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On the topology of complex univariate polynomials

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Abstract. To any univariate polynomial $f \in \mathbb{C}[z]$ we can associate the negative gradient flow of the smooth vector field $-\nabla|f|^2$ on the complex plane. The maximal invariant set of that flow on any sufficiently big disc $D_r \subset \mathbb{C}$ of radius $r \gg 1$ is an embedded directed graph Γ_f with the roots, and the critical points of f as vertices.

Recently, such graphs have been considered by N. A'Campo. He suggested to turn Γ_f into a rooted tree $\underline{\Gamma}_f$ by marking the vertex q which occurs as the limit point of the flow line $\gamma(t)$ which is asymptotic to the positive real axis for $t \to -\infty$ and observed that for every $d \in \mathbb{N}$, the partition of the space of monic polynomials of degree d into polynomials f with a prescribed rooted tree $\underline{\Gamma}_f = \underline{\Gamma}$ provides a real analytic stratification $\mathbb{C}[z]_d^{\mathrm{mon}} = \bigcup_{\underline{\Gamma}} V_{\underline{\Gamma}}$.

In this note, we introduce a Lyashko-Looijenga type map $\mathcal{L}\colon f \mapsto \prod_{c \in \mathrm{Crit}(f)}(u - f(c))$ in order to study the combinatorics and wall crossings for $\underline{\Gamma}_f$ as f varies in $\mathbb{C}[z]_d^{\mathrm{mon}}$ for any fixed degree d.

Keywords: Discriminants, Monodromy, Wall-crossings

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1 Introduction: Motivation and results

Let $f = z^d + a_{d-1} \cdot z^{d-1} + \cdots + a_1 \cdot z + a_0 \in \mathbb{C}[z]$ be a monic, complex, univariate polynomial. To any such f we can associate the smooth real valued function $|f|^2$ and its negative gradient flow

$$\phi\colon \mathbb{R}\times\mathbb{C}\to\mathbb{C},\quad (t,z)\mapsto \phi_t(z)$$

for the vector field $-\nabla|f|^2$ on the domain of f. Given that $|f|^2\colon\mathbb{C}\to\mathbb{R}$ is bounded from below and proper, it is evident that for $t\to\infty$, every flow line of ϕ eventually has to end up in one of the critical points of $|f|^2$. The latter are easily seen to be the roots p_i and the critical points c_j of f. The flow lines connecting pairs of such points are special as they are bounded; they form the edges of an embedded, planar graph $\Gamma_f\subset\mathbb{C}$ with the critical points of $|f|^2$ as vertices. In fact, this graph can be defined as the *maximal invariant subset*[1] of ϕ on any sufficiently big disc $D_r\subset\mathbb{C}$, $r\gg 0$.

Example 1.1. Consider the polynomial

$$f = z^3-2z-4 = (z-2)(z+1-i)(z+1+i).$$

The roots $\{2,-1\pm i\}$ and critical points $\left\{\pm\sqrt{\frac{2}{3}}\right\}$ are shown in Figure 1.1. They are connected by flow lines of the negative gradient flow, perpendicular to the level sets of $|f|^2$.

A'Campo has observed in [1] that Γ_f must always be a tree, i.e. simply connected; this is plausible since, almost by construction, Γ_f appears as a Euclidean neighborhood retract of the complex plane and must therefore be homotopy equivalent to $\mathbb{C}$. He turns Γ_f into a *rooted* tree $\underline{\Gamma}_f$ by marking the one vertex q that is the limit point of the unique flow line which is asymptotic to the positive real axis $\mathbb{R}_{>0}\subset\mathbb{C}$ for $t\to-\infty$.

Now as one varies the coefficients of f, the associated graph $\underline{\Gamma}_f$ and its embedding also change. A'Campo notes in [1, Theorem 7.2] that collecting all polynomials with a fixed rooted tree $\underline{\Gamma}$

$$V_{\underline{\Gamma}} = \left\{f\in\mathbb{C}[z] : \underline{\Gamma}_f = \underline{\Gamma}\right\}$$

[1]For a continuous flow $\varphi:\mathbb{R}\times X\to X$ the maximal invariant set $\mathrm{Inv}(N)$ of a subset $N\subset X$ is defined as $\mathrm{Inv}(N)=\{x\in X:\varphi(t,x)\in N\,\forall t\in\mathbb{R}\}$.

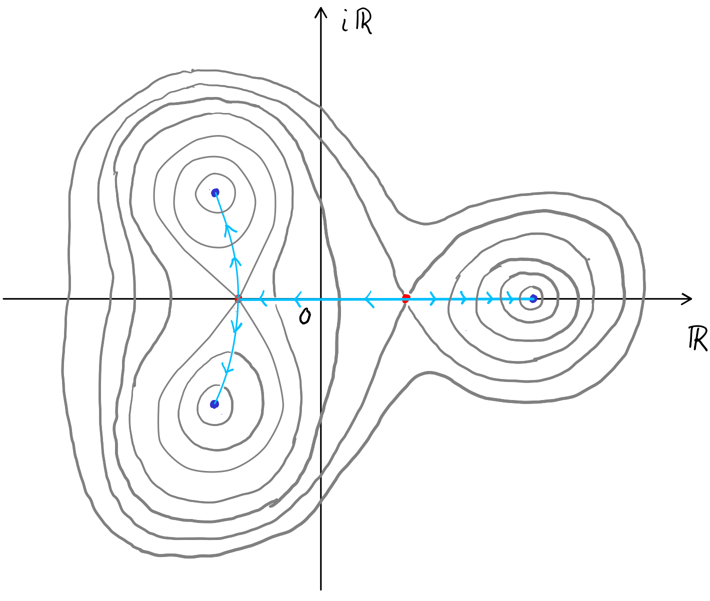

Figure 1.1: The graph Γ_f for $f = z^3 - 2z - 4$

induces a real analytic stratification[2] on the space $\mathbb{C}[z]_d^{\mathrm{mon}} \cong \mathbb{C}^d$ of monic polynomials of any fixed degree d and also on the complement of the discriminant $\mathbb{C}[z]_d^{\mathrm{mon}} \setminus \Delta$. This will be referred to as the Γ-stratification. Furthermore, he showed that the top dimensional strata are contractible and conjectured that this would also be the case for the remaining ones. One natural question which arises in this context is "Which graphs appear as Γ_f for some polynomial f?"; we answer this in Theorem 3.5.

In this note we use a *Lyashko-Looijenga*-type map to study the trees Γ_f for monic polynomials f; see e.g. [5], [6], and [7]. For a fixed degree d

[2]Unless otherwise specified, in this note a (complex/real, analytic/algebraic) *stratification* $\{V_i\}_{i\in I}$ of a topological space X will always mean a decomposition of X into a disjoint set of locally closed (complex/real, analytic/algebraic) subsets V_i. In particular, we do not require any additional conditions on adjacent strata.

it is given by

$$\mathcal{L}\colon \mathbb{C}[z]_d^{\mathrm{mon}} \to \mathbb{C}[u]_{d-1}^{\mathrm{mon}}, \quad f \mapsto h(u) = \prod_{c \in \mathrm{Crit}(f)} (u - f(c)),$$

where the critical points $c \in \mathrm{Crit}(f)$ are counted with multiplicity; i.e. $f(c)$ is a root of h with multiplicity ν whenever c is a root of the derivative $\partial_z f$ with that same multiplicity. It turns out that the restriction $\tilde{\mathcal{L}}$ of $\mathcal{L}$ to the *depressed*[3] polynomials

$$\tilde{\mathcal{L}}\colon \mathbb{C}[z]_d^{\mathrm{dep}} \to \mathbb{C}[u]_{d-1}^{\mathrm{mon}}$$

is a finite branched covering map of degree d^{d-2}, which is ramified over the *discriminant locus* $\Delta \subset \mathbb{C}[u]$ of polynomials with at least one root of multiplicity > 1; see Propositions 2.3 and 2.5.

Using this map, we can give our characterization of all trees Γ that can possibly occur as $\Gamma = \Gamma_f$ for some polynomial f in Theorem 3.5. It seems difficult to prove A'Campo's assertion that all strata in the Γ-stratification are contractible. However, the map $\mathcal{L}$ allows us to introduce another *complex algebraic* stratification, called the $\mathscr{S}'$-stratification, for which we can compute the homotopy groups of all strata. Moreover, the $\mathscr{S}'$-stratification posesses a natural real analytic refinement with *contractible* strata which is also a refinement of A'Campo's Γ-stratification.

One by-product that might be of interest and will be treated in the final Section 4, is the following. The *Galois group*

$$\mathrm{Gal}\,(\mathbb{C}(z)/\mathbb{C}(f))$$

of the field extension $\mathbb{C}(f) \subset \mathbb{C}(z)$ of rational function fields depends only on the stratum of f in the $\mathscr{S}'$-stratification. Therefore, if one had explicit equations for the strata and a precomputed list of their Galois groups, this would yield a rather cheap method to determine the Galois group of any given f by merely evaluating the defining equations of the strata on the coefficients of f. This result is wrapped up in Corollary 4.3 and

[3] A monic polynomial f is called *depressed* if its next-to-leading coefficient is zero.

we illustrate how to use it in Example 4.5. It remains to be determined whether this can be of practical use in computational Galois theory.

