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Nonlinearities with zeros for Laplacian, $p-$Laplacian and Poly-Laplacian

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Dedicated to Professor Pedro Ubilla
on the occasion of his 60th birthday

Abstract. In this survey we present a collection of results dealing with positive solutions for elliptic problems, with a nonlinearity $\lambda\, h(x,u)$ which admits one or more zeros.

We use a combination of several methods, such as variational methods, the sub and supersolutions method, comparison principles, a-priori estimates, truncation, Green's function and the properties of concave functions. We consider problems with the p-Laplacian operator and the poly-Laplacian. We study existence, nonexistence, multiplicity, and the behavior with respect to λ of positive solutions, showing that in many cases the solutions tend to converge to the zero of the nonlinearity and also some consequences of this behavior.

Keywords: Multiplicity of positive solutions; nonlinearity with zeros; $p-$Laplacian; poly-Laplacian; asymptotic behavior of solutions; Liouville–type theorems.

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1 Introduction

The aim of this survey is to present several results dealing with multiplicity of positive solutions for elliptic problems, with a nonlinearity which admits one or more zeros.

I started to work on this theme in 2008, when Professor Pedro Ubilla, to whom this survey is dedicated, invited a team of young researchers for a 6 months stay at his institution USACH, in Santiago - Chile. They were Leonelo Iturriaga, Justino Sanchez and myself. We spent the semester in a nice office at USACH, having stimulating conversations, learning many things, sometimes obtaining some progress, but also having time for the casual eating at a Chilean restaurant or a pool game at the end of the day. The immediate result was a first paper, [36], but it was also the start of a collaboration that went on for many years, with more papers on the same and other matters, also involving other mathematicians.

We will consider problems in the general form

$$\begin{cases} -\mathcal{Q}u = \lambda h(x,u) & \text{in} \quad \Omega, \\ \mathcal{B}u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{G_λ}$$

where $\mathcal{Q}$ will be either the p-Laplacian, or the poly-Laplacian (including of course the usual linear second order Laplacian), $\mathcal{B}$ a suitable boundary condition, $\lambda > 0$ a real parameter, Ω a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and where $h(x,t)$ is a nonnegative nonlinearity which is zero for some positive value of t, may be depending on x.

If $\mathcal{Q}$ is the Laplacian (or poly-Laplacian), it is known that a necessary condition for a positive solution u of Problem (G_λ) is

$$\int_\Omega (\lambda h(x,u) - \lambda_1 u)\, \phi_1 = 0\,, \tag{1.1}$$

where $\lambda_1, \phi_1 > 0$ are the first eigenvalue and eigenfunction of the operator. This means that the term in parentheses must change sign. This is not a sufficient condition but is a very useful guideline when looking for hypotheses that guarantee the existence of positive solutions. It turns out that also for the p-Laplacian, the number and the position of the inter-crossings of $\lambda h(x,t)$ with $\lambda_1 t^{p-1}$ is an indication of the number of positive solutions that we can expect to obtain and of the behavior of their norm. Often one expects to be able to obtain as many positive solutions as the number of these intersections.

In view of this fact, it becomes clear why it is interesting to study nonlinearities with zeros: if for instance h does not depend on $x \in \Omega$, $h(0) = 0$ and it grows at infinity faster than $\mathcal{Q}$, then for any further zero of h we may expect two positive solutions, at least when λ is large enough. In particular, we expect to be able to obtain existence for every $\lambda > 0$ if h has at least one zero at a positive value.

In the first work [36], $\mathcal{Q}$ is the p-Laplacian, the zero of h is allowed to depend on x and the growth at infinity is p-superlinear but subcritical. We first give conditions for the existence of a solution, then we obtain also a second solution under additional conditions, and finally we study the asymptotic behavior of the solutions as $\lambda \to \infty$, actually, it turns out that in this kind of problems the solutions tend, in a suitable sense, to converge to the zero of h.

In the works [31, 33] we consider h not depending on $x \in \Omega$. This allows us to use moving planes techniques and obtain that the solutions converge to the zero in a stronger sense, which permits to perform a truncation and then to consider even supercritical growth rates at infinity for h. Moreover, in [33], we consider the case of multiple zeros and actually obtain the expected optimal result of two solutions for every zero of the nonlinearity, for λ large enough.

Later, in [37], we considered Problem (G_λ) with the Laplacian operator, in the radial case in an annulus. In this setting, taking advantage of ODEs techniques, we are able to obtain results similar to those in [36, 31], but under weaker assumptions.

Finally, in [35] we turned our attention to the case where $\mathcal{Q}$ is the higher order operator poly-Laplacian. In this case we observe that a different behavior arises. Actually, the multiplicity result can be proved in less generality and in fact, we prove that in certain situations where two solutions exist with the Laplacian operator, no solution exists for the higher order operator.

The techniques used to deal with our problems are a mixture of variational methods and topological methods. Fundamental tools will be also a-priori estimates and some Liouville-type theorems. When working in dimension one we will also work with the Green's function and exploit properties of concave functions.

In the Sections 2, 3, 4 and 5, we will present, respectively, the four results mentioned above. We will prioritize the exposition and comparison of the results, while we will mostly give only a general and sometimes qualitative idea of the proofs (which can always be found in the original papers), trying to emphasize the motivations of the hypotheses that are imposed and of the techniques used.

A simple model for the kind of nonlinearity we are going to study is

$$h(x,u) = u^q |a(x) - u|^r, \tag{1.2}$$

where the function a describes the zero of h and the exponents q, r will need to be chosen properly in order to satisfy the hypotheses that will be provided.

2 Superlinear Nonlinearity with a variable zero

In the paper [36] we studied existence, multiplicity, and the behavior with respect to the parameter $\lambda > 0$, of positive solutions of the Problem

$$\begin{cases} -\Delta_p u = \lambda h(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

where $p > 1$ and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian operator.

We will assume that h grows as u^{p-1} near 0 and has a $p-$superlinear growth at infinity.

Problems with superlinear nonlinearities at infinity which have different behaviors at the origin have been extensively studied. See for example [1, 16, 19] for the Laplacian and [2, 54] for the $p-$Laplacian. In most of these works, the nonlinearity is strictly positive; however, as we already mentioned, the characteristics of the problem are quite different when the nonlinearity has a positive zero. In [40], this type of problems was considered for the Laplacian operator. Using topological degree arguments and under additional technical conditions which ensure a-priori bounds, it was shown that there exist two positive solutions: one solution lies strictly below the zero, while the other has a maximum greater than it. In [41], again the existence of two positive solutions was shown: one as a minimal positive solution, the other as the limit of a suitable gradient flow.

Here, we mainly use variational techniques to show the existence of at least one positive solution for every $\lambda > 0$, and at least two positive solutions for λ greater than the first eigenvalue of a certain nonlinear weighted eigenvalue problem for the $p-$Laplacian. Due to the dependence on x and since we are not requiring the convexity of the domain, these results improve those of [40, 41] even when $p = 2$. In order to obtain a second solution, we have to show that the first one is strictly below the zero of h, which we denote by $a(x)$. For this, additional hypotheses will have to be assumed (see Section 2.2).

Finally, for the study of the asymptotic behavior of the solutions when $\lambda \to \infty$, we need to obtain both a-priori estimates and a new Liouville–type theorem involving nonlinearities with zeros (see Theorem 2.8). We also need to extend to the p-Laplacian a result due to Redheffer (see [57, Theorem 1]). The results obtained allow us to show that the solutions u_λ of Problem (P_λ) satisfy

$$\lim_{\lambda \to \infty} u_\lambda(x) = a(x), \qquad \text{for every } x \in \Omega\,.$$

2.1 Existence results

In order to prove the existence of a solution for Problem (P_λ) we will assume the following hypotheses.

(H_1) The function $h : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ is continuous and $h(x, 0) = 0$.

(H_2) There exist a weakly $p-$superharmonic function $a \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ (that is, $-\Delta_p\, a \geq 0$ in the weak sense) and positive constants a_0, A_0 such that $a_0 \leq a(x) \leq A_0$ and

$$\begin{cases} h(x,t) = 0 & \text{if } t = a(x), \\ h(x,t) > 0 & \text{if } t \neq a(x),\ t > 0. \end{cases}$$

(H_3) There exist a function $b \in L^\infty(\Omega)$ and positive constants b_0, B_0 such that $b_0 \leq b(x) \leq B_0$ and

$$\lim_{u \longrightarrow 0^+} \frac{h(x,u)}{u^{p-1}} = b(x) \quad \text{uniformly in } x \in \Omega.$$

(M_1) There exists a continuous nondecreasing function $f_0 : \mathbb{R} \to \mathbb{R}$ such that $f_0(0) = 0$ and the map $s \mapsto h(x, s) + f_0(s)$ is nondecreasing for all $x \in \Omega$.

For small values of λ, we need the following hypothesis on the behavior of h at infinity.

(H_4) There exist $\rho > 0$ and $\sigma \in (p - 1, p^* - 1)$, where p^* denotes the critical Sobolev's exponent, given by $p^* = \frac{Np}{N-p}$ if $N > p$, and we may set $p^* = \infty$ if $N \leq p$, such that

$$\lim_{u \longrightarrow +\infty} \frac{h(x,u)}{u^\sigma} = \rho \ \text{ uniformly in } \ x \in \Omega.$$

The model (1.2) satisfies the conditions above with $r > 0$, $q = p - 1$ and $r + q + 1 < p^*$.

In view of hypothesis (H_3), we consider the nonlinear eigenvalue problem

$$(E_b) \qquad \begin{cases} -\Delta_p u = \lambda\, b(x)|u|^{p-2}u & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

and denote by $\lambda_{1,b}$ and $\phi_{1,b}$ its first eigenvalue and eigenfunction, for which we know that $\phi_{1,b} > 0$ with strictly negative (outward) normal derivative at the boundary, $\lambda_{1,b} > 0$ and we have the characterization

$$\int_\Omega |\nabla u|^p \geq \lambda_{1,b} \int_\Omega b(x)|u|^p \quad \text{for any } u \in W_0^{1,p}(\Omega)\,, \tag{2.1}$$

where the equality holds if and only if u is a multiple of $\phi_{1,b}$ (see for example [50, 4]).

We are now in the position to state our existence result.

