

Birationally equivalent Landau–Ginzburg models on cotangent bundles and adjoint orbits

Bruno Suzuki 

Instituto de Matemática e Estatística da Universidade de São Paulo, R. do Matão, 1010, São Paulo, Brazil

Abstract. We show that the Lie potential on the minimal semisimple adjoint orbit \mathcal{O}_n of $\mathfrak{sl}(n+1, \mathbb{C})$ coincides with toric potential on $T^*\mathbb{P}^n$. We then study the corresponding Landau–Ginzburg models in deformation families and give some examples of how the deformations affect the mirrors.

Keywords: LG models, birational equivalence.

2020 Mathematics Subject Classification: 14E05, 32G05.

1 Strategy

We show that the Lie theoretical potential on the minimal semisimple adjoint orbit \mathcal{O}_n of $\mathfrak{sl}(n+1, \mathbb{C})$ as defined in [15] coincides with standard potential on $T^*\mathbb{P}^n$ associated with the Hamiltonian action of a 1-dimensional algebraic torus \mathbb{C}^* . More precisely, we show that $(T^*\mathbb{P}^n, w)$ and (\mathcal{O}_n, w) are birationally equivalent. Having described this coincidence, we are then

The author was partially supported by MathAmSud ANID project MATH2020011. The author was supported by Grant 2021/11750-7 São Paulo Research Foundation - FAPESP, e-mail: obrunosuzuki@gmail.com

able to write a potential on the deformation family containing both $T^*\mathbb{P}^n$ and \mathcal{O}_n , so that it can be upgraded to a deformation of Landau–Ginzburg models. We then explore specific examples of Landau–Ginzburg models and give examples of how their deformations affect their mirrors. The proposed strategy here is to consider both deformations of varieties and deformations of the potential, combining them to describe deformations of LG models, obtaining families

$$\mathrm{LG} \rightsquigarrow \mathrm{LG}'$$

where LG and LG' behave very differently dualitywise. Indeed, we will have examples $\mathrm{LG}_0 = (T^*\mathbb{P}^1, w)$ that is a selfdual LG model defined on a toric variety which is not affine, deforming to $\mathrm{LG}_2 = (\mathcal{O}_1, w)$ which is a nonselfdual LG model defined on a nontoric affine variety.

Landau–Ginzburg models on adjoint orbits of semisimple Lie algebras $f_H: \mathcal{O}(H_0) \rightarrow \mathbb{C}$ were constructed by Gasparim–Grama–San Martin in [15, 16]. We consider here the minimal case (as in Definition 2.6) which we denote by LG_n . These Landau–Ginzburg models were shown to satisfy the conjecture of Katzarkov, Kontsevich and Pantev about three new Hodge theoretical invariants that take into consideration the potential, see [8, 18] and also [5] for some more features about Hodge diamonds of compactifications.

It would be desirable to carry out the study of all deformations families involving semisimple adjoint orbits of $\mathfrak{sl}(n+1, \mathbb{C})$. However, here we will carry out details only for the case of minimal orbits, presenting more details for the case of $\mathfrak{sl}(2, \mathbb{C})$, which is already nontrivial, as we shall see. Indeed, [4] showed that LG_2 has no projective mirrors, and furthermore, proved that the Fukaya–Seidel category of LG_2 is equivalent to a proper subcategory of the category of coherent sheaves on the second Hirzebruch surface, which however is not realised by any complex subvariety. As a consequence of [4] we know that there does not exist any projective variety whose category of coherent sheaves is equivalent to the Fukaya–Seidel category of LG_2 . However, in [14], using the intrinsic mirror symmetry

recipe of Gross and Siebert [19], Elizabeth Gasparim constructed in [14] a Landau–Ginzburg mirror for LG_2 which is geometrically extremely different from LG_2 . This nontrivial behaviour of duality of LG_2 will be contrasted to the selfduality of LG_0 .

2 Deformations

Definition 2.1. A **Landau–Ginzburg model** (LG) is a pair (X, w) formed by a variety X and a holomorphic function $w: X \rightarrow \mathbb{C}$ (or \mathbb{P}^1) called the superpotential.

To expand our families of examples, we will consider pairs (X, R) where $R: X \rightarrow \mathbb{C}$ is a rational function, defined as $R = f/g$, then this rational function can be considered as a map to \mathbb{P}^1 simply by making $R = [f : g]$, see [6]. Then, such a map can then be promoted to an LG model by blowing up the indeterminacy locus \mathcal{I} of R , that is the set of points where $f = g = 0$, and then R can be extended to a holomorphic map on \widetilde{X} , the blow-up of X at \mathcal{I} . Therefore in the examples that follow, we wish to consider rational potentials.

Definition 2.2. A **rational Landau–Ginzburg model** is a pair (X, R) formed by a variety X and a rational function $R: X \rightarrow \mathbb{P}^1$ called the superpotential.

From now on, by a LG model we mean a rational Landau–Ginzburg model (note that this notion includes all LG models that have a holomorphic potential).

Definition 2.3. A (commutative) **deformation family** of a Landau–Ginzburg model (X, w) is a smooth family of Landau–Ginzburg models (X_t, w_t) , with $t \in D \subset \mathbb{C}^n$ an open ball containing 0, such that $(X_0, w_0) = (X, w)$. We call X_t a **deformation** of X_0 , denoted by $(X_0, w_0) \rightsquigarrow (X_t, w_t)$.

For Landau–Ginzburg models considered in this text, the variety X will be a noncompact Calabi–Yau. This is exactly the case for which Katzarkov, Kontsevich and Pantev proved smoothness of the deformation spaces [21].

Example 2.4. Consider $T^*\mathbb{P}^1$ with coordinates $([1, x], y)$ and the Landau–Ginzburg models $\text{LG}_0 = (T^*\mathbb{P}^1, w_0)$, $\text{LG}_1 = (T^*\mathbb{P}^1, w_1)$ given by the functions $w_0 = x + y + \frac{y^2}{x}$ and $w_1 = 2x$.

Using the potential $w_t = 2x + t(x + y + \frac{y^2}{x})$ on $X_t = T^*\mathbb{P}^1$ we obtain the deformation

$$\text{LG}_0 \rightsquigarrow \text{LG}_1.$$

Definition 2.5. Let G be a Lie group and \mathfrak{g} its Lie algebra. The **adjoint orbit** of an element $H_0 \in \mathfrak{g}$ is

$$\mathcal{O}(H_0) = \text{Ad}(G) \cdot H_0 = \{gH_0g^{-1} : g \in G\}.$$

In all examples considered here H_0 is a diagonal matrix. Since we work with complex semisimple Lie groups G , the orbit $\text{Ad}(K) \cdot H_0$ of a compact subgroup $K < G$ is a **flag manifold** isomorphic to a homogeneous manifold G/P , where P is a parabolic subgroup of G .

Following [8, Def. 1.1], we have:

Definition 2.6. Let \mathcal{O}_n be the adjoint orbit of $H_0 = \text{Diag}(n, -1, \dots, -1)$ in $\mathfrak{sl}(n + 1, \mathbb{C})$, we call it the **minimal orbit**.