1.1 Preliminaries on polynomials and notation

Let $\mathbb{C}[z]$ be the ring of complex univariate polynomials in the variable z. We will consider this space as a direct limit of finite dimensional complex vector spaces

$$\mathbb{C}[z] = \bigcup_{n=0}^{\infty} \mathbb{C}[z]_{\leq d}, \quad \mathbb{C}[z]_{\leq d} := \{f \in \mathbb{C}[z] : \deg(f) \leq d\},$$

together with its induced topology[4]; thus, a set $U \subset \mathbb{C}[z]$ is (Zariski-)open if and only if it its intersection $U \cap \mathbb{C}[z]_{\leq d}$ is open for every d. Elements in $\mathbb{C}[z]$ are of the form

$$f(z) = a_d \cdot \prod_{i=1}^{d}(z - \lambda_i) = a_d \cdot z^d + a_{d-1} \cdot z^{d-1} + \cdots + a_1 \cdot z + a_0,$$

where $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ are the *roots* of f and $a = (a_0, a_1, \ldots, a_d)$ are the *coefficients*. The latter form a natural set of affine coordinates of $\mathbb{C}[z]_{\leq d}$ and we will therefore also write them as functions of f, i.e. $a_i = a_i(f)$.

Denote by

$$\mathbb{C}[z]_d^{\text{mon}} = \{f \in \mathbb{C}[z]_{\leq d} : a_d(f) = 1\}$$

the set of *monic* polynomials of degree d. This is an affine plane in $\mathbb{C}[z]$ with coordinates $a_0, \ldots, a_{d-1}$, closedly embedded into the open subset

$$\mathbb{C}[z]_d = \{f \in \mathbb{C}[z]_{\leq d} : a_d(f) \neq 0\} \subsetneq \mathbb{C}[z]_{\leq d}$$

of polynomials of degree *equal* to d.

The *discriminant set* is the set $\Delta \subset \mathbb{C}[z]$ of polynomials of any degree $d = \deg(f)$ such that f has a multiple root $\lambda_i = \lambda_j$ for some $i \neq j$. It is

[4]We will in the following use both the classical Euclidean and the Zariski topology; mostly in parallel. Doing so, we will frequently make use of the fact that every Zariski open set of a complex algebraic variety is in particular open in the classical topology

well known that that for every fixed number d the set $\Delta_d := \Delta \cap \mathbb{C}[z]_{\leq d}$ is the zero locus of the *discriminant polynomial*, a universal polynomial

$$\delta_d(a_0, a_1, \dots, a_d) \in \mathbb{Z}[a_0, \dots, a_d]$$

in the coefficients of f that can – up to sign – be given as the *resultant* of f and its derivative $\partial_z f$. For instance, for $d = 2$ we have $\delta_2(a_0, a_1, a_2) = a_1^2 - 4a_2 \cdot a_0$ and in case $d = 3$ we find

$$\delta_3(a_0, a_1, a_2, a_3) = a_2^2 \cdot a_1^2 - 4a_3 \cdot a_1^3 - 4a_2^3 \cdot a_0 - 27a_3^2 \cdot a_0^2 + 18a_0 \cdot a_1 \cdot a_2 \cdot a_3.$$

Setting $a_3 = 0$ in δ_3 gives an idea of the nesting of discriminant polynomials:

$$\delta_3(a_0, a_1, a_2, 0) = a_2^2 \cdot \delta_2(a_0, a_1, a_2).$$

Hence, we do not expect *one single, universal* discriminant polynomial $\delta(a_0, a_1, a_2, \dots)$ in infinitely many variables from which all the δ_d can be derived by substing all but finitely many variables with zero. However, the discriminant set Δ is a closed set in $\mathbb{C}[z]$ with $\Delta \cap \mathbb{C}[z]_{\leq d}$ of real codimension 2 for every degree d.

Let

$$\phi : \mathbb{C}^d \to \mathbb{C}[z]_d^{\text{mon}}, \quad (\lambda_1, \dots, \lambda_d) \mapsto f = \prod_{i=1}^{d} (z - \lambda_i)$$

be the map taking an enumerated set of roots to its associated monic polynomial. In the coordinates $a_i(f)$ as above we find

$$a_i(\phi(\lambda_1, \dots, \lambda_d)) = -(-1)^i \sigma_i(\lambda_1, \dots, \lambda_d) = -(-1)^i \cdot \sum_{I \subset \{1,\dots,d\}, |I| = i} \prod_{j \in I} \lambda_j,$$

where σ_i are the *elementary symmetric polynomials*. Note that by definition of Δ we find

$$\phi^{-1}(\Delta_d) = \{(\lambda_1, \lambda_2, \dots, \lambda_d) : \lambda_i = \lambda_j \text{ for some } i \neq j\} =: D$$

to be the *big diagonal* in $\mathbb{C}^d$.

Lemma 1.2. *The restriction*

$$\phi\colon \mathbb{C}^d \setminus D \to \mathbb{C}[z]_d^{\mathrm{mon}} \setminus \Delta$$

is a submersion with $d!$ *points in every fiber.*

Proof. Consider the Jacobian matrix of ϕ at a point $\lambda = (\lambda_1, \dots, \lambda_d)$ outside the big diagonal:

$$\frac{\partial a_i(\phi(\lambda))}{\partial \lambda_j} = -a_i\left(\prod_{k\neq j}(z-\lambda_k)\right)$$

for $i = 0, \dots, d-1$ and $j = 1, \dots, d$, $j \neq k$. Since the roots λ_j are pairwise distinct, the polynomials $\partial_j f := \prod_{k\neq j}(z-\lambda_k)$ for $j = 1, \dots, d$ form a basis of the vector space $\mathbb{C}[z]_{d-1}$. It is now easy to see that for this reason, the above Jacobian matrix must have full rank at λ. The second assertion follows from the fact that the polynomial $f = \phi(\lambda)$ does not depend on the enumeration of the roots λ_j. □

For an arbitrary $\xi \in \mathbb{C}$ we let

$$T_\xi : \mathbb{C}[z] \to \mathbb{C}[z], \quad (T_\xi(f))(z) := f(z-\xi)$$

be the *translation operator*. Note that this restricts to a $\mathbb{C}$-linear isomorphism on every subspace $\mathbb{C}[z]_{\leq d}$ and $\mathbb{C}[z]_d^{\mathrm{mon}}$. Using these operators, we can eliminate the next-to-leading coefficient $a_{d-1}(f)$ of any monic polynomial $f \in \mathbb{C}[z]_d^{\mathrm{mon}}$ via a so-called *Tschirnhaus transformation*

$$\tau : \mathbb{C}[z]_d^{\mathrm{mon}} \to \mathbb{C}[z]_d^{\mathrm{dep}}, \quad f \mapsto T_{a_{d-1}(f)/d}(f),$$

i.e. by choosing $\xi = \frac{1}{d}a_{d-1}$. The codomain of τ is the set of *depressed* polynomials

$$\mathbb{C}[z]_d^{\mathrm{dep}} := \{f \in \mathbb{C}[z]_d^{\mathrm{mon}} : a_{d-1}(f) = 0\}$$

and geometrically the Tschirnhaus transformation simply moves the origin to the "center of mass" of the roots of f. In the following, we will also refer to τ as the *Tschirnhaus projection.* It has a natural section given by the inclusion of the depressed polynomials and thus

$$\mathbb{C}[z]_d^{\mathrm{mon}} \cong \mathbb{C} \times \mathbb{C}[z]_d^{\mathrm{dep}}$$

as complex manifolds in a canonical way.