Theorem 2.1.

(i) Under hypotheses (H_1)–(H_3) and (M_1), there exists a positive solution $u(x)$ of Problem (P_λ) which satisfies $u(x) \leq a(x)$, for every $\lambda > \lambda_{1,b}$.

(ii) Under hypotheses (H_1)–(H_4), there exists a positive solution $u(x)$ of Problem (P_λ), for every $\lambda \in (0, \lambda_{1,b})$.
Moreover, if also the following hypothesis holds,

$$(H_5) \qquad h(x,t) < b(x)t^{p-1}, \ \textit{for any} \quad x \in \Omega \quad \textit{and} \quad t \in (0, a(x)),$$

then, for some $x_0 \in \Omega$, we have $u(x_0) > a(x_0)$.

(iii) Under hypotheses (H_1)–(H_5), there exists a positive solution $u(x)$ of Problem (P_λ) for $\lambda = \lambda_{1,b}$.

As is common when looking for positive solutions, we define the auxiliary function

$$\widetilde{h} : \mathbb{R} \to [0, \infty) : \begin{cases} \widetilde{h}(x,s) = h(x,0) = 0 & \text{for} \quad s \leq 0, \\ \widetilde{h}(x,s) = h(x,s) & \text{for} \quad s > 0, \end{cases}$$

that is, $\widetilde{h}(x,s) = h(x,s^+)$ where $s^+ = \max\{0,s\}$.

The solutions of Problem (P_λ) with the new function $\widetilde{h}$ are then non-negative solutions of the original Problem (P_λ). Moreover, since $h(x,s) \geq 0$, any nontrivial solution is, in fact, strictly positive, by the strong maximum principle in [55, Theorem 1.1]. Moreover, observe that, by hypotheses (H_1) and (H_4), all weak solutions of Problem (P_λ) are of class $\mathcal{C}^{1,\alpha}$ for some $\alpha \in (0,1)$ (see [29, 39]), and the same holds for the eigenfunction $\phi_{1,b}$.

Let us give a overview of the Proof of Theorem 2.1.

For $\lambda > \lambda_{1,b}$ a solution can be obtained by the method of sub and supersolutions (see [10]), which can be applied in view of the monotonicity condition (M_1). In fact, by hypothesis (H_2), $a(x)$ is always a supersolution of Problem (P_λ), which is also not a solution because $a(x) \geq a_0 > 0$, while a strict subsolution can be obtained in view of hypothesis (H_3), in the form $\varepsilon\phi_{1,b}$ with $\varepsilon > 0$ small enough that $\lambda_{1,b}\, b(x)\ (\varepsilon\phi_{1,b})^{p-1} < \lambda h(x,\varepsilon\phi_{1,b})$.

For $\lambda < \lambda_{1,b}$ a solution can be obtained by showing that the functional

$$J_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R} : u \mapsto J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \lambda \int_\Omega \widetilde{H}(x,u)\,, \qquad (2.2)$$

where $\widetilde{H}(x,t) = \int_0^t \widetilde{h}(x,s)\,ds$, has a strict local minimum at the origin and satisfies the hypotheses of the the mountain pass theorem, in view of the p-superlinear and subcritical behavior of h stated in hypothesis (H_4). Moreover, under hypothesis (H_5) it is easy to show that the solution u must exceed a somewhere, otherwise, using (2.1), one obtains the contradiction

$$\int_\Omega |\nabla u|^p = \lambda \int_\Omega h(x,u)u < \lambda \int_\Omega b(x)|u|^p \leq \frac{\lambda}{\lambda_{1,b}} \int_\Omega |\nabla u|^p < \int_\Omega |\nabla u|^p\,.$$

Finally, a solution for $\lambda = \lambda_{1,b}$ can be obtained as the limit of solutions with $\lambda \nearrow \lambda_{1,b}$ (here hypothesis (H_5) is required to guarantee that the limit is nontrivial).

2.2 The second solution

The proof that a second positive solution exists in the case $\lambda > \lambda_{1,b}$ is somewhat more complicated, in fact we will first need to apply several comparison results, in order to be able to show that the first solution lies strictly below the function $a(x)$. For this, we also need the following monotonicity hypothesis, which is stronger than (M_1):

(M_2) there exists a constant $k > 0$ such that, for all $x \in \Omega$ the map $s \mapsto h(x,s) + k\, s^{p-1}$ is increasing.

The fact that the first solution lies strictly below $a(x)$ is not obvious, actually, it is known (see for instance in [15, 58, 59]) that, at least in regions where a is constant or harmonic, one may have a so called "flat core" or "coincidence set", that is, an open set where, for λ large, the solution coincides with a. This phenomenon is connected with the shape of a and with the behavior of the nonlinearity near to it.

For this reason, we will need to assume one of the following conditions.

(a) $p = 2$ (semilinear case).

(b) $a(x) \equiv \overline{a}$, with $\overline{a}$ a positive constant, and there exists a constant $C > 0$ such that $h(x,t) \le C|\overline{a} - t|^{p-1}$ for $t \le \overline{a}$.

(c) $-\Delta_p\, a \in L^\infty(\Omega)$ and there exists $\varepsilon > 0$ such that $-\Delta_p\, a(x) > \varepsilon$ a.e. $x \in \Omega$.

(d) $a \in \mathcal{C}^1$ and $\nabla a \neq 0$ in Ω.

In particular, the second condition in the case (b) is complementary to the hypothesis which guarantees the existence of flat core solutions. Also, conditions (c) and (d) clearly avoid the existence of flat horizontal regions for the function a.

Our multiplicity result is the following.

Theorem 2.2. *Under hypotheses* (H_1)–(H_4) *and* (M_2)*, if at least one of the conditions* ($a - b - c - d$) *holds, then there exist at least two positive solutions* $u_1 \le u_2$ *of Problem* (P_λ) *for* $\lambda > \lambda_{1,b}$*, where* $u_1 < a$.

One first have to prove the following Lemma.

Lemma 2.3. *Assume that the hypotheses of Theorem 2.1 point (i) and hypothesis* (M_2) *hold. If moreover one of the conditions* $(a-b-c-d)$ *holds true, let* u *be a solution of Problem* (P_λ) *satisfying* $0 < u \le a$. *Then* $u < a$.

The proof of this lemma relies on the application of several known comparison results: [55, Theorem 1.1] for the cases (a) and (b), [5, Theorems 2.4 and 2.6] (see also [61]) for the case (c) and [12, Theorem 1.4] for the case (d).

At this point, the second solution is obtained by using variational techniques. Following the lines of [17] we first show that the solution of Theorem 2.1-(i) is a minimum of a suitable functional, and then we use this fact in order to obtain a second solution from the first one. This is done as follows. As in the previous Section, there exists an $\varepsilon_\lambda > 0$ such that $\varepsilon_\lambda \phi_{1,b} < a$, which are a subsolution and a supersolution respectively. Applying [17, Proposition 3.1], we obtain a solution u_1 which minimizes the functional J_λ in (2.2), in the set $X = \{u \in W_0^{1,p}(\Omega) : \varepsilon_\lambda \phi_{1,b} \le u \le a\}$. However, by Lemma 2.3 and using again the comparison principle in [5, Theorems 2.4 and 2.6], we obtain that

$$\varepsilon_\lambda \phi_{1,b} << u_1 < a\,, \tag{2.3}$$

where $u << v$ means that $u < v$ in Ω and also $\frac{\partial u}{\partial n} > \frac{\partial v}{\partial n}$ on $\partial\Omega$, where n is the outward normal. Now it follows from (2.3) that X contains a $\mathcal{C}_0^1(\overline{\Omega})$ neighborhood of u_1. Consequently, u_1 is a local minimizer of J_λ in the $\mathcal{C}_0^1(\overline{\Omega})$ topology. Applying the results of [24] (see also [7]), we see that u_1 is also a local minimizer of J_λ in $W_0^{1,p}(\Omega)$.

The proof then consists in obtaining a second solution of Problem (P_λ) in the form $u_1 + w$, where w is a nontrivial solution of the Problem

$$\begin{cases} -\Delta_p (u_1 + w) = \lambda \widetilde{h}(x, u_1 + w^+) & \text{in } \Omega\,, \\ w = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.4}$$

Observe that if $w \in W_0^{1,p}(\Omega)$ solves Problem (2.4), then $w \ge 0$. The solution is finally obtained by showing that the associated functional satisfies the conditions of the mountain pass theorem.

2.3 Asymptotic behavior of the solutions

May be the most interesting result in [36] is the study of the asymptotic behavior of the solutions u_λ as λ tends to 0 or to ∞.

As noticed in the introduction, a rule of thumbs to study the existence and the behavior of positive solutions is to consider the intersections of the nonlinearity $\lambda h(x,t)$ with, in our case, the function $\lambda_{1,b}t^{p-1}$. For λ small there is only one intersection, which also goes to infinity as $\lambda \to 0$. For $\lambda > \lambda_{1,b}$, there is also an intersection below the zero and as λ grows larger, both intersections approach the zero of the nonlinearity. The two theorems below show that, with suitable additional conditions, the solutions follow a similar behavior.

Theorem 2.4. *Under hypotheses* (H_1) *and* (H_3)*, if* $\{u_\lambda\}$ *is a family of positive solutions of Problem* (P_λ)*, then* $\|u_\lambda\|_\infty \to \infty$ *when* $\lambda \to 0$.

Theorem 2.5. *Under hypotheses* (H_1)*–*(H_3)*, and the following:*

(H_4^*) hypothesis (H_4) *holds with* $\sigma \in (p-1, p_*-1)$*, where* p_* *denotes the Serrin's exponent given by* $p_* = \frac{(N-1)p}{N-p}$ *if* $N > p$*, and again we may set* $p_* = \infty$ *if* $N \le p$*,*

(H_7) there exists $\gamma > 0$ *such that* $h(x,t) \ge \gamma|t-a(x)|^\sigma$ *for* $t \ge a(x)$*,*

if $\{u_\lambda\}$ *is a family of positive solutions of Problem* (P_λ)*, and if there exists an* $\varepsilon > 0$ *such that* $\varepsilon\phi_{1,b} \le u_\lambda$ *for every* u_λ *in the family, then* $u_\lambda \to a$ *pointwise in* Ω *when* $\lambda \to +\infty$.