The minimal adjoint orbit \mathcal{O}_n is diffeomorphic to the cotangent bundle of the projective space \mathbb{P}^n [16, Thm. 2.1]. \mathcal{O}_n is a nontoric Calabi–Yau manifold.

Example 2.7. Let $\text{LG}_2 = (\mathcal{O}_1, f_H)$, where \mathcal{O}_1 is the semisimple adjoint orbit of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, obtained by choosing $H_0 = \text{Diag}(1, -1)$.

Lemma 6.3 describes \mathcal{O}_1 as a deformation of $T^*\mathbb{P}^1$, explicitly presenting the deformation family. We put on \mathcal{O}_1 the potential given by $f \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = 2x$. Observe that this potential coincides with the potential w defined on

$T^*\mathbb{P}^1$ in Example 2.4, forming LG_1 . Defining the same potential $2x$ on every element of the family, we obtain a deformation of Landau–Ginzburg models

$$\text{LG}_1 \rightsquigarrow \text{LG}_2.$$

We will present a more detailed approach to this deformation family in Example 6.5, including a partial compactification.

Remark 2.8. The choice of potential on \mathcal{O}_1 comes from the following construction. We choose $H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and then the adjoint orbit

$$\mathcal{O}_1 = \text{Ad}(\text{SL}(2))H_0 = \{gH_0g^{-1} : g \in \text{SL}(2)\}$$

is the (smooth) affine hypersurface in \mathbb{C}^3 given by the equation $x^2 + yz - 1 = 0$. The potential $f_H: \mathcal{O}_1 \rightarrow \mathbb{C}$ defined by $A \mapsto \langle H, A \rangle$ gives simply $f_H(A) = 2x$.

As a real manifold, \mathcal{O}_1 is diffeomorphic to $T^*\mathbb{P}^1$. Now we recall Lemma 2.1 of [4] which shows the existence of a Lagrangian submanifold of the adjoint orbit of $\mathfrak{sl}(2)$ that is diffeomorphic to \mathbb{P}^1 . Observe this is not granted a priori since the symplectic structure of $\mathcal{O}(H_0)$ is not that of $T^*\mathbb{P}^1$. However, \mathcal{O}_1 is affine, hence it does not contain an algebraic subvariety isomorphic to \mathbb{P}^1 .

Example 2.9. Observe that in Example 2.4 we varied the potential while keeping the underlying manifold fixed, whereas in Example 2.7 we varied the manifold while keeping the potential. Combining them, we obtain a deformation that changes both, namely

$$\text{LG}_0 \rightsquigarrow \text{LG}_2.$$


We then wish to compare the mirrors of LG_0 and LG_2 , and we will see that they behave very differently. Since $T^*\mathbb{P}^1$ is a toric variety, we can use toric duality for LG_0 .


3 Toric duality

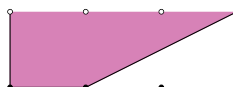
Definition 3.1. An n -dimensional variety X is called **toric** if it satisfies the following two properties:

- there exists an action of the torus $(\mathbb{C}^*)^n \times X \rightarrow X$, and
- the Zariski closure of X is the torus: $\overline{(\mathbb{C}^*)^n} = X$.

For basic notions of toric varieties see [13]. The polytope representing a toric variety is the convex figure obtained by drawing the complement of the torus, i.e., $\overline{X} - (\mathbb{C}^*)^n$. More precisely, it is the image of \overline{X} by the moment map, which is, by the theorems of Atiyah and Guillemin–Sternberg, the convex hull of the image of the fixed points of the torus action.

Example 3.2. $\mathbb{P}^1 \times \mathbb{P}^1$ is depicted by a square .

Example 3.3. \mathbb{P}^2 is depicted by a triangle .

Example 3.4. $T^*\mathbb{P}^1$ is depicted by .

For the case when X is a toric variety, Clarke [12] showed that one can state a generalised version of the Homological Mirror Symmetry conjecture of Kontsevich [22] as a duality between LG models. He also showed that this construction generalises the dualities due to Batyrev–Borisov [3], Berglund–Hübsch [9], Givental [14], and Hori–Vafa [20]. Given a toric LG model (X, f) , the construction in [12] produces a dual LG model (X^\vee, f^\vee) .

We recall some definitions from [12]. Associated to a toric Landau–Ginzburg model is its linear data:

1. the linear data Div , and
2. the linear data Mon .

[12] uses a more refined version of this data, which includes the compactified Kähler class and for duality requires the kopasectic condition, but we will not need such details here. The dual toric Landau–Ginzburg model is obtained by exchanging Div and Mon, that is

$$\text{Div}(X) = \text{Mon}(f^\vee), \quad \text{Div}(X^\vee) = \text{Mon}(f).$$

The matrix Mon describes the infinitesimal action on monomials, for details see [12]. The matrix Div encodes the divisor of the character-to-divisor map. In particular, the Chow group of X is the cokernel of $\text{Div}(X)$. In our examples the rows of Div will represent the divisors of X and the rows of Mon will represent the monomials appearing in the potential f .

Definition 3.5. If there is an isomorphism $(X^\vee, f^\vee) \simeq (X, f)$, then the LG model is called **selfdual**.

We now give some elementary examples of toric duality. Using the work of Clarke in [12], Callander, Gasparim, Jenkins and Silva [11] considered mirror symmetry for Landau–Ginzburg models defined over vector bundles on projective spaces, showing that both the cotangent bundle of \mathbb{P}^1 and the resolved conifold admit potentials that form a self-mirror LG model.

Example 3.6 ($\text{LG}_0 \Leftrightarrow \text{LG}_0$). For the case of $T^*\mathbb{P}^1$ using the potential $x + y + \frac{y^2}{x}$ one obtains a LG model that is dual to itself [11, Prop. 6.2] as depicted in Figure 3.1. Self-duality of this LG model is verified by simply pointing out that in this case the toric data is

$$\text{Mon} = \text{Div} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Example 3.7. (A nontrivial duality) We now see a nontrivial duality $(Y, g) = (X, f)^\vee$. Consider the Landau–Ginzburg models $(X, f) = (\mathbb{P}^2, x + y + x^{-1} + y^{-1})$ and $(Y, g) = (\mathbb{P}^1 \times \mathbb{P}^1, x + y + x^{-1}y^{-1})$ as depicted in Figure 3.2.

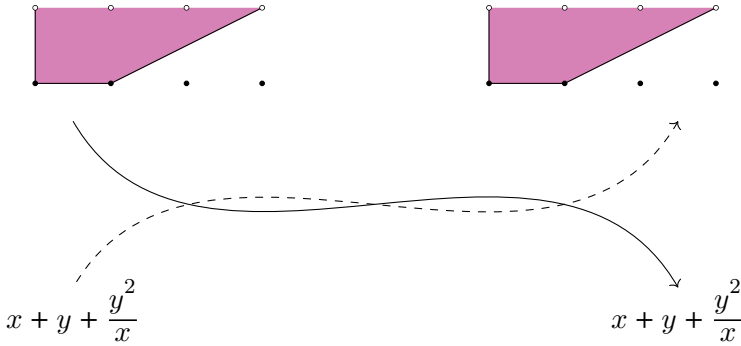


Figure 3.1: Selfdual LG_0 model

Since the Div matrix is given by the inward normals of the moment polytope, we have:

$$\text{Div}_X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \text{Div}_Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To see that (Y, g) is dual to (X, f) just observe that $\text{Mon}_g = \text{Div}_X$, and $\text{Div}_Y = \text{Mon}_f$.