2 The Lyashko-Looijenga map

We define the *Lyashko-Looijenga* map

$$\mathcal{L}\colon \mathbb{C}[z] \to \mathbb{C}[u]$$

as follows. For $f \in \mathbb{C}[z]$ of arbitrary degree d let $\partial_z f$ be its derivative and $c(f) = (c_1(f), \ldots, c_{d-1}(f))$ some enumeration of the roots of $\partial_z f$, counted with multiplicities. The c_i are the critical points of f and we set $v_i(f) = f(c_i(f))$ to be the associated critical values in the same order. Then on every degree d the map $\mathcal{L}$ is given by

$$\mathcal{L} : \mathbb{C}[z]_d \to \mathbb{C}[u]_{d-1}^{\mathrm{mon}}, \quad f \mapsto h = \prod_{i=1}^{d-1} (u - v_i(f)). \tag{2.1}$$

We will denote the restriction of $\mathcal{L}$ to the depressed polynomials by

$$\tilde{\mathcal{L}}\colon \mathbb{C}[z]_d^{\mathrm{dep}} \to \mathbb{C}[u]_{d-1}^{\mathrm{mon}}, \quad f \mapsto h = \mathcal{L}(f).$$

Given only this definition, $\mathcal{L}$ is merely a set-theoretic map. However, we have the following:

Proposition 2.1. *For every number d the map*

$$\mathcal{L}\colon \mathbb{C}[z]_d^{\mathrm{mon}} \to \mathbb{C}[u]_{d-1}^{\mathrm{mon}}, \quad f \mapsto h = \mathcal{L}(f)$$

is polynomial over $\mathbb{Q}$ in the coefficients of f and h; i.e. there exist polynomials $Q_l(\underline{a}) \in \mathbb{Q}[a_{d-1}, a_{d-2}, \ldots, a_0]$ such that $f = z^d + a_{d-1} \cdot z^{d-1} + \cdots + a_1 \cdot z + a_0$ is taken to

$$\mathcal{L}(f) = u^{d-1} + Q_{d-2}(\underline{a}) \cdot u^{d-2} + \cdots + Q_1(\underline{a}) \cdot u + Q_0(\underline{a}).$$

Furthermore, every $Q_l(\underline{a})$ *is quasihomogeneous of degree* $d \cdot (d-l-1)$ *for the weights* $\deg a_i = d - i$.

Before we give the proof, let us illustrate this phenomenon "manually" in low degrees:

Example 2.2. Suppose $d = 2$ and let $f = z^2 + a_1 \cdot z + a_0$ be an arbitrary monic polynomial. Then $c_1(f) = -a_1/2$, $v_1(f) = -a_1^2/4 + a_0$, and hence $\mathcal{L}(f) = u + a_1^2/4 - a_0$. We find that $4 \cdot Q_0(a_1, a_0)$ is simply the *discriminant* of the quadratic polynomial. This is not too surprising as one can check manually that f has a double root if and only if it has a critical value $v = 0$.

For $d = 3$ and $f = z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0$ we find the solutions of

$$\partial_z f = 3z^2 + 2a_2 \cdot z + a_1 = 0$$

to be

$$c_{1,2}(f) = -\frac{a_2}{3} \pm \sqrt{\frac{a_2^2}{9} - \frac{a_1}{3}}$$

such that

$$v_{1,2}(f) = \pm \left(\frac{2a_1}{3} - \frac{2a_2^2}{9} \right) \sqrt{\frac{a_2^2}{9} - \frac{a_1}{3}} + \left(\frac{2a_2^3}{27} - \frac{a_2 \cdot a_1}{3} + a_0 \right)$$

and therefore

$$\begin{aligned} \mathcal{L}(f) &= u^2 + \left(-\frac{4}{27} a_2^3 + \frac{2}{3} a_2 \cdot a_1 - 2a_0 \right) \cdot u \\ &\quad + \left(\frac{4}{27} a_2^3 \cdot a_0 - \frac{1}{27} a_2^2 \cdot a_1^2 - \frac{2}{3} a_2 \cdot a_1 \cdot a_0 + \frac{4}{27} a_1^2 + a_0^2 \right). \end{aligned}$$

Again, the constant term of $\mathcal{L}(f)$ is a rational multiple of the discriminant of f.

Proof. (of Proposition 2.1) Let $R = \mathbb{Q}[\alpha_0, \alpha_1, \ldots, \alpha_{d-1}]$ and

$$F = z^d + \sum_{i=0}^{d-1} \alpha_i \cdot z^i$$

the "tautological polynomial" of degree d in $R[z]$. We let $\deg \alpha_i = d - i$ be the weights of the variables α_i and set $\deg z = 1$ so that F is quasi-homogeneous of degree d. Let $\xi = (\xi_1, \dots, \xi_k)$ be a further set of indeterminates of weight 1 and

$$G = \prod_{j=1}^{k} (z - \xi_j) = z^k + \sum_{j=0}^{k-1} b_j(\xi_1, \dots, \xi_k) \cdot z^j$$

the "universal polynomial" in z with roots $\xi_1, \dots, \xi_k$. We define

$$H = \prod_{j=1}^{k} (u - F(\xi_j)) \in R[\xi][u]$$

to be the "evaluation of F on the roots of G". Clearly, H is symmetric in the ξ_j and therefore the coefficients of H can be written as polynomials $P_l(\beta_0, \dots, \beta_{k-1}) \in R[\beta_0, \dots, \beta_{k-1}]$ in the coefficients $b_j(\xi)$ by virtue of the elementary theorem on symmetric functions:

$$H = u^k + \sum_{l=0}^{k-1} P_l(b_0(\xi), b_1(\xi), \dots, b_{k-1}(\xi)) \cdot u^l.$$

Since the polynomials $b_j(\xi)$ are homogeneous of degree $k - j$, we assign the same weights to the corresponding variables β_j. We obtain a new polynomial

$$H' = u^k + \sum_{l=0}^{k-1} P_l(\beta_0, \dots, \beta_{k-1}) \cdot u^l \in R[\beta_0, \dots, \beta_{k-1}][u].$$

As F was quasi-homogeneous of degree d, the coefficient of u^l in H is quasi-homogeneous of degree $d \cdot (k - l)$. We infer that the same must hold for the polynomials $P_l(\beta)$.

Now we choose $k = d - 1$ and we turn R into an $R[\beta_0, \dots, \beta_{d-2}]$ algebra via the homomorphism

$$\psi : R[\beta_0, \dots, \beta_{d-2}] \to R, \quad \beta_l \mapsto \frac{l+1}{d} \alpha_{l+1},$$

as suggested by taking the formal derivative $\frac{1}{d}\partial_z F$. Note that this morphism preserves the degrees for the given weights. Then

$$\psi(H') = u^k + \sum_{l=0}^{d-2} P_l\left(\frac{1}{d}\alpha_1, \frac{2}{d}\alpha_2, \ldots, \frac{d-1}{d}\alpha_{d-1}\right) \cdot u^l \in R[u]$$

has coefficients $Q_l(\alpha_0, \ldots, \alpha_{d-1}) = P_l(\alpha_1, \ldots, \alpha_{d-1})$ in $R = \mathbb{Q}[\alpha_0, \ldots, \alpha_{d-1}]$. Note that the coefficients of the P_l were already polynomials in the α_i and the β_j are substituted by variables of the same degree so that indeed $Q_l(\underline{a})$ is quasi-homogeneous of degree $d(d-l-1)$. These Q_l are the sought for polynomials since now for any *complex* polynomial $f \in \mathbb{C}[z]$ with coefficients $a_i = a_i(f)$ we have

$$u^k + \sum_{l=0}^{d-2} Q_l(a_0, a_1, \ldots, a_{d-1}) \cdot u^l = \mathcal{L}(f),$$

as can easily be verified by substitution and tracking the above construction backwards. $\square$

The above proof can be turned into an algorithm to describe $\mathcal{L}$ explicitly. However, the complexity of this problem increases rapidly with the degree of f. For instance, for $d = 5$ we find