Remark 2.6. Observe that the solutions obtained in Theorem 2.2 satisfy the estimate $\varepsilon_\lambda\phi_{1,b} \le u_\lambda$ required above, moreover, one can see that the value of ε_λ may be chosen independent of λ when it is large enough (far from $\lambda_{1,b}$). $\triangleleft$

Remark 2.7. Remember that, in Section 2.2, in order to achieve the multiplicity result, we had to put ourselves in the situation where the first solution cannot touch the function $a(x)$. Theorem 2.5 shows that, nevertheless, the solution eventually converges to $a(x)$, though only pointwise. $\triangleleft$

An important tool used in the proof of the preceding results is the following Liouville–type theorem for a nonnegative function with zeros.

Theorem 2.8. *Let $f : [0, \infty) \to [0, \infty)$ be a continuous function satisfying the following four assumptions:*

(f_1) *There exists $\overline{a} > 0$ such that*

$$\begin{cases} f(t) = 0 & \text{if } t = 0 \text{ or } t = \overline{a} \,, \\ f(t) > 0 & \text{if } t \neq \overline{a}, \ t > 0. \end{cases}$$

(f_2) *There exist constants $\gamma > 0$ and $\sigma \in (p-1, p_*-1)$ such that $f(t) \geq \gamma(t-\overline{a})^\sigma$, for $t > \overline{a}$.*

(f_3) *There exists a constant $\overline{b} > 0$ such that $\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = \overline{b}$.*

(f_4) *There exists a constant $\Lambda > 0$ such that $0 \leq f(t) \leq \Lambda\,(t^\sigma + 1)$, for $t \geq 0$.*

Then any $\mathcal{C}^1$ weak solution of the Problem

$$-\Delta_p w = f(w), \qquad w \geq 0, \qquad \text{in } \mathbb{R}^N, \tag{2.5}$$

is either the constant function $w \equiv 0$, or else $w \equiv \overline{a}$.

The proof of this theorem relies on an extension of a result contained in [57]. Both proofs are detailed in [36, Section 5] and will not be reported here.

The proof of Theorem 2.4 is rather easy, it is based on the idea that for λ small the nonlinearity intersects $\lambda_{1,b}t^{p-1}$ at large values of t.

Proof of Theorem 2.4. ($\lambda \to 0$). Suppose by contradiction that there exists $C > 0$ such that $u_\lambda \leq C$, and let D be such that $h(x,u)/u^{p-1} \leq D\,b_0$ for $0 < u \leq C$. If $\lambda D < \lambda_{1,b}$, then

$$\int_\Omega |\nabla u_\lambda|^p = \lambda \int_\Omega h(x, u_\lambda) u_\lambda \leq \lambda D\, b_0 \int_\Omega u_\lambda^p < \lambda_{1,b} \int_\Omega b(x) u_\lambda^p\,,$$

which contradicts the characterization of $\lambda_{1,b}$ in (2.1). □

The proof of Theorem 2.5 requires a-priori estimates for the possible solutions of Problem (P_λ). Such estimates are stated in the following Lemma.

Lemma 2.9. *Suppose assumptions (H_1), (H_3), and (H_4^*) hold.*

(1) Given $\widetilde{\lambda} > 0$, there exists a constant $D_{\widetilde{\lambda}}$ such that if $u \in \mathcal{C}^1(\overline{\Omega})$ is a positive solution of Problem (P_λ) with $\lambda > \widetilde{\lambda}$, then $\|u\|_\infty \leq D_{\widetilde{\lambda}}$.

(2) If λ is bounded, then the estimate extends to the $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ norm, for some $\alpha \in (0,1)$.

Proof. The full proof will be omitted here, but can be found in [36].

One assumes the existence of a sequence $\{(u_n, \lambda_n)\}_{n\in\mathbb{N}}$, where $\lambda_n > \widetilde{\lambda}$ and u_n is a $\mathcal{C}^1$ positive solution of Problem (P_{λ_n}), such that $S_n = \max_{\overline{\Omega}} u_n = u_n(x_n) \xrightarrow[n\to\infty]{} \infty$, being $\{x_n\} \subset \Omega$ a sequence of points where the maximum is attained.

Defining $w_n(y) = S_n^{-1} u_n(A_n y + x_n)$ for a suitable sequence $A_n \searrow 0$ and taking limit, one deduces the existence of a function w defined either in $\mathbb{R}^N$ or in a half–space, that satisfies, in the weak sense, the relations

$$\begin{cases} 4w^\sigma \geq -\Delta_p w \geq w^\sigma, \qquad w > 0\,, \\ w(0) = \max w = 1\,. \end{cases} \tag{2.6}$$

This contradicts the Liouville–type theorems in [48, Theorem 2.1] (in the case of $\mathbb{R}^N$) and [44, Theorem 3.1] (in the case of the half–space); we thus obtain assertion (1).

Assertion (2) is now a consequence of the regularity theorems of [39]. $\square$

Proof of Theorem 2.5. ($\lambda \to \infty$). Consider a sequence $\{(u_n, \lambda_n)\}_{n\in\mathbb{N}}$ with u_n a $\mathcal{C}^1$ solution of (P_{λ_n}), $\varepsilon\phi_{1,b} \leq u_n$, and $\lambda_n \to +\infty$. By the item (1) of Lemma 2.9 we have that $\|u_n\|_\infty$ is bounded.

Fix a point $x_0 \in \Omega$ and let $\delta_0 = \mathrm{dist}(x_0, \partial\Omega)$. Proceeding similarly to the proof of Lemma 2.9, define this time $w_n(y) = u_n(A_n y + x_0)$ with $A_n \searrow 0$ in such a way that $\lambda_n A_n^p = 1$.

By taking limit one obtains the existence of $w \in \mathcal{C}^1(\mathbb{R}^N)$, which is a weak solution of the Problem

$$-\Delta_p w = h(x_0, w), \qquad w \geq 0, \qquad \text{in } \mathbb{R}^N. \tag{2.7}$$

Now the hypotheses (H_1)–(H_4) and (H_7), allow to apply the Liouville–type Theorem 2.8, and conclude that either $w \equiv 0$ or $w \equiv a(x_0)$, the first possibility being excluded by the estimate $\varepsilon\phi_{1,b} \leq u_n$.

One then concludes that $u_n(x_0) \to a(x_0)$ for every $x_0 \in \Omega$ concluding the proof. □

Remark 2.10. Comparing the proofs of Lemma 2.9 and of Theorem 2.5, we see that in 2.9 we centered the blow-up at the maximum point of the solution. By doing this, if the maximum point converges to the boundary, the limiting problem (2.6) may be defined in a half-space instead of $\mathbb{R}^N$. Instead in Theorem 2.5 we had to center the blow-up at a fixed point, due to the lack of an equivalent to Theorem 2.8 in the case of the half-space.

If we have the information that the maxima stay bounded away from the boundary, then a stronger result can be obtained by centering the blow-up at the maximum point an then obtaining a result for the limit of $\|u_\lambda\|_\infty$. ◁

3 Problems involving critical and supercritical nonlinearities with multiple zeros

In this section we present some results that were obtained in [31, 33], concerning the existence of multiple positive $\mathcal{C}^1(\overline{\Omega})$-weak solutions of the Problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q_λ}$$

where f is a continuous nonnegative function which has zeros $\{z_0 = 0 < z_1 < z_2 < \ldots < z_k\}$ and now we assume also that Ω is convex. In [33] we proved the existence of at least $2k$ or $2k+1$ (depending on the behavior of f at the origin) positive solutions for λ large. No hypotheses are made about the behavior at infinity of f. The particular case with only one positive zero and a growth near the origin at least as u^{p-1} was studied in [31].

Comparing with the results in [36], here we are removing the dependence on $x \in \Omega$ in the nonlinearity and also assuming the convexity of Ω. On the other hand, we are completely removing any requirement on the behavior at infinity of the nonlinearity (recall that in [36] we had to assume a growth for h subcritical and even below the Serrin exponent p_* when dealing with the asymptotic behavior results: see hypotheses (H_4) and (H_4^*)).

Critical and supercritical problems present several additional difficulties. It is known from [51, 52] that if the domain Ω is star-shaped and the nonlinearity is $|u|^{r-2}u$ with r greater or equal to the critical exponent p^*, then no nontrivial solution exists. A solution can be recovered either by considering more topologically complex domains, or by perturbing the nonlinearity. In this second direction several authors considered nonlinearities with any growth at infinity but which behave like $|u|^{q-2}u$ with $q \in (p, p^*)$ near zero; for instance, [11], [45] and [32] assume this type of condition, then they truncate the nonlinearity and look for estimates on the possible solutions. These estimates allow to prove that the solutions are below the truncation point for suitable values of λ, and then a solution of the original problem is obtained.

If f has a zero, the results of Section 2.3 show that solutions of a suitable truncated problem converge to the zero, at least pointwise, when $\lambda \to \infty$. This behavior suggests that also for this problem, truncation procedures like those in [11, 45, 32] could be used to prove the existence of two solutions when considering critical or supercritical nonlinearities. Unfortunately, the pointwise convergence is not enough to guarantee a suitable control on the L^∞ norm of the solutions.

Assuming that Ω is convex and that f is independent from $x \in \Omega$, it becomes possible to use suitable monotonicity results (such as those obtained by the moving planes method, see [8, 21]) which give a better knowledge of the geometry of the solutions. This allows to estimate the L^∞ norm when $\lambda \to \infty$ and then to complete the truncation argument (see also Remark 2.10).

For the results of this section we assume the following hypotheses on the nonlinearity f:

(F_1) $f : [0,T] \to \mathbb{R}$ is a continuous function and there exists a set $\{z_0 = 0 < z_1 < z_2 < \ldots < z_k\} \subseteq [0,T)$ such that f is locally Lipschitz continuous in $(0,T]$, $f(0) = f(z_1) = \cdots = f(z_k) = 0$ and $f(x) > 0$ for $x \in (0,T] \setminus \{z_1; \ldots; z_k\}$.

(F_2) There exist $c_j > 0$ and $\sigma_j \in (p-1, p_* - 1)$ such that

$$\lim_{t \to z_j} \frac{f(t)}{|t - z_j|^{\sigma_j}} = c_j\,, \quad j = 1, \ldots, k.$$

(F_3) There exists $L > 0$ such that the map $t \mapsto f(t) + Lt^{p-1}$ is increasing for $t \in [0,T]$.