Example 3.8. (Hyperkähler families) The birational isomorphism considered here could be considered in a much broader context. In fact, if instead of considering cotangent bundles of projective spaces, we considered cotangent bundles of generalised flag manifolds, then we could consider families of hyperkähler structures; we recall the definition.

A hyperkähler structure on the Riemannian manifold (M, g) is expressed using three almost complex structures I, J, K satisfying

$$I^2 = J^2 = K^2 = IJK = -1,$$

such that any linear combination $aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$ is a Kähler structure on M , whose corresponding complex structures may be

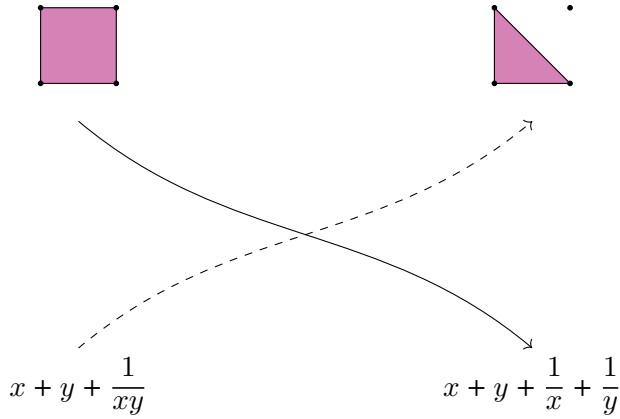


Figure 3.2: Nontrivial toric duality

regarded as a deformation of M . All hyperkähler manifolds are Ricci-flat and consequently Calabi–Yau manifolds.

Families of hyperkähler structures containing both cotangent bundles of flag manifolds and adjoint orbits were studied by Kronheimer [24], Biquard [10], and Kovalev [23], we briefly recall some of their results. Let G be a complex semisimple Lie group with Lie algebra \mathfrak{g} and let H denote the stabiliser of an element of \mathfrak{g} . Kovalev’s theorem [23, Thm. 1.1] describes a family of hyperkählerian structures on G/H . In particular, this family contains a complex structure for which the quotient G/H is isomorphic to a semisimple adjoint orbit of G . More explicitly, the adjoint orbit $\text{Ad}(G) \cdot H_0$ occurs as a deformation of the cotangent bundle of the flag manifold $\text{Ad}(K) \cdot H_0$, where K is a compact subgroup of G .

Many interesting examples can be obtained using the adjoint orbit $\mathcal{O}(H_0)$ of an element of the Cartan subalgebra $H_0 \in \mathfrak{h} \subset \mathfrak{g}$. Suppose we take $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, hence we would have a flag manifold

$$\mathbb{F} = \text{Ad}(\text{SU}(n)) \cdot H_0$$

of $\text{SU}(n)$ and a diffeomorphism

$$\mathcal{O}(H_0) = \text{Ad}(\text{SL}(n)) \cdot H_0 \simeq T^*\mathbb{F}$$

to the cotangent bundle of \mathbb{F} , where the flag manifold is isomorphic a homogeneous manifold of the form

$$\mathbb{F} = \text{SU}(n)/\text{S}(\text{U}(d_1) \times \text{U}(d_2) \times \cdots \times \text{U}(d_k))$$

with $d_1 + d_2 + \cdots + d_k = n$, see [2].

4 LG models from Lie theory

In this section we develop the description of the Lie theoretical potential on adjoint orbits in local coordinates, so that it can be compared with the toric potential. We first recall some results from [15] and [16]. Let \mathfrak{g} be a complex semisimple Lie algebra with Cartan subalgebra \mathfrak{h} , and $\mathfrak{h}_{\mathbb{R}}$ the real subspace generated by the roots of \mathfrak{h} . Let Π be the set of all roots of \mathfrak{g} . An element $H \in \mathfrak{h}$ is called regular if $\alpha(H) \neq 0$ for all $\alpha \in \Pi$.

Choose $H_0 \in \mathfrak{h}$ and a regular element $H \in \mathfrak{h}_{\mathbb{R}}$, the **height function** $f_H: \mathcal{O}(H_0) \rightarrow \mathbb{C}$ with respect to H , defined by

$$f_H(x) = \langle H, x \rangle, \quad x \in \mathcal{O}(H_0),$$

has a finite number of isolated singularities and defines a symplectic Lefschetz fibration on $\mathcal{O}(H_0)$, [15, Thm. 3.1]. This height function will give rise to the expression of what we call the Lie potential.

We now describe the potential on the adjoint orbit in detail, writing its description in each coordinate chart by a polynomial map. The construction of this intermediate step also clarifies why the potential gives rise to a Lefschetz fibration, which is by definition locally a quadratic polynomial map.

Given $H_0 \in \mathfrak{h}_{\mathbb{R}}$, let

$$\mathfrak{n}_{H_0}^+ = \sum_{\alpha(H_0) > 0} \mathfrak{g}_{\alpha} \quad \text{and} \quad \mathfrak{n}_{H_0}^- = \sum_{\alpha(H_0) < 0} \mathfrak{g}_{\alpha}$$

be the sums of the eigenspaces of $\text{ad}(H_0)$ associated to the positive and negative eigenvalues, respectively. The subspaces $\mathfrak{n}_{H_0}^\pm$ are nilpotent subalgebras. If G is a Lie group with Lie algebra \mathfrak{g} then the subgroups $N_{H_0}^\pm = \exp \mathfrak{n}_{H_0}^\pm$ are closed and the set

$$N_{H_0}^- Z_{H_0} N_{H_0}^+ = N_{H_0}^- N_{H_0}^+ Z_{H_0}$$

is open and dense in G where Z_{H_0} is the centraliser of H_0 in G and the product

$$(y, x, h) \in N_{H_0}^- \times N_{H_0}^+ \times Z_{H_0} \mapsto yxh \in N_{H_0}^- N_{H_0}^+ Z_{H_0}$$

is a diffeomorphism. Consider now the adjoint orbit $\mathcal{O}(H_0) = \text{Ad}(G)H_0 = G/Z_{H_0}$. The set $\text{Ad}(N_{H_0}^- N_{H_0}^+ Z_{H_0})H_0$, denoted just by $N_{H_0}^- N_{H_0}^+ Z_{H_0} \cdot H_0$ satisfies

$$N_{H_0}^- N_{H_0}^+ Z_{H_0} \cdot H_0 = N_{H_0}^- N_{H_0}^+ \cdot H_0$$

since $Z_{H_0} \cdot H_0 = H_0$. Given that $N_{H_0}^- N_{H_0}^+ Z_{H_0}$ is open and dense in G it follows that $N_{H_0}^- N_{H_0}^+ \cdot H_0$ is open and dense in $\mathcal{O}(H_0)$. Moreover, the map

$$(y, x) \in N_{H_0}^- \times N_{H_0}^+ \mapsto yx \cdot H_0 \in N_{H_0}^- N_{H_0}^+ \cdot H_0$$

is a diffeomorphism.