$$\begin{aligned}
& \mathcal{L}(z^5 + a_4 \cdot z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0) \\
= \quad & u^4 + (-4a_0 + \frac{4}{5}a_1a_4 + \frac{6}{5}a_2a_3 - \frac{16}{25}a_2a_4^2 \\
& -\frac{18}{25}a_3^2a_4 + \frac{64}{125}a_3a_4^3 - \frac{256}{3125}a_4^5) \cdot u^3 \\
& +(6a_0^2 - \frac{12}{5}a_0a_1a_4 - \frac{18}{5}a_0a_2a_3 + \frac{48}{25}a_0a_2a_4^2 + \frac{54}{25}a_0a_3^2a_4 - \frac{192}{125}a_0a_3a_4^3 \\
& +\frac{768}{3125}a_0a_4^5 + \frac{16}{25}a_1^2a_3 - \frac{2}{125}a_1^2a_4^2 + \frac{18}{25}a_1a_2^2 - \frac{82}{125}a_1a_2a_3a_4 \\
& +\frac{32}{625}a_1a_2a_4^3 - \frac{36}{125}a_1a_3^3 + \frac{204}{625}a_1a_3^2a_4^2 - \frac{192}{3125}a_1a_3a_4^4 - \frac{36}{125}a_2^3a_4 \\
& +\frac{33}{125}a_2^2a_3^2 + \frac{112}{625}a_2^2a_3a_4^2 - \frac{128}{3125}a_2^2a_4^4 - \frac{126}{625}a_2a_3^3a_4 + \frac{144}{3125}a_2a_3^2a_4^3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{108}{3125}a_3^5-\frac{27}{3125}a_3^4a_4^2)\cdot u^2\\
&+(-4a_0^3+\frac{12}{5}a_0^2a_1a_4+\frac{18}{5}a_0^2a_2a_3-\frac{48}{25}a_0^2a_2a_4^2-\frac{54}{25}a_0^2a_3^2a_4\\
&+\frac{192}{125}a_0^2a_3a_4^3-\frac{768}{3125}a_0^2a_4^5-\frac{32}{25}a_0a_1^2a_3+\frac{4}{125}a_0a_1^2a_4^2\\
&-\frac{36}{25}a_0a_1a_2^2+\frac{164}{125}a_0a_1a_2a_3a_4-\frac{64}{625}a_0a_1a_2a_4^3+\frac{72}{125}a_0a_1a_3^3\\
&-\frac{408}{625}a_0a_1a_3^2a_4^2+\frac{384}{3125}a_0a_1a_3a_4^4+\frac{72}{125}a_0a_2^3a_4-\frac{66}{125}a_0a_2^2a_3^2\\
&-\frac{224}{625}a_0a_2^2a_3a_4^2+\frac{256}{3125}a_0a_2^2a_4^4+\frac{252}{625}a_0a_2a_3^3a_4-\frac{288}{3125}a_0a_2a_3^2a_4^3\\
&-\frac{216}{3125}a_0a_3^5+\frac{54}{3125}a_0a_3^4a_4^2+\frac{64}{125}a_1^3a_2-\frac{32}{625}a_1^3a_3a_4+\frac{36}{3125}a_1^3a_4^3\\
&-\frac{204}{625}a_1^2a_2^2a_4-\frac{112}{625}a_1^2a_2a_3^2+\frac{746}{3125}a_1^2a_2a_3a_4^2-\frac{144}{3125}a_1^2a_2a_4^4\\
&-\frac{24}{3125}a_1^2a_3^3a_4+\frac{6}{3125}a_1^2a_3^2a_4^3+\frac{126}{625}a_1a_2^3a_3-\frac{24}{3125}a_1a_2^3a_4^2\\
&-\frac{356}{3125}a_1a_2^2a_3^2a_4+\frac{16}{625}a_1a_2^2a_3a_4^3+\frac{72}{3125}a_1a_2a_3^4-\frac{18}{3125}a_1a_2a_3^3a_4^2\\
&-\frac{108}{3125}a_2^5+\frac{72}{3125}a_2^4a_3a_4-\frac{16}{3125}a_2^4a_4^3-\frac{16}{3125}a_2^3a_3^3+\frac{4}{3125}a_2^3a_3^2a_4^2)\cdot u^1\\
&+a_0^4-\frac{4}{5}a_0^3a_1a_4-\frac{6}{5}a_0^3a_2a_3+\frac{16}{25}a_0^3a_2a_4^2+\frac{18}{25}a_0^3a_3^2a_4-\frac{64}{125}a_0^3a_3a_4^3\\
&+\frac{256}{3125}a_0^3a_4^5+\frac{16}{25}a_0^2a_1^2a_3-\frac{2}{125}a_0^2a_1^2a_4^2+\frac{18}{25}a_0^2a_1a_2^2-\frac{82}{125}a_0^2a_1a_2a_3a_4\\
&+\frac{32}{625}a_0^2a_1a_2a_4^3-\frac{36}{125}a_0^2a_1a_3^3+\frac{204}{625}a_0^2a_1a_3^2a_4^2-\frac{192}{3125}a_0^2a_1a_3a_4^4\\
&-\frac{36}{125}a_0^2a_2^3a_4+\frac{33}{125}a_0^2a_2^2a_3^2+\frac{112}{625}a_0^2a_2^2a_3a_4^2-\frac{128}{3125}a_0^2a_2^2a_4^4\\
&-\frac{126}{625}a_0^2a_2a_3^3a_4+\frac{144}{3125}a_0^2a_2a_3^2a_4^3+\frac{108}{3125}a_0^2a_3^5-\frac{27}{3125}a_0^2a_3^4a_4^2\\
&-\frac{64}{125}a_0a_1^3a_2+\frac{32}{625}a_0a_1^3a_3a_4-\frac{36}{3125}a_0a_1^3a_4^3+\frac{204}{625}a_0a_1^2a_2^2a_4\\
&+\frac{112}{625}a_0a_1^2a_2a_3^2-\frac{746}{3125}a_0a_1^2a_2a_3a_4^2+\frac{144}{3125}a_0a_1^2a_2a_4^4+\frac{24}{3125}a_0a_1^2a_3^3a_4\\
&-\frac{6}{3125}a_0a_1^2a_3^2a_4^3-\frac{126}{625}a_0a_1a_2^3a_3+\frac{24}{3125}a_0a_1a_2^3a_4^2+\frac{356}{3125}a_0a_1a_2^2a_3^2a_4\\
&-\frac{16}{625}a_0a_1a_2^2a_3a_4^3-\frac{72}{3125}a_0a_1a_2a_3^4+\frac{18}{3125}a_0a_1a_2a_3^3a_4^2+\frac{108}{3125}a_0a_2^5
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{72}{3125}a_0a_2^4a_3a_4+\frac{16}{3125}a_0a_2^4a_4^3+\frac{16}{3125}a_0a_2^3a_3^3-\frac{4}{3125}a_0a_2^3a_3^2a_4^2\\
&+\frac{256}{3125}a_1^5-\frac{192}{3125}a_1^4a_2a_4-\frac{128}{3125}a_1^4a_3^2+\frac{144}{3125}a_1^4a_3a_4^2-\frac{27}{3125}a_1^4a_4^4\\
&+\frac{144}{3125}a_1^3a_2^2a_3-\frac{6}{3125}a_1^3a_2^2a_4^2-\frac{16}{625}a_1^3a_2a_3^2a_4+\frac{18}{3125}a_1^3a_2a_3a_4^3\\
&+\frac{16}{3125}a_1^3a_3^4-\frac{4}{3125}a_1^3a_3^3a_4^2-\frac{27}{3125}a_1^2a_2^4+\frac{18}{3125}a_1^2a_2^3a_3a_4\\
&-\frac{4}{3125}a_1^2a_2^3a_4^3-\frac{4}{3125}a_1^2a_2^2a_3^3+\frac{1}{3125}a_1^2a_2^2a_3^2a_4^2
\end{aligned}
$$

Starting from degree $d = 7$, I have been running into severe difficulties to produce these expressions, using several gigabytes of RAM for caching the intermediate results. However, I have not yet tried to accomplish this with any optimized algorithm.

We wish to describe $\mathcal{L}$ further:

Proposition 2.3. *The restriction*

$$\tilde{\mathcal{L}}\colon \mathbb{C}[z]_d^{\mathrm{dep}} \to \mathbb{C}[u]_d^{\mathrm{mon}}$$

is a finite algebraic map of degree d^{d-2}.

In order to prove that proposition, we need an easy preparatory lemma.

Lemma 2.4. *Suppose $f \in \mathbb{C}[z]_d^{\mathrm{mon}}$ has only one critical value $v = 0$. Then $f = (z-c)^d$ for some constant $c \in \mathbb{C}$.*

Proof. Let $c_1, c_2, \ldots, c_r$ be the pairwise distinct roots of the derivative $f' = \partial_z f$ and $\nu_1, \ldots, \nu_r$ its multiplicities: $f' = h_j \cdot (z-c_j)^{\nu_j}$, $(z-c_j) \nmid h_j$. Then, by assumption, every c_j is also a root of f. If we let $\mu_j > 0$ be its multiplicity as a root of f, then we see from

$$
\begin{aligned}
f' &= \partial_z\left((z-c_j)^{\mu_j}\cdot g(z)\right)\\
&= \mu_j\cdot(z-c_j)^{\mu_j-1}g(z)+(z-c_j)^{\mu_j}g'(z)\\
&= (z-c_j)^{\mu_j-1}(\mu_j\cdot g(z)+(z-c_j)g'(z))
\end{aligned}
$$

that we must have $\nu_j = \mu_j - 1$ since $(z - c_j)$ does not divide $g(z)$. On the other hand, we have

$$d = \deg f \geq \sum_{j=1}^{r} \mu_j = \sum_{j=1}^{r} (\nu_j + 1) = \deg f' + r = d - 1 + r.$$

Thus $r = 1$ and $c = c_1$ is the only root of both f and f'. □

Proof. (of Proposition 2.3) We have already established that $\mathcal{L}$ is quasi-homogeneous. Thus, to prove global finiteness of $\tilde{\mathcal{L}}$, it is sufficient to verify that claim for the induced analytic germ at the origin. As $\mathbb{C}[z]_d^{\mathrm{dep}} = \{a_{d-1} = 0\}$ is the hyperplane cut out by a monomial of weight 1, we may equally well consider the map

$$\begin{array}{rcl} \tilde{\mathcal{L}}^{\mathrm{ext}} \colon (\mathbb{C}^d, 0) & \to & (\mathbb{C}^{d-1} \times \mathbb{C}, 0), \\ (a_{d-1}, a_{d-2}, \ldots, a_0) & \mapsto & (Q_{d-2}(\underline{a}), \ldots, Q_1(\underline{a}), Q_0(\underline{a}), a_{d-1}). \end{array}$$

According to Lemma 2.4, the preimage of $h = u^{d-1}$ under $\tilde{\mathcal{L}}^{\mathrm{ext}}$ consists of only one single polynomial $f = z^d$. It follows from the Weierstrass finiteness theorem that $\tilde{\mathcal{L}}^{\mathrm{ext}}$ is a finite analytic map. For dimensional reasons, the components of that map form a regular sequence, so that it is flat. Exploiting the quasi-homogeneity, we may globalize that local assertion.