About the behavior of the nonlinearity near the origin, we will assume one of the following two hypotheses:

(F_4) There exists $\sigma_0 \in (p-1, p_* - 1)$ such that

$$\lim_{t \to 0^+} \frac{f(t)}{t^{\sigma_0}} = 1;$$

or

(F_5) $\displaystyle\liminf_{t \to 0^+} \frac{f(t)}{t^{p-1}} \geq 1.$

The model (1.2) with $a \equiv z_1$, $q \in (0, p_* - 1)$ and $r \in (p-1, p_* - 1)$ satisfies the conditions above with $k = 1$, however, the condition $r < p_* - 1$ is required only to satisfy (F_2), but the growth at infinity need not to be bounded.

Our main results are the following

Theorem 3.1. *Assume that the hypotheses* (F_1) *through* (F_3) *hold and* Ω *is convex smooth and bounded. If also hypothesis* (F_4) *holds, then there exists* $\lambda^* > 0$ *such that the Problem* (Q_λ) *has at least* $2k+1$ $\mathcal{C}^1$*-weak positive solutions* $v_{0,\lambda}$, $u_{j,\lambda}$, $v_{j,\lambda}$, $j = 1, \dots, k$, *for* $\lambda > \lambda^*$.

Moreover, these solutions satisfy, when $\lambda \to \infty$,

$$\|v_{0,\lambda}\|_\infty \to 0^+\,, \quad \|u_{j,\lambda}\|_\infty \to z_j^- \quad \textit{and} \quad \|v_{j,\lambda}\|_\infty \to z_j^+\,, \quad j = 1, \dots, k\,.$$

Theorem 3.2. *Assume that the hypotheses* (F_1) *through* (F_3) *hold and* Ω *is convex smooth and bounded. If also hypothesis* (F_5) *holds, then there exists* $\lambda^* > 0$ *such that the Problem* (Q_λ) *has at least* $2k$ $\mathcal{C}^1$*-weak positive solutions* $u_{j,\lambda}$, $v_{j,\lambda}$, $j = 1, \dots, k$, *for* $\lambda > \lambda^*$.

Moreover, these solutions satisfy, when $\lambda \to \infty$,

$$\|u_{j,\lambda}\|_\infty \to z_j^- \quad \textit{and} \quad \|v_{j,\lambda}\|_\infty \to z_j^+\,, \quad j = 1, \dots, k\,.$$

Remark 3.3. In fact, it will be clear from the proofs that we might state our result with more details: under the hypotheses of the Theorem 3.1 (resp. 3.2), there exist $\Lambda_j\,:\; j = 0, \dots, k$ such that for $\lambda > \Lambda_j$ Problem (Q_λ) has at least $2j+1$ (resp. $2j$) $\mathcal{C}^1$-weak positive solutions.

Moreover, it follows directly by the main theorems, that we may also consider a function f with an infinite number of zeros, all satisfying a condition as in (F_2), obtaining an arbitrary number of solutions, for sufficiently large λ. An example could be $f(t) = |\sin(t)|^\sigma$ with $\sigma \in (p-1, p_*-1)$. $\lhd$

3.1 Some required tools

As already mentioned, we will need to obtain L^∞ estimates for our solutions. This is done by refining the blow-up argument as described in Remark 2.10 and then relies on two main results: one is a Liouville-type theorem in $\mathbb{R}^N$, the second, which guarantees that the blow-up procedure actually leads to a problem in $\mathbb{R}^N$, is contained in the following Lemma, which is a consequence of the results in [21].

Lemma 3.4. *Assume that* Ω *is a convex and smooth bounded domain, then there exists* $\delta_\Omega > 0$ *(depending only on* Ω*), such that if* $h : [0, +\infty) \to [0, \infty)$ *is a continuous function which is positive and locally Lipschitz continuous in* $(0, \infty)$*, then the* $\mathcal{C}^1(\overline{\Omega})$*-weak solutions* u *of*

$$\begin{cases} -\Delta_p u = h(u), & u \geq 0, \quad \textit{in} \quad \Omega\,, \\ u = 0 & \qquad\quad \textit{on} \quad \partial\Omega\,, \end{cases}$$

satisfy the property that there exists a point $x \in \Omega$ *such that* $dist(x, \partial\Omega) \geq \delta_\Omega$ *and* $u(x) = \|u\|_\infty$.

In order to apply Lemma 3.4, we need to impose the Lipschitz condition on f, and the convexity of Ω. Moreover, we cannot apply it directly to our problem because f is not strictly positive. For this reason we will need to solve first an auxiliary problem (see Problem $(Q_{j,\lambda,\tau})$ in Section 3.2), where a positive perturbation is added, and then to shrink the perturbation and obtain actual solutions of (Q_λ) which maintain the property that their maxima are bonded away from the boundary, so that the blow-up argument can be used to obtain the L^∞ estimates.

For the Liouville-type theorem, in [31] we used the same Theorem 2.8 from Section 2, while in [33] we used the following Lemma, which allows to consider a nonlinearity with several zeros and is a consequence of [20, Theorem 3.12].

Lemma 3.5. *Assume that* $f : [0, \infty) \to [0, \infty)$ *is a continuous function,* $N > p$ *and* u *is a nonconstant* $\mathcal{C}^1(\overline{\Omega})$*−weak solution of*

$$-\Delta_p u \geq f(u), \qquad u \geq 0, \qquad \textit{in } \mathbb{R}^N, \tag{3.1}$$

with $\gamma := \inf_{\mathbb{R}^N} u$.

Then $f(\gamma) = 0$ *and*

$$\liminf_{t \to \gamma^+} \frac{f(t)}{(t-\gamma)^{p_*-1}} = 0\,. \tag{3.2}$$

Actually, by the hypotheses (F_2), (F_4) and (F_5), no zero of f satisfies condition (3.2), so that every solution of (3.1) must be constant.

3.2 Proof of the results

The first step in order to prove our theorems is to build a family of auxiliary problems

$$(Q_{j,\lambda,\tau}) \qquad \begin{cases} -\Delta_p u = \lambda f_j(u) + \tau (u^+)^{p-1} & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases}$$

where $\tau \geq 0$ and f_j (for $j = 0, \dots, k$) is defined by truncating f at a point $R_j \in (z_j, z_{j+1})$ (or above the last zero z_k) and redefining it above R_j as a power of exponent σ_j:

$$f_j(t) = \begin{cases} f(t^+), & \text{if } t \leq R_j\,, \\ \frac{f(R_j)}{R_j^{\sigma_j}} t^{\sigma_j}, & \text{if } t > R_j\,. \end{cases} \tag{3.3}$$

The reason for this construction is twofold: the truncation hides the behavior at infinity of f and then allows us to make no assumptions on it, while the added term $\tau(u^+)^{p-1}$ allows us to apply Lemma 3.4 when $\tau > 0$.

As in Section 2, the nontrivial solutions of the Problem $(Q_{j,\lambda,\tau})$ are positive and $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover, if u is a solution of Problem $(Q_{j,\lambda,\tau})$ and $u < R_j$, then it is also a solution of Problem $(Q_{i,\lambda,\tau})$ for $i > j$. In the case $\tau = 0$, it is also a solution of Problem (Q_λ).

We start with some lemmas. First of all, we state the following a-priori estimates: their proof is analogous to that of Lemma 2.9, as it depends only on the growth at infinity of the nonlinearity.

Lemma 3.6. *Under hypotheses* (F_1), (F_2) *and* (F_4) *or* (F_5), *the conclusions of Lemma 2.9 hold for any weak solution* $u \in \mathcal{C}^1(\overline{\Omega})$ *of Problem* $(Q_{j,\lambda,\tau})$ *with* $j \in \{0, 1, \dots, k\}$, $\lambda > \widetilde{\lambda}$ *and* $\tau \geq 0$.

We also need suitable supersolutions for the Problems $(Q_{j,\lambda,\tau})$ with $j = 1, \dots, k$. Actually, the constant functions z_j are always supersolutions if $\tau = 0$, but we aim for a family of supersolutions which are near to these constants and are still supersolutions when τ is positive (and small). For this purpose, let $n := \|e\|_\infty$ where $e \in W_0^{1,p}(\Omega)$ is the solution of

$$\begin{cases} -\Delta_p e = 1 & \text{in } \Omega\,, \\ e = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Lemma 3.7. *Under hypothesis* (F_2), *for any* $\lambda > 0$, $j \in \{1, \dots, k\}$ *there exist* $\tau^*_{j,\lambda}, \delta_{j,\lambda} > 0$ *such that* $\overline{u}_{j,\xi} = z_j + \xi + \frac{\delta_{j,\lambda}}{4n} e$ *is a supersolution for* ($Q_{j,\lambda,\tau}$) *for any* $\xi \in [-\delta_{j,\lambda}, \delta_{j,\lambda}/2]$ *and* $\tau \in [0, \tau^*_{j,\lambda})$. *Moreover, we may choose* $\delta_{j,\lambda}$ *as nonincreasing functions of* λ *and such that* $z_j - \delta_{j,\lambda} > R_{j-1}$.

Proof. Fix j and λ (we omit the indexes in the parameters). Set $\delta > 0$ such that $\lambda f(t) < \frac{1}{2}\left(\frac{|t-z_j|}{4n}\right)^{p-1} \le \frac{1}{2}\left(\frac{\delta}{4n}\right)^{p-1}$ for $|t - z_j| \le \delta$, which is possible by (F_2). Since this estimate still holds for lower values of λ we deduce that δ may be chosen as a nonincreasing function of λ. For small enough τ, also $\tau u^{p-1} < \frac{1}{2}\left(\frac{\delta}{4n}\right)^{p-1}$ near z_j. So $v := \overline{u}_{j,\xi}$ as defined satisfies $|v - z_j| \le \delta$ and

$$-\Delta_p v = \left(\frac{\delta}{4n}\right)^{p-1} > \lambda f_R(v) + \tau v^{p-1}\,. \qquad \square$$

At this point, in view of Lemma 3.4, we give the following definition.