On the other hand, $\exp: \mathfrak{n}_{H_0}^\pm \rightarrow N_{H_0}^\pm$ is a diffeomorphism. Therefore,

$$(Y, X) \in \mathfrak{n}_{H_0}^- \times \mathfrak{n}_{H_0}^+ \mapsto e^{\text{ad}(Y)} e^{\text{ad}(X)} \cdot H_0 \in N_{H_0}^- N_{H_0}^+ \cdot H_0$$

defines a coordinate chart of $\mathcal{O}(H_0)$, around H_0 (with open and dense image).

The goal here is to describe the Lefschetz fibration on $\mathcal{O}(H_0)$ defined by $f_H(x) = \langle H, x \rangle$ in this coordinate chart. A similar description also exists for coordinate charts around the singularities wH_0 , $w \in \mathcal{W}$. As described in the lecture notes of San Martin [26], the expressions for f_H in these coordinate charts are polynomial functions, in fact, homogeneous polynomials of degree 2.

Theorem 4.1. [26] *The map f_H is polynomial.*

Proof. If $X \in \mathfrak{n}_{H_0}^+$ then $[X, H_0] = -\text{ad}(H_0)X \in \mathfrak{n}_{H_0}^+$ and since $\mathfrak{n}_{H_0}^+$ is nilpotent, there exists $n \geq 1$ such that

$$e^{\text{ad}(X)}H_0 = H_0 + g^+(X)$$

where $g^+ : \mathfrak{n}_{H_0}^+ \rightarrow \mathfrak{n}_{H_0}^+$ is the polynomial map

$$g^+(X) = \sum_{k=1}^n \frac{1}{k!} \text{ad}(X)^k(H_0).$$

Now, if $Y \in \mathfrak{n}_{H_0}^-$, then similarly $[Y, H_0] = -\text{ad}(H_0)Y \in \mathfrak{n}_{H_0}^-$. Moreover, $\mathfrak{n}_{H_0}^-$ is nilpotent and $\text{ad}(Y)$ is nilpotent. Thus, there exists $m \geq 1$ such that

$$e^{\text{ad}(Y)}e^{\text{ad}(X)}H_0 = H_0 + g^-(Y) + \sum_{j=0}^m \sum_{k=1}^n \frac{1}{j!} \frac{1}{k!} \text{ad}(Y)^j \text{ad}(X)^k(H_0) \quad (4.1)$$

where $g^- : \mathfrak{n}_{H_0}^- \rightarrow \mathfrak{n}_{H_0}^-$ is the polynomial map

$$g^-(Y) = \sum_{j=1}^m \frac{1}{j!} \text{ad}(Y)^j(H_0).$$

These expressions show that $f_H(e^{\text{ad}(Y)}e^{\text{ad}(X)}H_0)$ is polynomial in X and Y given that f_H is the restriction of the linear map $x \mapsto \langle H, x \rangle$.

Now, observe that $f_H(g^-(Y)) = 0$ and consequently

$$f_H(e^{\text{ad}(Y)}e^{\text{ad}(X)}H_0) = \langle H, H_0 \rangle + \langle H, * \rangle,$$

where $*$ is the part of the double summation in Equation (4.1) belonging to \mathfrak{h} . □

Example 4.2. Let $\mathfrak{g} = \mathfrak{sl}(n + 1, \mathbb{C})$ and choose H_0 such that the flag \mathbb{F}_{H_0} is the projective space \mathbb{P}^n . We take

$$H_0 = \text{Diag}(n, -1, \dots, -1).$$

In such a case, if $X \in \mathfrak{n}_{H_0}^+$, then

$$X = \begin{pmatrix} 0_{1 \times 1} & v \\ 0 & 0_{n \times n} \end{pmatrix},$$

with $v \in \mathbb{R}^n$, whereas $Y \in \mathfrak{n}_{H_0}^-$ if and only if $Y^T \in \mathfrak{n}_{H_0}^+$. If $X \in \mathfrak{n}_{H_0}^+$, then

$$e^{\text{ad}(X)} H_0 = H_0 + [X, H_0] = H_0 - X.$$

Consequently, if $Y \in \mathfrak{n}_{H_0}^-$, then

$$\begin{aligned} e^{\text{ad}(Y)} e^{\text{ad}(X)} H_0 &= H_0 - X + Y - [Y, X] + \frac{1}{2} [Y, Y - [Y, X]] \\ &= -X + (H_0 - [Y, X]) + \left(Y - \frac{1}{2} [Y, [Y, X]] \right) \end{aligned}$$

because $[Y, H_0] = Y$, given that the last term has already been assembled into the sum $\mathfrak{n}_{H_0}^+ \oplus \mathfrak{z}_{H_0} \oplus \mathfrak{n}_{H_0}^-$ where \mathfrak{z}_{H_0} is the centraliser of H_0 .

Now, if $H = \text{Diag}(\lambda_1, \dots, \lambda_{n+1})$, then H is orthogonal to both $\mathfrak{n}_{H_0}^+$ and $\mathfrak{n}_{H_0}^-$ (Cartan–Killing); thus

$$\begin{aligned} f_H \left(e^{\text{ad}(Y)} e^{\text{ad}(X)} H_0 \right) &= \langle H, H_0 \rangle - \langle H, [Y, X] \rangle \\ &= \langle H, H_0 \rangle + \langle H, [X, Y] \rangle. \end{aligned}$$

We write

$$X = \begin{pmatrix} 0_{1 \times 1} & \mathbf{x} \\ 0 & 0_{n \times n} \end{pmatrix} \quad Y = \begin{pmatrix} 0_{1 \times 1} & 0 \\ \mathbf{y}^T & 0_{n \times n} \end{pmatrix}$$

with $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. Since the Cartan–Killing product is $\langle A, B \rangle = 2(n + 1)\text{tr}(AB)$ for matrices in $\mathfrak{sl}(n + 1)$, it follows that

$$\begin{aligned} f_H \left(e^{\text{ad}(Y)} e^{\text{ad}(X)} H_0 \right) &= \\ &= 2(n + 1) [\text{tr}(HH_0) + (\lambda_1 - \lambda_2) x_1 y_1 + \dots + (\lambda_1 - \lambda_{n+1}) x_n y_n] \end{aligned}$$

which is a degree 2 polynomial.

Similar calculations go through if we replace H_0 by $w \cdot H_0$: if $w = (1j)$, then $\mathfrak{n}_{w \cdot H_0}^+$ consists of matrices having nonzero entries only on row j (0 in

the diagonal), whereas $\mathfrak{n}_{w \cdot H_0}^-$ consists of nonzero entries only in column j . Thus, if (x_1, \dots, x_n) and (y_1, \dots, y_n) are the entries on row and column j , respectively, then

$$\begin{aligned} f_H \left(e^{\text{ad}(Y)} e^{\text{ad}(X)} (w \cdot H_0) \right) &= \\ &= 2(n+1) [\text{tr}(H(w \cdot H_0)) + (\lambda_j - \lambda_1)x_1y_1 + \dots + (\lambda_j - \lambda_{n+1})x_ny_n]. \end{aligned}$$

Example 4.3. Let $H = \text{Diag}(1, 0, -1)$, $H_0 = \text{Diag}(2, -1, -1) \in \mathfrak{sl}(3, \mathbb{C})$ and choose $w = (123) \in \mathcal{W}$. Then $w \cdot H_0 = \text{Diag}(-1, 2, -1)$ and $w^2 \cdot H_0 = \text{Diag}(-1, -1, 2)$, so that

$$\begin{aligned} \langle H, H_0 \rangle &= 2(n+1)\text{tr}(HH_0) = 18, \\ \langle H, w \cdot H_0 \rangle &= 2(n+1)\text{tr}(H(w \cdot H_0)) = 0, \\ \langle H, w^2 \cdot H_0 \rangle &= 2(n+1)\text{tr}(H(w^2 \cdot H_0)) = -18. \end{aligned}$$

Note that the constant term of the expression changes according to the choice of critical points.