In order to see that the degree of $\tilde{\mathcal{L}}$ is d^{d-2}, recall that any *homogeneous* complete intersection morphism

$$\phi \colon (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0), \quad a \mapsto b = (\phi_1(a), \ldots, \phi_d(a)),$$

with $\phi_i(a)$ homogeneous of some degree e_i, has degree $e_1 \cdot e_2 \cdots e_d$. Substituting $a_i = \alpha_i^{d-i}$ in $\tilde{\mathcal{L}}^{\mathrm{ext}}$ in new variables α_i we obtain a such a homogeneous map of degree $\prod_{j=0}^{d-1} d \cdot (d - j - 1)$. To arrive at the degree of $\tilde{\mathcal{L}}^{\mathrm{ext}}$, we have to divide by the degree $d!$ of the substitution map. As the degrees of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^{\mathrm{ext}}$ coincide, we are done. □

After investigating the algebraic side, we now come to the preliminary version of our main result for the topological considerations[5].

[5] We shall later establish a more general statement in Proposition 3.7.

Proposition 2.5. *For every number d the following holds.*

i) $\mathcal{L}$ factors through the Tschirnhaus projection as

$$\mathcal{L} = \tilde{\mathcal{L}} \circ \tau;$$

ii) the restriction of $\tilde{\mathcal{L}}$ to the complement

$$\tilde{\mathcal{L}} \colon \mathbb{C}[z]_d^{\mathrm{dep}} \setminus \tilde{\mathcal{L}}^{-1}(\Delta) \to \mathbb{C}[u]_{d-1}^{\mathrm{mon}} \setminus \Delta$$

is a finite topological covering map (of degree d^{d-2}).

Proof. The first statement is easy and follows directly from the T-invariance of $\mathcal{L}$: On one hand, the roots and the critical points of $T_\xi(f)$ are those of f, but translated by $\xi \in \mathbb{C}$. On the other hand, the critical values of $T_\xi(f)$ and f are the same and, hence, so are $\mathcal{L}(T_\xi(f))$ and $\mathcal{L}(f)$.

In order to verify the second claim, we first study the full map $\mathcal{L}$ but restricted to

$$\Omega'_d = \mathbb{C}[z]_d^{\mathrm{mon}} \setminus \mathcal{L}^{-1}(\Delta),$$

the set of monic polynomials f with pairwise distinct critical values $v_i(f)$, $i = 1, \dots, d-1$. Then, necessarily, also the critical points $c_1(f), \dots, c_{d-1}(f)$ are pairwise distinct. Recall from Lemma 1.2 that at any such point $c = (c_1, \dots, c_{d-1})$ the map $\phi = \phi_{d-1} : (c_1, \dots, c_{d-1}) \mapsto \prod_{i=1}^{d-1}(u - c_i)$ is locally invertible. Writing

$$\mathrm{diag} : \mathbb{C}[z]_{\leq d} \times \mathbb{C}^k \to \mathbb{C}^k, \quad (g, (w_1, \dots, w_k)) \mapsto (g(w_1), \dots, g(w_k)),$$

and choosing some local inversion of ϕ_{d-1} we can locally compose $\mathcal{L}$ as

$$\mathcal{L} = \phi_{d-1} \circ \mathrm{diag} \circ \left(\mathrm{id}, \phi_{d-1}^{-1} \circ \partial_z\right). \tag{2.2}$$

Now consider the Jacobian matrix of $\mathcal{L}$: We can write the partial derivative of the n-th component $\mathcal{L}_n$ of $\mathcal{L}$ with respect to the i-th coordinate a_i as

$$\frac{\partial \mathcal{L}_n}{\partial a_i} = \sum_{j=1}^{d-1} \frac{\partial (\phi_{d-1})_n}{\partial v_j} \cdot \left(\frac{\partial\, \mathrm{diag}_j}{\partial g} \cdot \frac{\partial g}{\partial a_i} + \sum_{k=1}^{d-1} \frac{\partial\, \mathrm{diag}_j}{\partial w_k} \cdot \frac{\partial (\phi_{d-1}^{-1} \circ \partial_z)_k}{\partial a_i} \right).$$

By construction,

$$\frac{\partial \operatorname{diag}_j}{\partial w_k}(f,c) = \frac{\partial f}{\partial z}(c_j) = 0$$

for any critical point $c = c(f) = (c_1, \ldots, c_{d-1}) = \phi_{d-1}^{-1}(\partial_z f)$, so that the sum over k vanishes. The other term simplifies to

$$\frac{\partial \operatorname{diag}_j}{\partial g}(f, c(f)) \cdot \frac{\partial f}{\partial a_i}(c_j(f)) = \begin{cases} (c_j(f))^i & \text{if } i < d; \\ 0 & \text{if } i = d \end{cases}$$

for all $j = 1, \ldots, d-1$ and $i = 1, \ldots, d$. Thus, the Jacobian matrix of $\mathcal{L}$ at $f \in \Omega'_d$ reads

$$\operatorname{Jac}(\mathcal{L})(f) = \operatorname{Jac}(\phi_{d-1})(v_1(f), \ldots, v_{d-1}(f)) \cdot V'(c_1(f), \ldots, c_{d-1}(f))$$

where the $v_j(f)$ are the critical values of f and V' is the *Vandermonde matrix* for the critical points $c_i(f)$, extended to a $(d-1) \times d$-matrix by adding one zero column.

Since all critical points and values are pairwise distinct, both matrices have full rank $d-1$ so that $\mathcal{L}$ is a submersion at every $f \in \Omega'_d$. Due to the T-invariance of $\mathcal{L}$, the 1-dimensional kernel of $\operatorname{Jac}(\mathcal{L})(f)$ must then locally be generated by the action of $(\mathbb{C}, +)$ via the translation operator T. Choosing appropriate coordinates of the domain around f we find that also the component of the fiber $\mathcal{L}^{-1}(\{\mathcal{L}(f)\})$ that is passing through f, must in fact be the $(\mathbb{C}, +)$-orbit of f and it meets $\mathbb{C}[z]_d^{\mathrm{dep}}$ transversally in the unique point $\tau(f)$. The second claim of Proposition 2.5 follows straightforwardly. □

Given the previous Proposition, it is natural to ask:

"How do the fibers of $\mathcal{L}$ degenerate as we approach a point in the boundary divisor $\mathcal{L}^{-1}(\Delta)$ of Ω'_d in $\mathbb{C}[z]_d^{\mathrm{mon}}$?"

Again, we can provide some insights from manual computations in low degrees:

Example 2.6. Let $d = 3$ and consider an arbitrary *depressed* polynomial $f = z^3 + a_1 \cdot z + a_0 \in \mathbb{C}[z]_3^{\mathrm{dep}}$. Then $c_{1,2}(f) = \pm\sqrt{\frac{-a_1}{3}}$ and $v_{1,2}(f) = a_0 \mp 2\sqrt{\frac{-a_1}{3}}^3$, so that

$$h = \tilde{\mathcal{L}}(f) = u^2 - 2a_0 \cdot u + \left(a_0^2 + \frac{4}{27}a_1^3\right).$$

Observe that a_0 is uniquely determined by the coefficients of h and $h \in \Delta$ if and only if $a_1 = 0$. The number points in the fiber $\tilde{\mathcal{L}}^{-1}(\{h\})$ over a point $h \notin \Delta$ is $r = r(3) = 3$ corresponding to the three possible choices for a_1. Moreover, the monodromy action of the fundamental group $\mathbb{Z}$ of $\mathbb{C}[u]_2^{\mathrm{mon}} \setminus \Delta$ on the fibers of $\tilde{\mathcal{L}}$ is given by taking the cyclic generator to

$$(a_0, a_1) \mapsto \left(a_0, e^{\frac{2\pi i}{3}} \cdot a_1\right).$$

As an example, suppose $a_0 = -4$ and $a_1 = -2$ so that f is the polynomial from Example 1.1. As a base point $* \in \mathbb{C}[u]_2^{\mathrm{mon}} \setminus \Delta$ we may choose $\mathcal{L}(f) = u^2 + 8u + \frac{2^4 5^2}{3^3}$ and the loop

$$\gamma \colon [0,1] \to \mathbb{C}[u]_2^{\mathrm{mon}} \setminus \Delta, \quad t \mapsto u^2 + 8u + 16 - \frac{32}{27}e^{2\pi i t}$$

as a generator of the fundamental group. Starting from f, this path lifts to the non-closed path

$$\tilde{\gamma} \colon [0,1] \to \mathbb{C}[z]_3^{\mathrm{dep}}, \quad t \mapsto z^3 - 2e^{\frac{2\pi i}{3}t}z - 4$$

which ends at

$$\tilde{\gamma}(1) = z^3 - 2\zeta_3 z - 4$$

with $\zeta_3 = e^{\frac{2\pi i}{3}}$ a third root of unity. Repeating this procedure, we eventually find that the fiber over $*$ is

$$\tilde{\mathcal{L}}^{-1}(\{*\}) = \left\{z^3 - 2\zeta_3^k z - 4 : k = 0, 1, 2\right\}$$

with the monodromy acting by cyclic permutation.