Definition 3.8. We say that a family of nonnegative functions defined in Ω satisfies the δ_Ω-property if for every u in the family there exists a point $x \in \Omega$ such that $dist(x, \partial\Omega) \ge \delta_\Omega$ and $u(x) = \|u\|_\infty$, where δ_Ω is given in Lemma 3.4.

Remark 3.9. As already observed, under the hypotheses (F_1) and (F_2), if Ω is convex, then Lemma 3.4 implies that the family of the $\mathcal{C}^1(\overline{\Omega})$-weak solutions of the Problems ($Q_{j,\lambda,\tau}$) with $j \in \{0, , \dots, k\}$ and $\lambda, \tau > 0$, satisfies the δ_Ω-property. In the case $j = 0$ this is true also for $\tau = 0$. ◁

The following Lemma performs the estimate that was described in Remark 2.10 and is crucial for our argument: it states that if we know that a family of solutions of the Problems ($Q_{j,\lambda,\tau}$) satisfies the δ_Ω-property, then their infinity norm must converge to the set of the zeros of f_j, when $\lambda \to \infty$.

This fact will be used later in order to prove that, for λ large, the solutions we obtain are distinct, and also in order to prove that they stay below the point where f_j is truncated, so that they are in fact solutions of the original Problem (Q_λ).

Lemma 3.10. *Assume hypotheses (F_1), (F_2) and (F_4) or (F_5). For a given $j \in \{0, \dots, k\}$, if u_{j,λ_n,τ_n} are solutions of the corresponding Problems (Q_{j,λ_n,τ_n}) which satisfy the δ_Ω-property, and $\lambda_n \to \infty$ while $\tau_n \geq 0$ is bounded, then the sequence $\|u_{j,\lambda_n,\tau_n}\|_\infty$ has the limit set, for $n \to \infty$, contained in $\{0, z_1, \dots, z_j\}$.*

Proof. The proof follows the lines of that of Theorem 2.5, but this time the blow-up is centered at a point $x_n \in \Omega$ such that $d_n := dist(x_n, \partial\Omega) \geq \delta_\Omega$ and $u_n(x_n) = \|u_n\|_\infty$.

As a result, the rescaled solutions converge to a solution of the limiting problem

$$- \Delta_p w = f_j(w), \qquad w \geq 0\,, \qquad \text{in } \mathbb{R}^N, \tag{3.4}$$

which is always defined in R^N since x_n stays away from the boundary. In view of the Hypotheses (F_2), (F_4) and (F_5), we can apply Lemma 3.5 to conclude that w is constant, that is, w must be a zero of f_j.

This proves that $w_n(0) = u_n(x_n) = \|u_n\|_\infty \to z \in \{0, z_1, \dots, z_j\}$. $\square$

In the following Lemma we obtain two solutions of $Q_{j,\lambda,\tau}$ with $\tau > 0$, one below z_j and the other one exceeding it; the requirement is that we first have a subsolution whose infinity norm is between z_{j-1} and z_j.

The second solution is obtained by a topological degree argument, adapting a result obtained, for $p = 2$, by de Figueiredo and Lions in [18], see also [34] for the general case. The reason to use a topological argument instead of the variational one used in Section 2 is that here we need to exploit the family of supersolutions from Lemma 3.7 in order to guarantee a separation between the two solutions that will allow to distinguish them even after taking limit for $\tau \to 0$.

Observe that Lemma 3.7 relies on hypothesis (F_2), which in fact avoids the formation of flat cores (see in Section 2.2 and compare with condition (b) there), actually, if flat cores could occur, then it would become difficult to separate the two solutions and obtain the multiplicity result.

Lemma 3.11. *Assume hypotheses* (F_1), (F_2) *and* (F_3)*; fix* $j \in \{1, \dots, k\}$ *and* $\lambda > 0$*. Suppose that the Problems* $(Q_{j,\lambda,\tau})$*,* $\tau \geq 0$ *have a common subsolution* $\underline{u} > 0$*, satisfying* $z_{j-1} < \|\underline{u}\|_\infty < R_{j-1}$*.*

Then for a given $\tau_0 \in (0, \tau^*_{j,\lambda})$*, Problem* (Q_{j,λ,τ_0}) *has 2 solutions* u_{j,λ,τ_0}*,* v_{j,λ,τ_0} *satisfying*

$$z_{j-1} < \|\underline{u}\|_\infty \leq \|u_{j,\lambda,\tau_0}\|_\infty < z_j - \delta_{j,\lambda}/4 < z_j + \delta_{j,\lambda}/4 < \|v_{j,\lambda,\tau_0}\|_\infty . \quad (3.5)$$

Sketch of the proof. One solution is obtained by the sub and supersolution method: for $\tau \in (0, \tau^*_{j,\lambda})$, we have the supersolution $\widetilde{u} = \overline{u}_{j,-\delta_{j,\lambda}} = z_j - \delta_{j,\lambda} + \frac{\delta_{j,\lambda}}{4n} e$ from Lemma 3.7; by construction, we have $\underline{u} < R_{j-1} < \widetilde{u} < z_j - \delta_{j,\lambda}/2$. Then the sub and supersolutions method gives a solution u_{j,λ,τ_0} with the claimed properties.

The argument for the second solution is rather technical and can be seen in details in [31, 33]. One considers $X = \mathcal{C}^1_0(\Omega)$, a set of the form

$$\mathcal{O} = \{u \in X : \|u\|_X < B_\lambda, \, u >> \underline{u}\} ,$$

and proves, by using comparison principle and the a-priori estimates, that the Leray-Shauder degree in $\mathcal{O}$ of the operator corresponding to Problem (Q_{j,λ,τ_0}) is well defined, independent of τ and then eventually zero since $f_j \geq 0$ and then no positive solution exists if $\tau > \lambda_1$.

However, one proves that the degree is 1 in

$$\mathcal{O}' = \{u \in \mathcal{O} : \, u < \overline{u} \text{ in } \Omega\} ,$$

where $\overline{u}$ is the supersolution $\overline{u}_{j,0} > z_j$ from Lemma 3.7.

Then, applying the excision property, it follows that (Q_{j,λ,τ_0}) has a solution $v_{j,\lambda,\tau_0} \in \mathcal{O} \setminus \overline{\mathcal{O}'}$; in particular, since $v_{j,\lambda,\tau_0} \geq \overline{u}$ we obtain the last estimate $\|v_{j,\lambda,\tau_0}\|_\infty > z_j + \delta_{j,\lambda}/4$. □

Lemma 3.11 allows us to obtain the following Lemma, which will be used to iteratively produce two solutions of Problem (Q_λ), starting with a given solution.

Lemma 3.12. *Assume hypotheses (F_1), (F_2), (F_3), hypothesis (F_4) or (F_5), and the convexity of Ω. Suppose that, for some $j \in \{1, \dots, k\}$, there exists $\Lambda_{j-1} > 0$ such that there exists a solution v_λ of $(Q_{j,\lambda,0})$ for any $\lambda > \Lambda_{j-1}$, satisfying $z_{j-1} < \|v_\lambda\|_\infty < R_{j-1}$.*

Then there exists $\Lambda_j \geq \Lambda_{j-1}$ such that for $\lambda > \Lambda_j$, $(Q_{j,\lambda,0})$ has two more solutions $u_{j,\lambda,0}$ and $v_{j,\lambda,0}$, satisfying

$$R_{j-1} \leq \|u_{j,\lambda,0}\|_\infty \leq z_j - \delta_{j,\lambda}/4 < z_j + \delta_{j,\lambda}/4 \leq \|v_{j,\lambda,0}\|_\infty < R_j \,. \tag{3.6}$$

Moreover $\|u_{j,\lambda,0}\|_\infty \to z_j^-$ and $\|v_{j,\lambda,0}\|_\infty \to z_j^+$ as $\lambda \to \infty$.

Proof. First we fix $\lambda' > \Lambda_{j-1}$, and we observe that $v_{\lambda'}$ is a subsolution for Problem $(Q_{j,\lambda,\tau})$ for any $\tau \geq 0$ and $\lambda > \lambda'$.

Then we apply Lemma 3.11 with this j and this subsolution. As a result we get two solutions $u_{j,\lambda,\tau}$ and $v_{j,\lambda,\tau}$ of $(Q_{j,\lambda,\tau})$ satisfying

$$z_{j-1} < \|v_{\lambda'}\|_\infty \leq \|u_{j,\lambda,\tau}\|_\infty < z_j - \delta_{j,\lambda}/4 < z_j + \delta_{j,\lambda}/4 < \|v_{j,\lambda,\tau}\|_\infty \,, \tag{3.7}$$

for every $\lambda > \lambda'$ and $\tau \in (0, \tau^*_{j,\lambda})$. Since Ω is convex, these solutions satisfy the δ_Ω-property by Remark 3.9.

Now, for a fixed value of $\lambda > \lambda'$, by the a-priori estimates in Lemma 3.6 we can take the $\mathcal{C}^1$ limit as $\tau \searrow 0$ and obtain nonnegative weak solutions of $(Q_{j,\lambda,0})$, which we denote by $u_{j,\lambda,0}, v_{j,\lambda,0}$. By taking limit in (3.7) we obtain part of the estimates in (3.6) and in particular we see that $u_{j,\lambda,0}$ and $v_{j,\lambda,0}$ are distinct. Moreover, as limits of functions satisfying the δ_Ω-property, also $u_{j,\lambda,0}$, $v_{j,\lambda,0}$ satisfy it.

Then, we can apply Lemma 3.10 and we deduce that $\|u_{j,\lambda,0}\|_\infty \to z_j^-$ and $\|v_{j,\lambda,0}\|_\infty \to z_j^+$ as $\lambda \to \infty$ (actually z_j is the last zero of f_j and they could not converge to a lower one since v_λ already exceeds it).

Thus there exists Λ_j such that $R_{j-1} < \|u_{j,\lambda,0}\|_\infty , \|v_{j,\lambda,0}\|_\infty < R_j$ for $\lambda > \Lambda_j$, which completes the estimates in (3.6). □

At this point we have the ingredients to prove our main results.

Proof of Theorem 3.2. The proof is by induction.

By Hypotheses (F_5) we have, as in Section 2.1, that there exists $\varepsilon > 0$ such that $\varepsilon\phi_1 < R_0$ and $\varepsilon\phi_1$ is a subsolution for Problem $(Q_{1,\lambda,\tau})$ for any $\tau \geq 0$ and $\lambda > \overline{\lambda} > \lambda_1$.