Definition 4.4. We call the expression of f_H written on a chart around H_0 the **Lie potential** on $\mathcal{O}_n \subset \mathfrak{sl}(n+1, \mathbb{C})$. It is given by

$$\boxed{\mathfrak{f}_H(H_0) = \text{tr}(HH_0) + (\lambda_1 - \lambda_2)x_1y_1 + \dots + (\lambda_1 - \lambda_{n+1})x_ny_n.} \tag{4.2}$$

Example 4.5. Consider $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ and choose H_0 so that the flag \mathbb{F}_{H_0} is the Grassmannian $\text{Gr}_k(n+1)$. Calculations are similar to the ones for the projective space. The main difference is that the decomposition is given by blocks of size k and $n-k+1$, so that the elements of $\mathfrak{n}_{H_0}^+$ are matrices of the form

$$\begin{pmatrix} 0_{k \times k} & v \\ 0 & 0_{(n-k+1) \times (n-k+1)} \end{pmatrix}.$$

Similarly,

$$\begin{aligned} e^{\text{ad}(Y)} e^{\text{ad}(X)} H_0 &= H_0 - X + Y - [Y, X] + \frac{1}{2} [Y, Y - [Y, X]] \\ &= -X + (H_0 - [Y, X]) + \left(Y - \frac{1}{2} [Y, [Y, X]] \right) \end{aligned}$$

and $f_H(e^{\text{ad}(Y)}e^{\text{ad}(X)}H_0) = \langle H, H_0 - [Y, X] \rangle$ is a degree 2 polynomial.

These 2 examples can be easily generalised to all cases when the eigenvalues of $\text{ad}(H_0)$ are 0 and ± 1 . Analogous calculations show that

$$e^{\text{ad}(Y)}e^{\text{ad}(X)}H_0 = -X + (H_0 - [Y, X]) + \left(Y - \frac{1}{2} [Y, [Y, X]] \right)$$

and hence $f_H(e^{\text{ad}(Y)}e^{\text{ad}(X)}H_0) = \langle H, H_0 - [Y, X] \rangle$ is a degree 2 polynomial in the variables X and Y .

Remark 4.6. In further generality, given a complex Lie algebra, we may choose a set of positive roots Π^+ and simple roots $\Sigma \subset \Pi^+$ with corresponding Weyl chamber \mathfrak{a}^+ . A subset $\Theta \subset \Sigma$ defines a parabolic subalgebra \mathfrak{p}_Θ with parabolic subgroup P_Θ and a flag manifold $\mathbb{F}_\Theta = G/P_\Theta$. An element $H_\Theta \in \text{cl}\mathfrak{a}^+$ is characteristic for $\Theta \subset \Sigma$ if $\Theta = \{\alpha \in \Sigma : \alpha(H_\Theta) = 0\}$. Let $Z_\Theta = \{g \in G : \text{Ad}(g)H_\Theta = H_\Theta\}$ be the centraliser in G of the characteristic element H_Θ . The adjoint orbit $\mathcal{O}(H_\Theta) = \text{Ad}(G) \cdot H_\Theta \approx G/Z_\Theta$ of the characteristic element H_Θ is diffeomorphic to the cotangent bundle $T^*\mathbb{F}_\Theta$ [16, Thm. 2.1].

Remark 4.7. In the specific case of the minimal orbit, there is a very useful compactification to a product or projective spaces, which goes as follows. The diagonal action of $\text{SL}(n + 1, \mathbb{C})$ in $\mathbb{P}^n \times \text{Gr}(n, n + 1) \simeq \mathbb{P}^n \times \mathbb{P}^n$ has two orbits, an open one and a closed one. The open orbit is isomorphic to the adjoint orbit $\mathcal{O}_n = \text{Ad}(G) \cdot H_\Theta$ with $H_\Theta = \text{Diag}(n, -1, \dots, -1)$ and is formed by the pairs of transversal elements in $\mathbb{P}^n \times \text{Gr}(n, n + 1)$. The closed orbit is isomorphic to the flag $F_{\Theta \cap \Theta^*}$. Since $\Theta \cap \Theta^*$ is the complement of $\{\alpha_{12}, \alpha_{n, n+1}\}$ it follows that $F_{\Theta \cap \Theta^*} = F(1, n)$.

5 Toric potential vs Lie potential

Cotangent bundles of projective spaces and their Lagrangian skeleta were studied in [7] from the viewpoint of their duality with bundles on

collars of local surfaces. Starting with a Hamiltonian action of $\mathbb{T} = \mathbb{C}^*$ on $T^*\mathbb{P}^n$ expressed in the open chart $V_0 = \{x_0 \neq 0\}$ by

$$\mathbb{T} \cdot V_0 = \left\{ [1, t^{-1}x_1, \dots, t^{-n}x_n], (ty_1, \dots, t^ny_n) \right\}$$

we obtain a Hamiltonian vector field $T^*\mathbb{P}^n$

$$X(x_1, \dots, x_n, y_1, \dots, y_n) = (-x_1, \dots, -nx_n, y_1, \dots, ny_n),$$

as described in [7, Prop. 3.1]. Taking $T^*\mathbb{P}^n$ with the canonical symplectic form ω , the corresponding potential h therefore satisfies

$$dh(Z) = \omega(X, Z),$$

for all vector fields $Z \in \mathfrak{X}(T^*\mathbb{P}^n)$. In V_0 coordinates, writing

$$\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n), \mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n),$$

we obtain a differential equation

$$\begin{aligned} \left(\frac{\partial h}{\partial \mathbf{x}} \frac{\partial h}{\partial \mathbf{y}} \right) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} &= \sum_{i=1}^n dx_i \wedge dy_i ((-x_1, \dots, -nx_n, y_1, \dots, ny_n), (a_1, \dots, a_n, b_1, \dots, b_n)) \\ &= -2 \sum_{i=1}^n ix_i b_i + iy_i a_i, \end{aligned}$$

whose solutions are:

$$\mathbf{h}_c = c - 2x_1y_1 - \dots - 2nx_ny_n = c + \sum_{i=1}^n -2ix_iy_i, \tag{5.1}$$

for $c \in \mathbb{C}$. See [7] for the expression of h_c on the other charts. In Section 6 we will consider $n = 1$, in which case there are two charts.

Definition 5.1. We call \mathbf{h}_c a **toric potential** on $T^*\mathbb{P}^n$.