We can apply the above procedure to any depressed polynomial outside $\tilde{\mathcal{L}}^{-1}(\Delta)$. Interestingly, the monodromy action of the one generator of the fundamental group then extends to a map on the whole space which is induced by a symmetry in the complex z-plane:

$$\mathbb{C}[z]_3^{\mathrm{dep}} \to \mathbb{C}[z]_3^{\mathrm{dep}}, \quad z \mapsto e^{\frac{2\pi i}{3}} z.$$

From the explicit form of $\tilde{\mathcal{L}}$ above it is easy to see that the critical values of f are distinct if and only if the critical points are distinct and this is the case whenever $a_1 = 0$ so that f has an inflection point at $z = 0$. But all *monic* polynomials of degree three with an inflection point arise as

$$f = T_\xi(x^3 + c), \quad \xi, c \in \mathbb{C},$$

as can easily be verified by hand. Therefore, $\mathcal{L}^{-1}(\Delta)$ consists *entirely* of *all* those polynomials f with an inflection point, the restriction

$$\tilde{\mathcal{L}}|_{\tilde{\mathcal{L}}^{-1}(\Delta)} \colon \tilde{\mathcal{L}}^{-1}(\Delta) \to \Delta$$

is one-to-one, and $\tilde{\mathcal{L}}$ is branched of degree three over Δ.

3 Flow graphs and monodromy

Given a monic polynomial $f \in \mathbb{C}[z]_d^{\mathrm{mon}}$, there is a natural connection between its flow graph Γ_f and the configuration of its critical values. Namely, let $V = \{v_1, \ldots, v_n\} \subset \mathbb{C}$ be its (pairwise different) critical values, counted without multiplicity for the moment, and let $\star \subset \mathbb{C}$ be the union of the closed real line segments joining the origin with either one of the v_j. Then we find that

$$\Gamma_f \subset f^{-1}(\star)$$

must be lying over the set $\star$.

Example 3.1. For the polynomial $f = z^3 - 2z - 4$ from Example 1.1, the star $\star$ is completely real as the critical values

$$v^{\pm}(f) = -4 \mp 2\sqrt{\frac{2}{3}}^{3}$$

come to lie on the negative real axis. A schematic illustration of this configuration is given in Figure 3.1. Note that the vertical axis is necessarily complex and the roots and critical points of f are not really situated on one real axis.

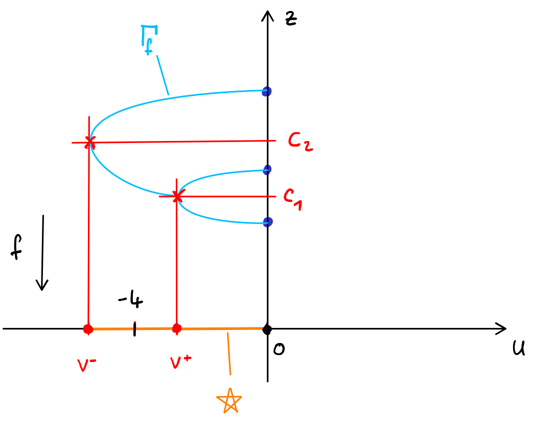

Figure 3.1: The branched covering map $f\colon \Gamma_f \to \star$

Any polynomial f is a finite covering which is branched over its critical values. We therefore have a natural monodromy action on the roots of f by the fundamental group of the complement of the critical values. The fact that f has a multiple root if and only if it has a critical value $v = 0$ suggests that this position is special for either one of the v_j. With a view towards the upcoming discussion, we give the following description of the monodromy.

Let $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{C} \setminus \{0\}$ be a set of pairwise different points $\neq 0$ and let $* \in \mathbb{C}$ be a real number close to $+\infty$. To every $w \in V$ we assign a real path γ_w as follows.

- Start from $*$ in the negative direction along the real axis;
- for every $v \in \mathbb{R}_{>0}$ within the interval $(0, *)$ make a small clockwise detour around v;

- when reaching the origin, make a counterclockwise detour around it until reaching the angle $\arg(w)$;
- follow the real line segment from $0 \in \mathbb{C}$ in the direction of w;
- for every $v \in V$, $v \neq w$ encountered along the way, take another counterclockwise detour around that point;
- when reaching w, encircle it counterclockwise and follow the previous path back to $*$.

We extend this set of paths by one more path γ_0 avoiding all points $v_j \in \mathbb{R}_{>0}$ in the above manner and encircling the origin. Altogether, these paths $\{\gamma_0\} \cup \{\gamma_v : v \in V\}$ provide a set of generators of the fundamental group $\pi_1(\mathbb{C}^* \setminus V, *)$ of the punctured complex plane.

As already indicated in the introductory paragraph, the set V will be the set of critical values of the polynomial f, together with the origin. Enumerate the roots of $f - *$ by $q_1, \dots, q_d$. By choice of $*$, we will assume them to be pairwise different and parallel transport along each one of the paths γ_w provides a monodromy representation

$$\sigma \colon \pi_1(\mathbb{C}^* \setminus V, *) \to \mathfrak{S}_d \tag{3.1}$$

where we identified the group of permutations of the roots with the symmetric group $\mathfrak{S}_d$. We will write $\sigma(w)$ as a shorthand notation for the permutation $\sigma(\gamma_w)$ associated to the critical value w and $\sigma(0)$ for image of the loop γ_0 around the origin.

Let $p_1, \dots, p_m$ be the roots of f. Note that these may be fewer than d since they may have higher multiplicities. We can relate them to the roots q_j of $f - *$ by first following γ_0 and then taking limits of the roots of $f - t$ for real $1 \gg t > 0$ and $t \to 0$. After renumeration we may assume that the first a_1 of the roots q_j converge to p_1, the next a_2 converge to p_2, etc. for some function $a \colon \mathbb{N} \to \mathbb{N}_0$ satisfying $\sum_i a_i = d$.

Lemma 3.2. *In the situation above the following holds for the graph Γ_f:*

i) Whenever a critical point c is also a root p of f, i.e. when the associated critical value $f(c) = 0$ is equal to zero, that root is a multiple root; we will count the point $p = c$ as one vertex of Γ_f.

ii) There is a connecting edge from a critical point c with non-zero critical value $v = f(c)$ to a root p of f if and only if one of the roots q of $f - *$ converging to p appears in a non-trivial cycle of $\sigma(v)$ and for every other critical value w on the real line segment from $0 \in \mathbb{C}$ to v one has that (q) is a full cycle in $\sigma(w)$.

iii) There is a connecting edge from a critical point c to another critical point c' if and only if $v' = f(c')$ lies on the real line segment from 0 to $v = f(c)$, both $\sigma(v)$ and $\sigma(v')$ have a non-trivial cycle with one common root q, and (q) is a full cycle of $\sigma(w)$ for all w on the real line segment between v and v'.

Proof. Suppose $c \in \mathbb{C}$ is a critical point of f with critical value $v = f(c) = 0$. Then $f = g \cdot (x - c)$ and $f' = h \cdot (x - c)$ for some polynomials g and h. But $f' = g' \cdot (x - c) + g = h \cdot (x - c)$ which shows that g is divisible by $(x - c)$. Hence, c is a root of f of multiplicity ≥ 2 and i) is proved.

Let $c \in \mathbb{C}$ be a critical point of f with critical value $v = f(c) \neq 0$ and let $q_1(t), \ldots, q_d(t)$ be the roots of $f - t$ for $t \neq v$ close to v. It is evident from the local normal form of complex analytic functions in one variable that there is precisely one cycle $(q_{j_1}, \ldots, q_{j_k})$ in the monodromy permutation $\sigma(v)$ of v, consisting of those points $q_j(t)$ converging to c. Consider the trajectories of these points as t approaches the origin along a straight real line starting from v.

Suppose w is another critical value of f encountered on that line. If there was a root $q_j(t)$ converging to a critical point over w, the real path swept out by $q_j(t)$ as t varies between v and w would already be an edge of the graph Γ_f by itself. In particular, this trajectory could not be part of an edge between c and any root p or to any other critical point. This shows that if $q_j(t)$ was to contribute to an edge between either c and another critical point c', it would have to converge to both c and c' but must not converge to any critical point over any critical value w in between v and v'. Consequently, $(q_j(t))$ would appear as a full cycle in all monodromy permutations for the latter. A similar argument holds for a root p in place of c'.