Then we can apply Lemma 3.11 with $j = 1$ and the subsolution $\underline{u} = \varepsilon\phi_1$. As a result we get, for every $\lambda > \overline{\lambda}$ and $\tau \in (0, \tau^*_{1,\lambda})$, solutions $u_{1,\lambda,\tau}$ and $v_{1,\lambda,\tau}$ of $(Q_{1,\lambda,\tau})$ satisfying $0 < \|\varepsilon\phi_1\|_\infty \leq \|u_{1,\lambda,\tau}\|_\infty < z_1 - \delta_{1,\lambda}/4 < z_1 + \delta_{1,\lambda}/4 < \|v_{1,\lambda,\tau}\|_\infty$. By reasoning as in the proof of Lemma 3.12 we may take the limit for $\tau \to 0$ and we obtain solutions $u_{1,\lambda,0}$ and $v_{1,\lambda,0}$ of $(Q_{1,\lambda,0})$ satisfying

$$0 < \|\varepsilon\phi_1\|_\infty \leq \|u_{1,\lambda,0}\|_\infty \leq z_1 - \delta_{1,\lambda}/4 < z_1 + \delta_{1,\lambda}/4 \leq \|v_{1,\lambda,0}\|_\infty \,, \tag{3.8}$$

which also satisfy the δ_Ω-property.

Then we may apply Lemma 3.10. Since $\|u_{1,\lambda,0}\|_\infty$ cannot tend to zero by (3.8), we deduce that $\|u_{1,\lambda,0}\|_\infty, \|v_{1,\lambda,0}\|_\infty \to z_1$, in fact, $\|u_{1,\lambda,0}\|_\infty \to z_1^-$ and $\|v_{1,\lambda,0}\|_\infty \to z_1^+$ by (3.8).

As a consequence, there exists Λ_1 such that $u_{1,\lambda,0}, v_{1,\lambda,0} < R_1$ for $\lambda > \Lambda_1$, which implies that they are also solutions of the Problems $(Q_{2,\lambda,0})$ and (Q_λ).

Thus we use $v_{1,\lambda,0}$ as the solution v_λ required in Lemma 3.12 and we apply this Lemma $k-1$ times, starting with $j = 2$. We obtain two more solutions at each step; in view of the estimate (3.6), these two solutions are always distinct from the previous ones, are also solutions of Problem (Q_λ), and the larger one will serve as v_λ for the next application of Lemma 3.12.

So we obtain a total of $2k$ positive solutions of Problem (Q_λ), for $\lambda > \Lambda_k$. The convergence result also comes from Lemma 3.12. □

Proof of Theorem 3.1. Theorem 3.1 can be proved in the same way, with the difference that, under condition (F_4), we do not have a subsolution but a first solution is obtained by applying the mountain pass theorem to

Problem $(Q_{0,\lambda,0})$. This solution is a solution of (Q_λ) for λ large enough, and can then be used as the solution v_λ to apply Lemma 3.12 k times, starting with $j = 1$, thus obtaining a total of $2k + 1$ positive solutions of Problem (Q_λ), for $\lambda > \Lambda_k$, and the convergence result. □

4 Problem in an annulus with a variable zero and local superlinearity

In this section we briefly describe the results in the paper [37], where we considered Problem (P_λ) in the radial and semilinear case ($p = 2$), when $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$ is the annulus with $0 < r_1 < r_2$, $N \geq 2$. In this case the problem can be written as the ordinary boundary value problem

$$\begin{cases} v''(t) + \lambda q(t) f(t, v(t)) = 0\,, & \text{for} \quad t \in (0,1)\,, \\ v(0) = v(1) = 0\,, \end{cases} \tag{R_λ}$$

where the function $q(t)$ is continuous and positive on the interval $[0, 1]$. In this setting, it will be possible to take advantage of some techniques available in dimension one, and in particular of the fact that any positive solution is concave, in order to obtain the same kind of results but under weaker assumptions than those considered in [36] and [31].

More precisely, we consider the following four assumptions:

(K_1) The function $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and there exists a continuous function $a : [0, 1] \to (0, +\infty)$, which is concave, such that $f(t, 0) = f(t, a(t)) = 0$ and $f(t, v) > 0$ if $0 < v < a(t)$.

(K_2) There exists a continuous function $b : [0, 1] \to (0, +\infty)$ such that

$$\lim_{v \to 0^+} \frac{f(t, v)}{v} = b(t) \text{ uniformly in } t \in [0, 1].$$

(K_3) There exist constants $0 < \alpha < \beta < 1$ such that

$$\lim_{v \to +\infty} \frac{f(t, v)}{v} = +\infty \text{ uniformly in } t \in [\alpha, \beta].$$

(M_3) (a) The function $f_v := \frac{\partial f}{\partial v}$ exists and is continuous in the set $\{(t,v) : t \in [0,1], v \in [0, a(t)]\}$,

(b) $f_v(t,v) < v^{-1} f(t,v)$ in the set $\{(t,v) : t \in (0,1), v \in (0, a(t))\}$.

The conditions (K_1-K_2) correspond directly to (H_1–H_3) from [36], while about the behavior at infinity of f we only assume hypothesis (K_3), which states that the nonlinearity is required to be superlinear, but only in a small interval (that is, in a small annulus in Ω), while no higher bound on the growth is imposed. Moreover, one does not need to ask the convexity of the domain nor that the nonlinear term is independent of $x \in \Omega$ (as we did in [31, 33]). The hypothesis (M_3) provides the monotonicity required to apply the sub and supersolutions method (as (M_1)-(M_2) in Section 2) and is also used to obtain the second solution, by a homotopy argument.

An example of a function f which satisfies our broader hypotheses, but not those in [36] and [31], could be

$$f(t,u) = \begin{cases} u(1-u)^2 & \text{for } u \le 1\,, \\ (e^{u-1} - u)\phi(t) & \text{for } u > 1\,, \end{cases}$$

where $\phi \in \mathcal{C}^0([0,1])$ is nonnegative, may be null in some set, but is positive in $[\alpha, \beta]$.

Under the hypotheses above, the main results in [37] state, as in [36], the existence of a positive solution of Problem (R_λ) for $0 < \lambda < \lambda_{1,qb}$, of two ordered positive solutions for every $\lambda > \lambda_{1,qb}$, and the asymptotic behaviors

$$\|v_\lambda\|_\infty \to +\infty \text{ when } \lambda \to 0^+$$

and (provided f has no other zero than a)

$$v_\lambda \to a \quad \text{pointwise in } (0,1) \text{ and } ||v_\lambda||_\infty \to ||a||_\infty\,, \qquad \text{when } \lambda \to \infty\,.$$

The proofs however require different techniques.

4.1 Sketch of the proof of the main results

We consider the Banach space $X = \mathcal{C}([0,1])$ endowed with the norm $\|v\|_\infty = \max_{t\in[0,1]} |v(t)|$ and, in view of the fact that if $u(t)$ is a nonnegative solution of Problem (R_λ) then it is a concave function, we define the cone

$$C = \{v \in X \ : \ v \text{ is concave and } \ v(0) = v(1) = 0\}$$

and the completely continuous operator $T_\lambda : C \longrightarrow C$

$$T_\lambda v(t) = \lambda \int_0^1 G(t,s)\, q(s) f(s, v(s))\, ds\,, \tag{4.1}$$

where $G(t,s)$ denotes the Green's function for the interval $(0,1)$. The nontrivial fixed points of T_λ correspond to the positive solutions of Problem (R_λ).

The results are obtained through fixed point theorems and degree theory, after careful estimates of (4.1). In particular, observe that the one dimensional Green's function satisfies

$$G(t,s) \le G(s,s) = s(1-s), \quad \text{for all } \ t, s \in [0,1], \tag{4.2}$$

while the concavity of the functions in C provides a way to compare their values in different points: the following Lemma is fundamental in our arguments.

Lemma 4.1. *Given a function v in the cone C and a point $p \in (0,1)$, the following estimates hold:*

$$(i) \quad v(t) \ge \begin{cases} \frac{t}{p}\, v(p) & t < p\,, \\ \frac{1-t}{1-p}\, v(p) & t > p\,, \end{cases} \qquad \textit{and} \qquad (ii) \quad v(t) \le \begin{cases} \frac{t}{p}\, v(p) & t > p\,, \\ \frac{1-t}{1-p}\, v(p) & t < p\,. \end{cases}$$

Moreover, for all $0 < t_0 < t_1 < 1$, we have

$$(iii) \qquad \min_{t\in[t_0,t_1]} v(t) \ge c_{t_0,t_1}\, ||v||_\infty\,,$$

where $c_{t_0,t_1} := \min\{t_0, 1-t_1\}$.

The use of topological methods usually requires a-priori estimates for the solutions, which cannot be obtained as in Section 2 under the weaker conditions we have here. The following Lemma shows how to exploit Lemma 4.1 in order to obtain a-priori estimates under the local-only superlinearity condition (K_3).