Theorem 5.2. For each $n \in \mathbb{N}$, there exist matrices $H, H_0 \in \mathfrak{sl}(n + 1, \mathbb{C})$ and a constant $c \in \mathbb{C}$ such that the Lie potential on the minimal adjoint orbit \mathcal{O}_n and the toric potential on the cotangent bundle $T^*\mathbb{P}^n$ coincide on dense open charts, that is $\mathbf{f}_H = \mathbf{h}_c$.

Proof. Take $H_0 = \text{Diag}(n, -1, \dots, -1) \in \mathfrak{sl}(n + 1, \mathbb{C})$,

$$H = \begin{cases} \text{Diag}(-n, \dots, -4, -2, 0, 2, 4, \dots, n) & \text{if } n \text{ is even,} \\ \text{Diag}(-n, \dots, -3, -1, 1, 3, \dots, n) & \text{if } n \text{ is odd} \end{cases}$$

and $c = -n^2 - n$.

On the Lie side, we have:

$$\mathbf{f}_H(H_0) = \text{tr}(HH_0) + (\lambda_1 - \lambda_2)x_1y_1 + \cdots + (\lambda_1 - \lambda_{n+1})x_ny_n,$$

where the eigenvalues of $H = \text{Diag}(\lambda_1, \dots, \lambda_{n+1})$ satisfy $\lambda_1 - \lambda_j = -2(j - 1)$, so that

$$\mathbf{f}_H(H_0) = -n^2 - n - 2x_1y_1 - \cdots - 2nx_ny_n.$$

On the toric side,

$$\begin{aligned} \mathbf{h}_c &= c - 2x_1y_1 - \cdots - 2nx_ny_n \\ &= -n^2 - n - 2x_1y_1 - \cdots - 2nx_ny_n. \end{aligned}$$

□

By Theorem 5.2 the value $c = -n^2 - n$ is preferable for our purposes here. Therefore, using coordinates $([1, x_1], y_1)$ for the cotangent bundle $T^*\mathbb{P}^1$, we write the potential

$$\mathbf{h}_{-2}: T^*\mathbb{P}^1 \rightarrow \mathbb{C}$$

as

$$\mathbf{h}_{-2} = -2x_1y_1 - 2.$$

Definition 5.3. Two Landau–Ginzburg models are called **birationally equivalent** if their domains are birationally equivalent varieties and their potentials coincide on Zariski open sets.

A rigorous version of our theorem may be stated as:

Theorem 5.4. *For each $n \in \mathbb{N}$, there exist matrices $H, H_0 \in \mathfrak{sl}(n + 1, \mathbb{C})$ and a constant $c \in \mathbb{C}$ such that the LG models (\mathcal{O}_n, f_H) and $(T^*\mathbb{P}^n, h_c)$ are birationally equivalent.*

Proof. By Theorem 5.2, the expressions of \mathbf{f}_H and \mathbf{h}_c for the potentials f_H and h_c coincide on Zariski open sets. □

6 Deformations and their mirrors

Example 6.1 ($\text{LG}_0 \rightsquigarrow \text{LG}_1$). We consider $T^*\mathbb{P}^1$ with coordinates $[1, x], y$ and the family of potentials

$$\text{LG}_t = t \left(x + \frac{1}{x} + \frac{y^2}{x} \right) + (1 - t)(2x).$$

Then we have that the initial LG model LG_0 is selfdual as shown in Example 3.6, while the final LG model is defined by $LG_1 = (T^*\mathbb{P}^1, 2x)$ which has the toric data

$$\text{Div} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \text{Mon} = (10).$$

Now, inverting the matrices by toric duality gives us the LG_1^\vee model

$$\text{Div} = (10) \quad \text{Mon} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}$$

defined on a variety with torus $(\mathbb{C}^*)^2$ and a single divisor, in this case a $\mathbb{C}^* \times \mathbb{C}$. We conclude that this family takes a selfdual LG model to another very far from selfdual.

$$\begin{array}{ccc} LG_0 & \rightsquigarrow & LG_1 \\ \parallel & & \not\parallel \\ LG_0^\vee & \rightsquigarrow & LG_1^\vee. \end{array}$$

Our next objective is to describe how duality works for the deformation family

$$LG_1 = (T^*\mathbb{P}^1, h_c) \rightsquigarrow LG_2 = (\mathcal{O}_2, f_H).$$

We will use the deformation of the Hirzebruch surfaces \mathbb{F}_2 to \mathbb{F}_0 in Lemma 6.2, extending it to a deformation of partially compactified Landau–Ginzburg models in Example 6.5

$$(\mathbb{F}_2, \mathbf{h}) \rightsquigarrow (\mathbb{F}_0, \mathbf{f}).$$

The Hirzebruch surface \mathbb{F}_2 embeds in $\mathbb{P}^1 \times \mathbb{P}^3$, as the set of points

$$([x_0, x_1], [y_0, y_1, y_2, y_3]) \in \mathbb{P}^1 \times \mathbb{P}^3$$

satisfying the equation

$$x_0(y_1, y_2) = x_1(y_2, y_3),$$

see [25]. We can cover \mathbb{F}_2 with four coordinate charts, $U = \{(z, u) \in \mathbb{C}^2\}$, $V = \{(\xi, v) \in \mathbb{C}^2\}$, $U' = \{(z, \mu) \in \mathbb{C}^2\}$, $V = \{(z, \eta) \in \mathbb{C}^2\}$ with the following changes of coordinates:

$$\xi = z^{-1}, \quad v = z^2 u, \quad \mu = u^{-1}, \quad \eta = v^{-1}.$$

We will make use of the following result:

Lemma 6.2. \mathbb{F}_2 deforms to \mathbb{F}_0 .



Figure 6.1: \mathbb{F}_2 deforms to \mathbb{F}_0

Proof. The deformation family \mathcal{M} taking \mathbb{F}_2 to \mathbb{F}_0 is given by

$$\{x_0(y_1, y_2) = x_1(y_2 + ty_0, y_3)\} = M_t \subset \mathbb{P}^1 \times \mathbb{P}^3.$$

In coordinates:

$$\begin{aligned} (z, u, t) &\mapsto ([1, z], [1, z^2u + tz, zu, u], t) \\ (z, \mu, t) &\mapsto ([1, z], [\mu, z^2 + tz\mu, z, 1], t) \\ (\xi, v, t) &\mapsto ([\xi, 1], [1, v, \xi v - t, \xi^2v - t\xi], t). \\ (\xi, \eta, t) &\mapsto ([\xi, 1], [\eta, 1, \xi - t\eta, \xi^2 - t\xi\eta], t). \end{aligned}$$

The deformation from \mathbb{F}_2 to \mathbb{F}_0 is given by

$$\{x_0(y_1, y_2) = x_1(y_2 + ty_0, y_3)\} = M_t \subset \mathbb{P}^1 \times \mathbb{P}^3,$$

$$\begin{aligned} x_0y_1 &= x_1y_2 + tx_1y_0 \\ x_0y_2 &= x_1y_3 \end{aligned}$$

□

Lemma 6.3. $T^*\mathbb{P}^1$ deforms to \mathcal{O}_2 .