To finish the proof, observe that there can never be two $q_j(t)$ converging to both c and c' (or p) as t varies on the real line segment between their critical values v and v': If this was the case, the graph Γ_f would have a cycle and could not be a tree anymore. □

We can use Lemma 3.2 to reconstruct the flow graph Γ_f from the monodromy representation σ from (3.1) of the roots of $f - *$. In fact, even more is true: The whole setup we are considering at this point suggests to consider the flow graph in the context of the well-known Riemann Existence Theorem:

Theorem 3.3. *([8, Proposition 4.9 and Corollary 4.10]) Let Y be a compact Riemann surface, let B be a finite subset of Y, and let $*$ be a base point of $Y \setminus B$. Then there is a 1-1 correspondence of*

- *isomorphism classes of (finite) holomorphic maps $F\colon X \to Y$ of degree d whose branch points lie in B and*
- *group homomorphisms $\rho\colon \pi_1(Y \setminus B, q) \to \mathfrak{S}_d$ with transitive image (up to conjugacy in $\mathfrak{S}_d$).*

In order to apply this theorem, we may compactify the codomain $\mathbb{C} \subset Y := \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and consider the set $B = \{0, v_1, \ldots, v_n, \infty\}$. For any monodromy representation $\rho\colon \pi_1(\mathbb{C} \setminus \{0, v_1, \ldots, v_n\}, *) \to \mathfrak{S}_d$ as above, let $F\colon X_\rho \to \mathbb{P}^1$ be the resulting finite map of compact Riemann surfaces and

$$f = F|_{F^{-1}(\mathbb{C})}\colon F^{-1}(\mathbb{C}) \to \mathbb{C}$$

the restriction of F to the preimage of the complement of $\{\infty\}$. Note that for the same reasons as before, there exists a maximal invariant set Γ_f for the negative gradient flow of $|f|^2$ on $U = F^{-1}(\mathbb{C})$, even though f is only a holomorphic map and not necessarily a polynomial as before. Again, this is a graph Γ whose vertices are the critical points of f and the preimages $p \in f^{-1}(\{0\})$. The graph can be constructed in a purely combinatorial manner from the homomorphism ρ, following the arguments in Lemma 3.2; we may therefore also write $\Gamma = \Gamma_\rho$. Moreover, the open set U retracts onto $\Gamma \subset U$, but Γ is not necessarily a tree anymore. Summarizing, we have the following:

Proposition 3.4. *Let $\rho\colon \pi_1(\mathbb{C} \setminus \{0, v_1, \ldots, v_n\}, *) \to \mathfrak{S}_d$ be a homomorphism with transitive image and $F\colon X \to \mathbb{P}^1$ the corresponding finite map of Riemann surfaces. Then the maximal invariant set $\Gamma \subset X$ is a tree if and only if $X \cong \mathbb{P}^1 = \mathbb{C} \cup \{\infty'\}$, $F^{-1}(\{\infty\}) = \{\infty'\}$, and*

$$f = F|_{X \setminus F^{-1}(\{\infty\})} \colon \mathbb{C} \to \mathbb{C}$$

as above is polynomial.

Proof. We have to show two implications; the second of which is starting with the assumption that $f\colon \mathbb{C} \to \mathbb{C}$ is polynomial and F its natural extension as a map $\mathbb{P}^1 \to \mathbb{P}^1$. In this case, it has already been show by A'Campo [1] that Γ_f is a tree.

For the converse, let $U = X \setminus F^{-1}(\{\infty\})$ be the complement of the fiber over ∞ and $f = F|_U \colon U \to \mathbb{C}$ the restriction of F. The complement of U is a finite set $F^{-1}(\{\infty\}) = \{q_1, \ldots, q_n\} \subset X$ and we can choose local orientation classes for X around these points so that the relative cohomology of the pair (X, U) becomes

$$H^k(X, U) = \begin{cases} 0 & \text{if } 0 \le k < 2; \\ \mathbb{Z}^n & \text{if } k = 2. \end{cases}$$

Now consider the associated long exact sequence:

$$0 \to H^1(X) \to H^1(U) \to H^2(X, U) \to H^2(X) \to 0$$

First of all, U retracts onto the graph Γ which provides us the with the zero at the right hand side. Second, when Γ is a tree, then its first homology group vanishes: $H^1(\Gamma) \cong H^1(U) = 0$. Thus, also $H^1(X) = 0$ and $X \cong \mathbb{P}^1$ must be the Riemann sphere, as there is no other compact Riemann surface of genus zero. Third, we must then also have $H^2(X, U) \cong H^2(X) \cong \mathbb{Z}$, so $\pi^{-1}(\{\infty\}) = \{\infty'\}$ consists of only one point. It follows that $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ restricts to an affine algebraic map

$$\mathbb{P}^1 \setminus \{\infty'\} \cong \mathbb{C} \to \mathbb{C} \cong \mathbb{P}^1 \setminus \{\infty\}, \quad z \mapsto u = f(z)$$

and therefore f is indeed the sought for polynomial. $\square$

3.1 Which graphs appear as flow graphs of polynomials?

By construction, the flow graph Γ_f of a univariate polynomial f is a directed, planar tree on a bipartite set of vertices, the *black ones* given by the roots of f and the *white ones* given by the critical points with non-zero critical values. As in Lemma 3.2, whenever a critical point c has critical value $f(c) = 0$, this point is also a multiple root and will be counted as the latter. Furthermore, we naturally have the following restrictions:

- There are no outgoing edges from black vertices; simply because $|f|^2$ is bounded from below by 0 and no flow lines can escape that bottom line.

- Every white vertex c has at least two outgoing edges. Suppose there was no outgoing edge. The only way this could happen would be if c was a local minimum of $|f|^2$. But locally around c we can write $f(x) = u \cdot (x - c)^k + v$ for $v = f(c)$ the critical value and some non-zero function $u(x)$, and we see that this can not be the case. Furthermore, $k \geq 2$ and hence there are at least two descending flow lines emanating from c.

- Any two incoming edges at a white vertex can not be neighbors (in the local orientation of edges at the vertex). Again, this follows from considerations of the local normal form of f at c as every flow line emanating from c must end up in either another critical point or a root of f and hence will contribute to the graph Γ_f.

Theorem 3.5. *For every planar, directed tree* Γ *on a bipartite set of vertices, satisfying the above criteria there exists a polynomial* $f \in \mathbb{C}[z]$ *such that* $\Gamma \cong \Gamma_f$.

Proof. Let Γ be a planar, directed tree satisfying the above conditions. We need to construct a polynomial f with $\Gamma_f = \Gamma$. To this end, we start with the black vertices, i.e. the roots $p_1, \dots, p_m$ of our sought for polynomial f. To each root p_i we assign the multiplicity ν_i given by the number of incoming edges of Γ at p_i. Let us denote by $e_{i,1}, \dots, e_{i,\nu_i}$ the incoming edges to p_i in some cyclic order and assign to every such edge a "virtual root" $q_j(p_i)$. The monodromy of f at the origin will permute every collection of these q_j cyclically.

Choose one root p and any white vertex c adjacent to p. To c we may associate any random point $v(c)$ which does not lie on the non-negative real axis. Again, the vertex c has incoming and outgoing edges. It is clear from the local normal form that the multiplicity of c as a critical point of f must be equal to the number of outgoing edges (with one of them ending up in p). Let $q_j(c)$ be a cyclic enumeration of those "virtual roots" associated to the outgoing edges of c. For the edge joining p with c we identify the virtual roots associated to the two ends.

We can proceed with this in a straightforward way. If p' was any other black vertex adjacent to c, we identify the corresponding virtual root of p' with the one of c for the joining edge.

If c' was another white vertex adjacent to c via some *outgoing* edge, we identify the virtual root of c with the one virtual root of c' for the next outgoing edge of c' in clockwise order and place the critical value for c' somewhere in the interior of the real line segment between $v(c)$ and the origin.

Similarly, if c' was adjacent to c via some *incoming* edge, we would do the same thing but placing $v(c')$ somewhere on the real ray from the origin to $v(c)$, but beyond $v(c)$ and identifying the roots in the mirrored way. Note that the non-existence of cycles in Γ assures that we can not run into a contradiction here.

Crawling through the whole graph Γ in this way and placing the critical values v with relative arguments in the correct order, we eventually produce a configuration of critical values with assigned monodromy actions on the set of virtual roots. Now the Riemann Existence Theorem 3.3 and Proposition 3.4 provide us with the desired polynomial f. $\square$

3.2 Stratifying the Lyashko-Looijenga map

The discriminant $\Delta \subset \mathbb{C}[u]^{\text{mon}}_{d-1}$ has a natural stratification indexed by the number of pairwise distinct roots of $h \in \mathbb{C}[u]$ with a given multiplicity. This is best understood by considering the following setup.