Lemma 4.2. *Suppose conditions (K_1) through (K_3) and (M_3) hold. Given $\widetilde{\lambda} > 0$, there exists a constant $D_{\widetilde{\lambda}} > 0$, such that any nontrivial fixed point of T_λ with $\lambda > \widetilde{\lambda}$ satisfies $\|u\|_\infty \leq D_{\widetilde{\lambda}}$.*

Proof. Suppose, for sake of contradiction, that $\{u_n\}$ is a sequence of nontrivial fixed points of T_λ with $\lambda > \widetilde{\lambda}$ and $\|u_n\|_\infty \to +\infty$. Then

$$u_n(t) \quad = \quad \lambda \int_0^1 G\,(t,s)\, q(s) [f(s, u_n(s))]\, ds\,. \tag{4.3}$$

Since $u_n \in C$, using the estimate (iii) in Lemma 4.1,

$$u_n(s) \geq c_{\alpha,\beta} \|u_n\|_\infty \quad \text{for } s \in [\alpha, \beta]. \tag{4.4}$$

Since the integrand in (4.3) is nonnegative and $\lambda > \widetilde{\lambda}$, we obtain

$$\begin{aligned} u_n(t) \quad &\geq \quad \lambda \int_\alpha^\beta G\,(t,s)\, q(s) u_n(s) \frac{f(s,u_n(s))}{u_n(s)}\, ds \\ &\geq \quad \widetilde{\lambda} \int_\alpha^\beta G\,(t,s)\, q(s) c_{\alpha,\beta} \|u_n\|_\infty \frac{f(s,u_n(s))}{u_n(s)}\, ds. \end{aligned}$$

By condition (K_3) and (4.4), for any $M > 0$ one has $\frac{f(s,u_n(s))}{u_n(s)} \geq M$ in $[\alpha, \beta]$ for n suitably large, then

$$u_n(t) \quad \geq \quad \left(M \widetilde{\lambda}\, c_{\alpha,\beta} \int_\alpha^\beta G\,(t,s)\, q(s)\, ds \right) \|u_n\|_\infty.$$

This leads to the contradiction

$$1 \geq \frac{u_n(1/2)}{\|u_n\|_\infty} \geq M \widetilde{\lambda}\, c_{\alpha,\beta} \left\{ \int_\alpha^\beta G\,(1/2, s)\, q(s)\, ds \right\}$$

for arbitrary $M > 0$. The assertion is then proved. □

Then, a solution for $\lambda < \lambda_{1,qb}$ is obtained by applying a Krasnosel'skii fixed point theorem in cones of expansion/compression (compare [3], [14], [13], [27], [38]). In particular, it is possible to prove, using (K_3) and the kind of arguments exploited in Lemma 4.2, that T_λ is expansive for large v, while by (K_2) it is compressive for small v and $\lambda < \lambda_{1,qb}$.

For $\lambda > \lambda_{1,qb}$, a first solution v_1 is simply the one form Theorem 2.1-(i), whose hypotheses are contained in (K_1), (K_2) and (M_3). The second solution is obtained by a degree argument that we describe below.

Consider the Problem

$$\begin{cases} -(v_1+u)''(t) &= \lambda q(t) f(t, v_1+u^+)\,, \quad \text{for} \quad 0<t<1\,, \\ u(0)=u(1) &= 0. \end{cases} \tag{R_+}$$

If $u \geq 0$ is a nontrivial solution of (R_+), then v_1+u is a second positive solution of (R_λ), which satisfies $v_1+u \geq v_1$.

For $\theta, \tau \in [0,1]$ and $\eta \geq 0$, we define the following parameterized family of operators:

$$\begin{cases} T_{\theta,\tau,\eta}u(t) = \lambda\theta \int_0^1 G(t,s)\, q(s) \left[\frac{f(s,v_1(s)+\tau u^+(s)) - f(s,v_1(s))}{\tau} + \eta\right] ds, \\ T_{\theta,0,\eta}u(t) = \lambda\theta \int_0^1 G(t,s)\, q(s) \left[f_v(s,v_1(s))u^+(s) + \eta\right] ds. \end{cases}$$

Observe that, by hypothesis (M_3)-a, $T_{\theta,\tau,\eta}$ is continuous with respect to the parameters θ, τ, η. Moreover, with this definition, a solution of (R_+) is a fixed point of $T_{1,1,0}$, since $-\lambda q(t) f(t,v_1) = v_1''$.

After obtaining a-priori estimates on the possible fixed points in the same way as described in Lemma 4.2, one shows that a suitable homotopy can be obtained joining $T_{1,1,\eta}$ with $T_{1,1,0}$, but one also observes that no fixed point with large η can exist, allowing to obtain that $deg(Id - T_{1,1,0}, B, 0) = 0$ in a suitable ball B. On the other hand, using $(M_3)-b$ to prove that $T_{\theta,0,0}u = u$ implies $u = 0$, one proves that

$$deg(Id - T_{\theta,0,0}, B, 0) = deg(Id, B, 0) = 1 \quad \text{for any } \theta \in [0,1]. \tag{4.5}$$

At this point, if $T_{1,1,0}$ had no nontrivial fixed points, then a homotopy could be obtained joining $T_{1,1,0}$ with $T_{1,0,0}$, providing a contradiction with the homotopy invariance of the degree.

The asymptotical behavior is also obtained by estimating (4.1) using (4.2) and Lemma 4.1, as in Lemma 4.2.

5 Poly-Laplacian and nonlinearities with zeros.

In the paper [35], we considered the following version of Problem (G_λ), with the poly-Laplacian operator with Navier conditions.

$$\begin{cases} (-\Delta)^k u = \lambda f(x,u) + \mu g(x,u) & \text{in} \quad \Omega, \\ (-\Delta)^i u = 0 & \text{on} \quad \partial\Omega, \ \ i = 0, .., k-1, \end{cases} \tag{$S^k_{\lambda,\mu}$}$$

where $k \in \mathbb{N}$, $\lambda, \mu \geq 0$ are two parameters and f, g are nonnegative functions for which we assume

(Z_1) there exists a continuous function $a(x) > a_0 > 0$ such that

$$\begin{cases} f(x,t) = 0 & \text{if } t \geq a(x), \\ g(x,t) = 0 & \text{if } 0 \leq t \leq a(x), \end{cases}$$

that is, the nonlinearity has a zero at $t = a(x)$ and then g describes the nonlinearity above the zero while f describes it below.

Equations with the bi-Laplacian or poly-Laplacian operator and several kinds of nonlinearities were studied in many works [23, 42, 53, 60, 56, 49, 30, 43, 6, 46, 47, 25, 22, 62, 64, 63]. Among them we emphasize those more related to our setting: [6] considered the problem of the existence of two positive solutions with a concave-convex nonlinearity similar to the one in [1]. A complement of this result was proved in [63]. Two positive solutions were also found in [62], in a situation similar to [16].

We describe below part of the results from [35], however we will assume here slightly stronger hypotheses in order to avoid technicalities.

5.1 Existence of two solutions for small μ

The operators $(-\Delta)^k$ are usually called poly-Laplacian and are the prototypes of linear elliptic operators of order $2k$. The boundary conditions assumed here are called Navier conditions and have the important property that, with them, ($S^k_{\lambda,\mu}$) becomes equivalent to a system of k equations with the Laplacian operator and Dirichlet boundary conditions. This fact has some important consequences, in particular that one can apply the maximum principle to each equation and obtain that if the right hand side of ($S^k_{\lambda,\mu}$) is nonnegative, then not only the solution u is nonnegative, but also $(-\Delta)^i u \geq 0$, for any $i \leq k$.

For the poly-Laplacian operator, the method of sub and supersolutions is only available when the nonlinearity is strictly increasing, as a consequence, we cannot use it in order to produce a first solution nor to bound it below the zero as we did in the previous sections. We will see in the last section that, in fact, in some situations this solution eventually exceeds the zero (see Remark 5.6).

On the other hand, Problem ($S^k_{\lambda,\mu}$) can be treated variationally: the natural working space, which we denote by $\mathbb{H}$, is a Hilbert space of functions which have square integrable weak derivatives up to the order k, for which the boundary conditions are enforced up to order $k-1$ and where one can use the norm $\|u\|_{\mathbb{H}} = \left\|\nabla^k u\right\|_{L^2}$. As for the case $k = 1$ there exists a critical exponent for the immersion of $\mathbb{H}$ in L^q, which in defined as $2^*_{N,k} = \frac{2N}{N-2k}$ if $N > 2k$ (and we may set $2^*_{N,k} = \infty$ if $N \leq 2k$).

For more details on the poly-Laplacian see in [35] and the specific book [26].

We first prove an existence and multiplicity result for ($S^k_{\lambda,\mu}$) when $\lambda > 0$ and μ lies in a region of the form $0 < \mu < \overline{\mu}(\lambda)$.

We assume the following additional hypotheses on the functions f, g and their primitives F, G.

(J_1) The functions $f, g : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ are continuous functions which satisfy $f(x, 0) = g(x, 0) = 0$ and g has subcritical growth.

(J_2) $$\lim_{t \longrightarrow 0^+} \frac{f(x,t)}{t} = +\infty \quad \text{uniformly.}$$

(J_3) $$\lim_{t\to+\infty} \frac{g(x,t)}{t} = +\infty \quad \text{uniformly.}$$

(J_4) There exist $\Theta > 2$ and $C > 0$ such that

$$\Theta G(x,t) - g(x,t)t \le C \quad \text{for } t \ge 0\,.$$

As a model, similar to (1.2), we can take

$$\begin{cases} f(x,t) = u^q\left[(a(x)-u)^+\right]^r\,, \\ g(x,t) = \left[(u-a(x))^+\right]^s\,, \end{cases} \tag{5.1}$$

with $q \in (0,1)$, $r > 0$ and $s \in (1, 2^*_{N,k} - 1)$.

Our existence and multiplicity result is the following.

Theorem 5.1. *Under Hypotheses (Z_1,J_1,J_2,J_3,J_4), there exists a function $M : (0,\infty) \to (0,\infty]$ such that the Problem ($S^k_{\lambda,\mu}$), $k \in \mathbb{N}$, has*

a) at least one positive solution for $\lambda > 0$ and $\mu = 0$,

b) at least two positive solutions for $\lambda > 0$ and $0 < \mu < M(\lambda)$.

In [35] we also considered the cases where instead of (J_2), different behaviors near the origin are assumed. As we have seen in the Sections 2 and 3 this results in a different number of solutions when λ is either small or large.

Sketch of the proof of Theorem 5.1. The functional

$$J_{\lambda,\mu}(u) = \frac{1}{2}\,\|u\|^2_{\mathbb{H}} - \lambda \int_\Omega F(x,u^+) - \mu \int_\Omega G(x,u^+) \tag{5.2}$$

is well defined and of class $\mathcal{C}^1$ in $\mathbb{H}$, moreover its nontrivial critical points are positive solutions of Problem ($S^k_{\lambda,\mu}$).

The conditions (J_2) and (Z_1) imply that the second term in (5.2) is dominant near the origin and then, given $\lambda > 0$, there exists u_0 such that $J_{\lambda,\mu}(tu_0) < 0$ for small t and any $\mu \ge 0$.