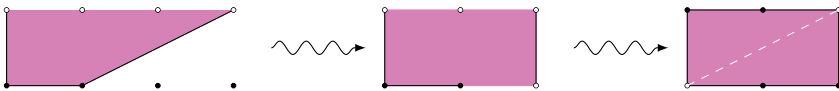


Figure 6.2: $T^*\mathbb{P}^1$ deforms to \mathcal{O}_2

Proof. We will use the family $\pi: \mathcal{M} \rightarrow \mathbb{C}$ given in Lemma 6.2 deforming the Hirzebruch surface \mathbb{F}_2 to \mathbb{F}_0 , written as the points $([x_0, x_1], [y_0, y_1, y_2, y_3], t) \in \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{C}$ satisfying the equation

$$x_0(y_1, y_2) = x_1(y_2 + ty_0, y_3).$$

Here $\pi^{-1}(0) \simeq \mathbb{F}_2$ and for $t \neq 0$ we have that $\pi^{-1}(t) \simeq \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

The orbit \mathcal{O}_2 may be written in coordinate charts as $U = \{(z, u) \in \mathbb{C}^2\}$ and $V = \{(\xi, v) \in \mathbb{C}^2\}$ with transition function

$$(\xi, v) \mapsto (z^{-1}, z^2u + tz),$$

where we have $z = \frac{x_1}{x_0}$, $\xi = \frac{x_0}{x_1}$, $u = y_3$, and $v = y_1$. The family $M \rightarrow \mathbb{C}$ deforming $T^*\mathbb{P}^1$ to \mathcal{O}_2 admits an embedding

$$j: M \rightarrow \mathcal{M} \tag{6.1}$$

given by

$$\begin{aligned} (z, u, t) &\mapsto ([1, z], [1, z^2u + tz, zu, u], t) \\ (\xi, v, t) &\mapsto ([\xi, 1], [1, v, \xi v - t, \xi^2v - t\xi], t). \end{aligned}$$

Note that in these coordinates, $t = 0$ and $y_1 = y_2 = y_3 = 0$ give the zero section of $T^*\mathbb{P}^1$.

Claim 6.4. The section at infinity σ_∞ is:

$$\begin{aligned} \sigma_\infty(z, t) &= ([1, z], [0, z^2, z, 1], t) \quad \text{on } U, \\ \sigma_\infty(\xi, t) &= ([\xi, 1], [0, 1, \xi, \xi^2], t) \quad \text{on } V. \end{aligned}$$

Proof of the claim. Note that

$$\begin{aligned} [1, z^2u + tz, zu, u] &= [1/u, z^2 + tz/u, z, 1] \\ [1, v, \xi v - t, \xi^2v - t\xi] &= [1/v, 1, \xi - t/v, \xi^2 - t\xi/v] \end{aligned}$$

So, by taking the limit when $u \rightarrow \infty$ and $v \rightarrow \infty$, i.e., $1/u \rightarrow 0$ and $1/v \rightarrow 0$ we obtain the section at infinity:

$$\begin{aligned} \lim_{u \rightarrow \infty} (z, u, t) &\mapsto ([1, z], [0, z^2, z, 1], t), \\ \lim_{v \rightarrow \infty} (\xi, v, t) &\mapsto ([\xi, 1], [0, 1, \xi, \xi^2], t). \end{aligned}$$

□

Example 6.5. ($\text{LG}_1 \rightsquigarrow \text{LG}_2$) We now revisit the family $\text{LG}_1 \rightsquigarrow \text{LG}_2$ presented in Example 2.7, commenting on the dual family.

Using Definition 5.1 with $c = -2$, $n = 1$ and coordinates $[1, x_1], y_1$ for the cotangent bundle $T^*\mathbb{P}^1$, the potential $\mathbf{h}_{-2}: T^*\mathbb{P}^1 \rightarrow \mathbb{C}$ can be written as

$$\mathbf{h}_{-2}([1, x_1], y_1) = -2x_1y_1 - 2, \quad \mathbf{h}_{-2}([0, 1], v_1) = -2.$$

As seen in Example 6.1, toric duality gives the dual LG_1^\vee as defined over a variety having a single divisor with a potential formed by 3 monomials, hence not isomorphic to LG_1 .

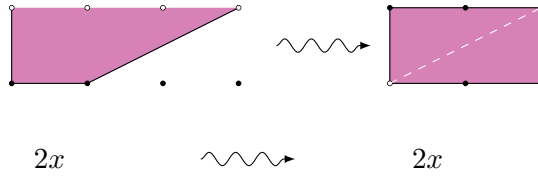


Figure 6.3: LG_1 TO LG_2

We now discuss the dual of the Landau–Ginzburg model $LG_2 = (\mathcal{O}_1, f_H)$, where \mathcal{O}_1 is the semisimple adjoint orbit of $\mathfrak{sl}(2, \mathbb{C})$, together with the potential f_H given by the height function $f_H(x, y, z) = 2x$. To start with, toric duality can not be applied in this case, because the orbit \mathcal{O}_1 is not a toric variety. So, an entire new recipe is needed.

The recipe for intrinsic mirrors by Gross and Siebert [19] produces the desired dual Landau–Ginzburg model, which was described in [14]. We summarise the construction.

The compactification mentioned in Remark 4.7, applied to the case when $n = 1$ gives $\mathbb{P}^1 \times \mathbb{P}^1 = \mathcal{O}_1 \cup \Delta$ where Δ is the diagonal. To construct the intrinsic mirror, we start with the pair $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$. Since Gross and Siebert work under the hypothesis of log geometry, we need a pair (X, D) , where D is an anticanonical divisor. In this case, an anticanonical divisor has type $(2, 2)$. So, we choose an additional divisor Δ' of type $(1, 1)$ such that $\Delta \cap \Delta' = \{p_1, p_2\}$ consists of 2 points.

Let \tilde{X} denote the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at these 2 points. We then study the pair

$$(\tilde{X}, D) \quad \text{with} \quad D = D_1 + D_2 + D_3 + D_4$$

where D_2 and D_4 are the proper transforms of Δ' and Δ respectively, and D_1 and D_3 are the exceptional curves obtained from blowing up the 2 points; thus $D_1^2 = D_3^2 = -1$.

Calculating theta functions following the Gross–Siebert intrinsic mirror symmetry recipe, [14] arrives at the variety given by

$$\vartheta_2 \vartheta_4 = \alpha \vartheta_1 + \beta \vartheta_1^{-1} + \gamma \tag{6.2}$$

with potential ϑ_2 .

