchhoff's theorem. Define the map:

$$m : \mathbb{S}^7 \times \mathbb{S}^7 \longrightarrow \mathbb{S}^7, \quad m(x,y) := \widetilde{\sigma}_x\,(y)\,/\|\widetilde{\sigma}_x\,(y)\,\|.$$

It follows from Kirchhoff's theorem this map is a well defined multiplication on $\mathbb{S}^7$ with e as two-sided identity. Thus every almost complex structure J on $\mathbb{S}^6$ defines an H-space structure on $\mathbb{S}^7$. Moreover, recall that we can always assume the given almost complex structure J is compatible with some Riemannian metric g on $\mathbb{S}^7$. Then, using Remark 3.6, we can define a multiplication on $\mathbb{R}^8$:

$$\hat{m} : \mathbb{R}^8 \times \mathbb{R}^8 \longrightarrow \mathbb{R}^8, \quad \hat{m}(x,y) := \widetilde{\sigma}_x\,(y)\,,$$

which satisfies the norm product rule $\|\hat{m}(x,y)\|^2 = \|x\|^2\|y\|^2$, and has a two-sided identity e. The multiplication $\hat{m}$ restricts to the H-space multiplication m on $\mathbb{S}^7$. By a celebrated theorem of Adams [1], the only spheres that admit an H-space structure are $\mathbb{S}^0, \mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$. However, the next theorem by Wallace shows that $\mathbb{S}^7$ does not admit an associative multiplication:

Theorem 4.4. *[22, Corollary 2] If a compact manifold M admits a continuous, associative multiplication with identity, then it is a topological group.*

In fact, combining the above theorem with von Neumann's solution of Hilbert's Fifth problem for compact groups [21] would imply that such M

has actually a Lie group structure. A stronger result in this direction was proved by James:

Theorem 4.5. *[10, Theorem 1.4] There exists no homotopy-associative multiplication on $\mathbb{S}^n$ unless $n = 1$ or 3.*

As we have already seen in (2.5), the non-associativity of the octonions causes the non-integrability of the almost complex structure induced on $\mathbb{S}^6$ by them. We would like to relate the (likely) non-existence of a complex structure on $\mathbb{S}^6$ with the lack of associative – or more generally homotopy-associative – multiplications on $\mathbb{S}^7$.

Question 2. Does the integrability of the almost complex structure J imply the (homotopy-)associativity of the multiplications m and $\hat{m}$?

Working with the multiplication $\hat{m}$ instead of m has the advantage of involving the additive structure of $\mathbb{R}^8$. As observed in Remark 2.8, we employed the alternativity of the octonions in establishing the connection between the Nijenhuis tensor and the associator. It would be reasonable to expect the answer to Question 2 to involve a similar requirement. In this regard, it is interesting to mention the following result, by Norman:

Theorem 4.6. *[17, Corollary 9.3] Any multiplication on a sphere satisfies the Moufang law, up to homotopy:*

$$(x \cdot y)\,(z \cdot x) = (x\,(y \cdot z))\,x.$$

Remark 4.7. In case J be the almost complex structure of $\mathbb{S}^6$ induced by the octonions, Question 1 is related to Question 2. This happens because J comes a priori from an ambient multiplication of $\mathbb{R}^8$, in other words, if we interpret J as a map $\widetilde{J} : \mathbb{S}^6 \longrightarrow SO(8,\mathbb{R})$, $x \longmapsto \widetilde{J}_x$ it extends as a linear map between $\mathbb{R}^8$ and $SO(8,\mathbb{R})$.

Remark 4.8. However, in the general case of an almost complex structure on $\mathbb{S}^6$, Questions 1 and 2 stand at different levels and are not immediately linked. Question 1 asks whether it is possible to integrate the parallelism, i.e., if there exists a multiplication the differential of which essentially induces the global frame. Question 2 asks directly if the multiplication induced by the global frame is (homotopy-)associative. This opens the possibility to combine analytic-homotopic tools, such as geometric flows, to study the problem in the same vein of Remark 4.2.

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