The hypotheses (J_3) and (J_4) are classical conditions that guarantee the (PS)-condition and (J_3) also implies that the last term in (5.2) is dominant far from the origin, then there exists $e \in \mathbb{H}$ such that, for any $\lambda \geq 0$ and $\mu > 0$,

$$J_{\lambda,\mu}(te) \to -\infty \qquad \text{when } t \to +\infty\,.$$

At this point what we need is to find a "range of mountains", that is, a sphere around the origin where the functional is positive: this is achieved by imposing the bound $\mu < M(\lambda)$, since the the second term in (5.2) is bounded by (Z_1) and the first one is coercive.

The described geometry allows to obtain, for $\lambda > 0$ and $0 < \mu < M(\lambda)$, a local minimum at a negative level inside the range of mountains and also a mountain pass solution at a positive level. When $\mu = 0$ one still has the minimum. $\square$

5.2 Existence for every $\mu > 0$

The proof of Theorem 5.1 shows why we had to split the nonlinearity in two terms with two different parameters: the geometry of the functional is similar to the one we had in Section 2, with the first solution being a minimum and then the second arising as a mountain pass solution, however, without the sub and supersolution method, we are not able to prove that the first solution stays below the zero, forcing us to balance the contribution of f and g in order to obtain the "range of mountains".

However, this result is not completely satisfying if one compares it with those described in the previous sections. Actually, one would expect to be able to obtain at least two solutions without the need to bound μ, as was the case in [36]. In other words, one would expect $M(\lambda) = \infty$ in Theorem 5.1.

This is in fact the case under some additional hypotheses: as stated in Theorem 5.2 below, in some cases existence and multiplicity can be extended to hold without the bound on μ.

On the other hand, in the next section we will show that in other cases existence is actually lost for large values of μ.

Theorem 5.2. *Under the hypotheses of Theorem 5.1, it is possible to guarantee that $M(\lambda) = +\infty$ in the following cases:*

(C1) *$N < 2k$ and the parameter $\lambda > 0$ is small enough;*

(C2) *$k = 1$, the parameter $\lambda > 0$ is small enough and both f, g satisfy a condition as (M_2) from Section 2;*

(C3) *$k = 1$, f, g satisfy (M_2) as above and a is weakly superharmonic as in (H_2) from Section 2.*

We observe that, in order to satisfy the condition (M_2), we have to take $r \geq 1$ in the model (5.1).

Sketch of the Proof of Theorem 5.2. The first step is always to prove that the first solution lies below $a(x)$.

In case ($C3$) one uses the fact that a is a supersolution and (since $k = 1$) follows the lines of the proof of Theorem 2.2. In fact, in this case the first solution stays below a for any value of $\lambda > 0$, and then one obtains the second solution for any value of $\mu > 0$.

In the other two cases, one does not need to assume that a is superharmonic but uses the fact that for λ small the first solution is also small, in order to bound it below a.

In the case ($C1$) we proceed as follows.

The condition $N < 2k$ implies that $\mathbb{H} \hookrightarrow L^\infty(\Omega)$. For $\lambda > 0$ and $\mu = 0$, as in the proof of Theorem 5.1 point (a), we obtain a solution $u_{\lambda,0}$ which is a local minimum at a negative level for $J_{\lambda,0}$. By the geometry of the functional one observes that $\|u_{\lambda,0}\|_{\mathbb{H}} \to 0$ as $\lambda \to 0$ and then, using the embedding in L^∞ we obtain that for λ small enough, $\|u\|_\infty < a_0 \leq a$ for every u in a small neighborhood $\mathcal{N}$ of $u_{\lambda,0}$.

As a consequence μ does not affect the value of $J_{\lambda,\mu}$ in $\mathcal{N}$ and so $u_{\lambda,0}$ is a first solution and also a local minimum at a negative level for every $\mu > 0$. The second solution is then obtained by applying the mountain pass theorem to the functional $J_{\lambda,\mu}$, in view of the estimates obtained in the proof of Theorem 5.1.

The case $(C2)$ is similar to the case $(C1)$, but instead of exploiting the low dimension one can exploit the better properties of the Laplacian operator. In fact, we can use the sub and supersolution method, Moser iteration (see [29, Proposition 1.3]) and minimization in the $\mathcal{C}_1$ topology (see [9]), in order to obtain as before, for λ small, a first solution $u_{\lambda,0}$ with a neighborhood which is not affected by the value of μ. □

5.3 Asymptotical behavior and nonexistence results

In view of the Theorems 5.1 and 5.2, a natural question is whether the bound on μ is only due to the technique we use to prove the existence and multiplicity result, or if it is possible that existence is actually lost for large μ.

It turns out that $M(\lambda)$ cannot be always $+\infty$. We will prove this nonexistence result for the one dimensional version of $(S^k_{\lambda,\mu})$

$$\begin{cases} (-1)^k\, u^{(2k)}(x) = \lambda f(x,u) + \mu g(x,u) & \text{in} \quad (-1,1), \\ u^{(2i)}(\pm 1) = 0 & i = 0,..,k-1\,, \end{cases} \tag{$\Sigma^k_{\lambda,\mu}$}$$

but it looks reasonable that a similar behavior should be found in higher dimension too.

We need to assume the additional condition, which is satisfied for instance by the model nonlinearity (5.1),

(J_5) (i) $f(x,\tau t) > \tau f(x,t) > 0$ for every $x \in \Omega,\ \tau \in (0,1),\ t \in (0, a(x))$;
(ii) $g(x,t) > 0$ for every $x \in \Omega,\ t > a(x)$.

Before going to the claimed nonexistence result we obtain, for Problem $(\Sigma^k_{\lambda,\mu})$, the following result of pointwise convergence to the zero, similar to Theorem 2.5.

Proposition 5.3. *For any $k \in \mathbb{N}$, if hypotheses (Z_1,J_1,J_2,J_3,J_5) hold, then along any two sequences $\mu_n, \lambda_n \to \infty$, the positive solutions u_{λ_n,μ_n} of Problem $(\Sigma^k_{\lambda,\mu})$, if exist, converge to $a(x)$ pointwise.*

Proof. We exploit some of those techniques we already used in Section 4. Actually, solutions of Problem ($\Sigma^k_{\lambda,\mu}$) in dimension 1 are concave and satisfy

$$u(x) \quad = \quad \int_{-1}^{1} \mathcal{G}_k(y,x)[\,\lambda f(y,u(y)) + \mu g(y,u(y))\,]\,dy\,, \qquad (5.3)$$

where $\mathcal{G}_k$ is the Green's Function for the poly-Laplacian with Navier conditions in the interval $(-1,1)$.

By careful estimates of (5.3) and the application of the properties of concave functions in $(-1,1)$ as those in Lemma 4.1, one first obtains a-priori estimates as in Lemma 4.2. Then, if one supposes for sake of contradiction that, as $\lambda,\mu\to\infty$, solutions stay far from a at a point p, then by Lemma 4.1 they also stay far from a in some neighborhood $\mathcal{N}$ of p. Then f,g are positive in $\overline{\mathcal{N}}$ and the right hand side of (5.3) becomes arbitrarily large, contradicting the a-priori estimates. $\square$

A consequence of Proposition 5.3 is that an obstacle to existence for μ,λ large can be obtained if the shape of the function a cannot be approximated by solutions of ($\Sigma^k_{\lambda,\mu}$). In particular we obtain the following nonexistence results.

Theorem 5.4. *Under hypotheses (Z_1,J_1,J_2,J_3,J_5), if $k\geq 2$ and $a\in\mathcal{C}^1[-1,1]$, then there exists $\Lambda_1>0$ and $N:(\Lambda_1,\infty)\to(0,\infty)$ such that Problem ($\Sigma^k_{\lambda,\mu}$) has no positive solution for $\lambda>\Lambda_1$ and $\mu>N(\lambda)$.*

Theorem 5.5. *Under hypotheses (Z_1,J_1,J_2,J_3,J_5), for any $k\in\mathbb{N}$, if moreover $a(x)$ is not a concave function, then the same conclusion of Theorem 5.4 holds.*

It is interesting to look at the case of the simple model nonlinearity (5.1) with $r\geq 1$ and constant $a(x)\equiv 1$: Theorem 5.2 says that $M(\lambda)=+\infty$ if $k=1$ or if $k>1$, $N<2k$ and λ is small. However, if $k>1$, the Theorem 5.4 shows that, in fact, Problem ($\Sigma^k_{\lambda,\mu}$) has no positive solutions at all for λ,μ large. This fact shows a quite different behavior of the higher order problem with respect to the second order one.

Remark 5.6. It is also possible to prove that, in the conditions of the Theorems 5.4-5.5, for λ large, those solutions that exist for $\mu < M(\lambda)$ are not bounded below the zero, which explains why they cease to exist when μ becomes too large. $\triangleleft$

In Theorem 5.5, the obstacle that leads to nonexistence is the fact that if $a(x)$ is not a concave function, then it cannot be approximated by solutions of Problem $(\Sigma^k_{\lambda,\mu})$, which are concave.
This of course is true also for $k = 1$, showing that the superharmonicity condition on a assumed in [36, 37] and in Theorem 5.2-($C3$) is crucial for the existence results.

In Theorem 5.4, the argument is as follows: as we already pointed out, if $k \geq 2$ then $u^{iv} \geq 0$ and then not only u but also $-u''$ is a concave function One first proves that concave functions u which, as $\mu, \lambda \to \infty$, approximate the regular function a in some subset $[\alpha, \beta] \subseteq (0, 1)$, maintain their second derivative bounded in $[\alpha, \beta]$. However, $u'' \to -\infty$ near the boundary as the solutions have to go from the boundary condition $u = 0$ up to the function $a \geq a_0 > 0$ and then stay close to a. This implies that, for μ, λ large, $-u''$ would have two distinct maxima, near the boundary, which is impossible for a concave function.

Observe that no contradiction arises if $k = 1$ and in fact the described behavior of convergence to the zero up to near the boundary and then go very fast to the boundary condition producing large second derivative is the expected one for the solutions in the second order case (see [28, 58, 59, 36]). In the case $k \geq 2$, on the other hand, the higher "stiffness" of the poly-Laplacian makes it impossible for the solution to jump from the boundary condition to near $a(x)$ without exceeding it. Actually, this fact is natural from a physical point of view, since the Laplacian models, in the one dimensional case, the deformation of a string, while the bi-Laplacian models a beam, which has a bending stiffness in addition to the tensile stiffness.

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