Changing to more standard algebraic coordinates $x := \vartheta_1, y := \vartheta_2$ and compactifying the direction ϑ_4 to \mathbb{P}^1 with coordinates $[u : v]$, our problem is then to study the surface $S \subset \mathbb{C}^* \times \mathbb{C} \times \mathbb{P}^1 = \{(x, y), [u : v]\}$ given by

$$S := \{uy = v(x + 1 + 1/x)\}$$

with potential

$$y = v(x + 1 + 1/x)/u.$$

The Landau–Ginzburg model $LG_2^\vee = (S, y)$ is the dual of LG_2 . Denoting by $D_{sg} LG_2^\vee$ the Orlov category of singularities of LG_2^\vee , [14, Sec.3.3] proves the equiva-

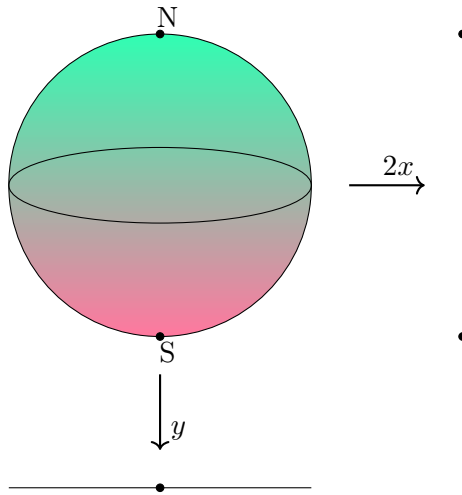


Figure 6.4: LG_2 and its mirror

lence

$$D_{sg} LG_2^\vee \cong \text{Fuk} LG_2 .$$

Note that here, LG_2^\vee is quite different from LG_2 , for instance, the potential y has the 2 singularities of LG_2^\vee occurring on the same fibre of the potential, unlike the potential $2x$ of LG_2 for which the singularities occur in different fibres, as described in Figure 6.

So that the mirror of the Landau–Ginzburg model geometrically resembles the rotation by 90 degrees of the original one.

Acknowledgements

I thank Elizabeth Gasparim, and upon request, give her the following absurdly overstated acknowledgement: without her help this work would have provoked “a catastrophe, a tragedy, a cataclysmic, apocalyptic, monumental calamity”, (Sir Humphrey Appleby, [27, 18:30]).

The author thanks Luiz A. B. San Martin for providing many useful suggestions and for the proof of Theorem 4.1.

The author was partially supported by MathAmSud ANID project MATH2020011. The author was supported by Grant 2021/11750-7 São Paulo Research Foundation -

FAPESP.

References

- [1] T. A. BURTON, L. HATVANI, Asymptotic stability of second order ordinary, functional, and partial differential equations, *J. Math. Anal. Appl.* **176**(1993), No. 1, 261–281.
- [2] A. ARVANITOEORGOS, Geometry of flag manifolds, *Int. J. Geometric Methods in Modern Phys.* **3** n.5-6 (2006) 957–974.
- [3] V. V. BATYREV, L. A. BORISOV, Dual cones and mirror symmetry for generalized Calabi–Yau manifolds in: *Mirror symmetry II*, *AMS/IP Stud. Adv. Math.*, 71–86, Amer. Math. Soc., Providence, RI, 1997.
- [4] E. BALLICO, S. BARMEIER, E. GASPARIM, L. GRAMA, L. A. B. SAN MARTIN, A Lie theoretical construction of a Landau–Ginzburg model without projective mirrors, *Manuscripta Math.* **158** n.1-2 (2018) 85–101.
- [5] E. BALLICO, B. CALLANDER, E. GASPARIM, Compactifications of adjoint orbits and their Hodge diamonds, *J. Algebra Appl.* **16** (2017) 1–17.
- [6] E. BALLICO, E. GASPARIM, L. GRAMA, L. A. B. SAN MARTIN, Some Landau–Ginzburg models viewed as rational maps, *Indagationes Math.* **28** (2017) 615–628.
- [7] E. BALLICO, E. GASPARIM, F. RUBILAR, B. SUZUKI, Lagrangian skeleta, collars and duality, in: *In Birational Geometry, Kähler-Einstein Metrics and Degenerations (Springer Proc. Math. Stat.)* **409** (2023).
- [8] E. BALLICO, E. GASPARIM, F. RUBILAR, L. A. B. SAN MARTIN, KKP conjecture for minimal adjoint orbits, *Eur. J. Math.*, accepted.
- [9] P. BERGLUND, T. HÜBSCH, A generalized construction of mirror manifolds *Essays on mirror manifolds*, 388–407, Int. Press, Hong Kong, 1992.
- [10] O. BIQUARD, Sur les équations de Nahm et la structure de Poisson des algèbres de Lie semi-simples complexes (in French) [On Nahm equations and the Poisson structure of complex semisimple Lie algebras], *Math. Ann.* **304** (1996) 253–276.
- [11] B. CALLANDER, E. GASPARIM, R. JENKINS, L. M. SILVA, Self-duality for Landau–Ginzburg models, *J. Geom. Symmetry Phys.* **35** (2014) 1–10.
- [12] P. CLARKE, Duality for Toric Landau–Ginzburg Models, *Adv. Theor. Math. Phys.* **21** (2008).
- [13] D. A. COX, J. B. LITTLE, H. K. SCHENCK, Toric varieties, *Graduate Studies in Mathematics* **124**, Am. Math. Soc., 2011.
- [14] E. GASPARIM, Intrinsic mirrors of adjoint orbits, *arXiv:2108.02471*.

- [15] E. GASPARIM, L. GRAMA, L. A. B. SAN MARTIN, Symplectic Lefschetz fibrations on adjoint orbits, *Forum Math.* **28** n.5 (2016) 967–980.
- [16] E. GASPARIM, L. GRAMA, L. A. B. SAN MARTIN, Adjoint orbits os semisimple Lie groups and Lagrangian submanifolds, *Proc. Edinburgh Math. Soc.* **59** (2016) 1–26.
- [17] E. GASPARIM, T. KÖPPE, F. RUBILAR, B.SUZUKI, Deformations of noncompact Calabi–Yau threefolds, *Proc. VI Latin American Congress of Mathematicians, Rev. Colomb. Mat.* (2017).
- [18] E. GASPARIM, R. RUBILAR, Deformations of Noncompact Calabi–Yau Manifolds, Families and Diamonds, *Contemp. Math.* **766** (2021) 117–132.
- [19] M. GROSS, B. SIEBERT, Intrinsic mirror symmetry and punctured Gromov–Witten invariants, *Algebraic geometry: Salt Lake City 2015, 199–230, (Proc. Sympos. Pure Math.)* **97** n.2, Amer. Math. Soc., Providence, RI (2018).
- [20] K. HORI, C. VAFA, Mirror symmetry, *arXiv:hep-th/0002222*.
- [21] L. KATZARKOV, M. KONTSEVICH, T. PANTEV, Bogomolov–Tian–Todorov theorems for Landau–Ginzburg models, *J. Differential Geom.* **105** n.1 (2017) 55–117
- [22] M. KONTSEVICH, Homological algebra of Mirror Symmetry, *Proc. International Congress of Mathematicians* (Zurich, 1994), Birkhäuser, Basel (1995) 120–139.
- [23] A. G. KOVALEV, Nahm’s equations and complex adjoint orbits, *Q. J. Math.* **47** (1996) 41–58.
- [24] P. B. KRONHEIMER, A hyper-kählerian structure on adjoint orbits of a semisimple complex group, *J. London Math. Soc.* **42** (1990) 193–208.
- [25] M. MANETTI, Lectures on deformations of complex manifolds (deformations from differential graded viewpoint), *Rend. Mat. Appl.* **24** n.7 (2004) 1–183.
- [26] L. A. B. SAN MARTIN, A função altura em órbitas adjuntas, personal communication (2016) (unpublished).
- [27] “The Official Visit.”, Yes Minister, created by A. Jay, J. Lynn, season 1, episode 2, BBC2 (1980).