Let $m \in \mathbb{N}$ be arbitrary and let $X \subset \operatorname{Aut}(\mathbb{C}^m)$ be the subgroup generated by the reflections about the top-dimensional components of the "big diagonal" in $\mathbb{C}^m$:

$$\mathbb{C}^m \to \mathbb{C}^m, \quad p \mapsto q \quad \text{with } q_k = \begin{cases} p_k & \text{if } k \neq i, j; \\ p_j & \text{if } k = i; \\ p_i & \text{if } k = j. \end{cases}$$

It is well known that the quotient $\mathbb{C}^m//X$ can be identified with the set of monic polynomials of degree m and the projection map is merely

$$\phi\colon \mathbb{C}^m \to \mathbb{C}^m//X \cong \mathbb{C}[u]^{\text{mon}}_m, \quad (p_1, \ldots, p_m) \mapsto \prod_{i=1}^{m} (u - p_i),$$

as discussed earlier: The map ϕ takes the big diagonal D to the discriminant Δ and is a finite topological covering away from these sets. Moreover, it is easy to see that the natural strata of the big diagonal map to those of Δ and these restrictions are finite topological covering maps, as well; even though of different multiplicity.

For us, it will be important to also consider the *extended discriminant*

$$\Xi := \Delta \cup H$$

where

$$H = \{h \in \mathbb{C}[u]^{\text{mon}}_{d-1} : h(0) = 0\}$$

is the set of polynomials with a zero root. We refine the natural stratification of $\mathbb{C}[u]$ given by the one of Δ and its complement as

$$\mathbb{C}[u]^{\mathrm{mon}}_{d-1} = \bigcup_{k=0}^{d-1} \bigcup_{a \in A^d_k} \mathscr{S}_{k,a}$$

where

$$A^d_k = \left\{ a\colon \mathbb{N} \to \mathbb{N}_0 : \sum_{i=1}^{\infty} i \cdot a_i = d-k-1 \right\}$$

and $\mathscr{S}_{k,a}$ is the set of polynomials $h(u)$ with

- a root of multiplicity k at the origin;
- $a(i)$ pairwise distinct roots $\neq 0$ of multiplicity i for every $i \in \mathbb{N}$.

We first need to establish that the $\mathscr{S}_{k,a}$ are indeed complex manifolds. To see this, let $h \in \mathscr{S}_{k,a}$ be arbitrary and set $m = \sum_i a_i$, the sum of the roots of h counted without multiplicity. Choose a partition of the set $\{1, 2, \ldots, m\}$ into disjoint sets B_i of cardinality $|B_i| = a_i$ and let $Y \subset X$ be the subgroup generated by those reflections respecting the partition. After renumeration of the roots, we may assume that B_i consists of the roots v_j with indices

$$\sum_{k<i} a_k < j \leq \sum_{k \leq i} a_k.$$

Then we obtain an intermediate quotient

$$\mathbb{C}^m \to \mathbb{C}^m//Y \to \mathbb{C}^m//X$$

which, according to the Chevalley-Shephard-Todd theorem [9], [4], is again smooth. Moreover, both projections are necessarily finite topological covering maps away from the big diagonal D and so are their restrictions to $(\mathbb{C} \setminus \{0\})^m \setminus D$ and its images.

We may define an analytic map $\lambda\colon \mathbb{C}^m//Y \to \mathscr{S}_{0,a}$ via local lifts

$$\begin{tikzcd}
(\mathbb{C} \setminus \{0\})^m \setminus D \arrow[d] \arrow[dr, "\lambda'"] & \\
((\mathbb{C} \setminus \{0\})^m \setminus D)//Y \arrow[r, "\lambda"'] & \mathscr{S}_{0,a}
\end{tikzcd}$$

with

$$\lambda'(v_1,\ldots,v_{a_1},v_{a(1)+1},\ldots,v_{a_1+a_2},\ldots) = \prod_{j=1}^{\infty} \prod_{\sum_{k<j} a_k < i \leq \sum_{k\leq j} a_k} (u-v_i)^j.$$

It is evident that λ is a bijective set theoretic map and, hence, an analytic isomorphism.

Corollary 3.6. *Every stratum $\mathscr{S}_{k,a}$ is a $K(G,1)$ (i.e. an Eilenberg-MacLane space) for G one of Brieskorn's generalized Braid groups.*

Proof. This follows directly from the above in conjunction with [3, Proposition 2]. □

With these preliminaries at hand, we are now in the position to prove the main result of this section.

Proposition 3.7. *The restriction of $\tilde{\mathcal{L}}\colon \mathbb{C}[z]_d^{\mathrm{dep}} \to \mathbb{C}[u]_{d-1}^{\mathrm{mon}}$ to the preimage of any stratum $\mathscr{S}'_{k,a} := \tilde{\mathcal{L}}^{-1}(\mathscr{S}_{k,a})$ is a finite covering map.*

Proof. We already know from Proposition 2.1 that $\tilde{\mathcal{L}}$ is an algebraic, finite, and *branched* covering. Hence, the restriction to $\mathscr{S}_{k,a}$ and its preimage is a topological covering map if and only if the number of preimages $f \in \tilde{\mathcal{L}}^{-1}(\{h\})$ is constant as h varies in $\mathscr{S}_{k,a}$. By virtue of λ' as above, we may locally identify such h with its sets of distinct non-zero roots $\{v_1,\ldots,v_m\}$, partitioned by their multiplicities.

According to Proposition 3.4, the number of preimages of any $h \in \mathscr{S}_{k,a}$ coincides with a certain number of admissible homomorphisms

$$\rho\colon \pi_1(\mathbb{C}\setminus\{0,v_1,\ldots,v_n\},*) \to \mathfrak{S}_d$$

where 0 and the v_i are the roots of h. Note that the multiplicities of the roots, i.e. the branching multiplicities of the maps F, impose further restrictions on the admissible choices of ρ but certainly, this number is finite and locally constant as we vary the v_i. □

Remark 3.8. We have already seen that the degree of the covering is d^{d-2} on the open stratum $\mathscr{S}_{0,(d,0,\dots)}$. However, I do not have a closed formula for the degree of $\tilde{\mathcal{L}}$ on $\mathscr{S}_{k,a}$ in general.

Corollary 3.9. *Every connected component $C \subset \mathscr{S}'_{k,a}$ of the $\mathscr{S}'$ stratification of $\mathbb{C}[z]_d^{\mathrm{mon}}$ is a $K(G, 1)$ for some finite subgroup $G \subset \pi_1(\mathscr{S}_{k,a})$.*

Proof. This follows directly from the long exact sequence in homotopy for the covering $C \to \mathscr{S}_{k,a}$. In particular, this yields a short exact sequence

$$1 \to \pi_1(C) \to \pi_1(\mathscr{S}_{k,a}) \to \pi_0(F_C) \to 1$$

where F_C is the fiber of that covering. We deliberately omitted the base points. □

Remark 3.10. It is, in principle, possible to compute the group G explicitly for every component $C \subset \mathscr{S}'_{k,a}$ as above by purely symbolic computations. To this end, let $f \in C$ be arbitrary and let $\rho = \rho(f)$ be its associated admissible representation in the sense of Lemma 3.2 and Proposition 3.4. According to Corollary 3.9, the fundamental group $\pi_1(\mathscr{S}_{k,a})$ is a generalized Braid group which comes with a standard set of generators. Using the description of ρ, it is not hard to write down the action of these generators on the set of admissible representations. This eventually allows one to find the orbit of f, i.e. its fiber $F_C = \tilde{\mathcal{L}}^{-1}(\{\tilde{\mathcal{L}}(f)\})$ in C, and the homomorphism $\pi_1(\mathscr{S}_{k,a}) \to \pi_0(F_C)$. It would be interesting to see whether these groups $G = \pi_1(C)$ satisfy any additional patterns.

3.3 A common refinement of the $\mathscr{S}'$- and the Γ-stratification

Fixing one connected component $C \subset \mathscr{S}'_{k,a}$ as above, we can observe that not all $f \in C$ have the same graph Γ_f. However, if we let $W \subset \mathscr{S}_{k,a}$ be the set of polynomials h such that all roots of h have pairwise different arguments $\neq 0$, then it is easy to see from Lemma 3.2 that the graph Γ_f is an invariant of every connected component of $\mathcal{L}^{-1}(W)$.

The components of W merely form the top dimensional strata of a real analytic refinement of the $\mathscr{S}$-stratification that we shall now describe. Fix

k and $a\colon \mathbb{N} \to \mathbb{N}_0$ as above and let m be the total number of non-zero roots of $h \in \mathscr{S}_{k,a}$. For any sequence of integers $b : 0 \leq b_0 < b_1 < \cdots < b_r = m$ we let $W_b \subset \mathscr{S}_{k,a}$ be the set of polynomials h such that

- b_0 roots of h have argument equal to $\varphi_0 = 0$;
- $b_i - b_{i-1}$ roots of h have argument $2\pi > \varphi_i > \varphi_{i-1}$.

Note that the W_b's are usually not connected, but have components depending on the decoration of the configuration of roots with the multiplicities.

It is evident that every W_b is a real analytic manifold and so are their preimages $\mathcal{L}^{-1}(W_b)$.

Definition 3.11. We define the strata of the W'-stratification of $\mathbb{C}[z]_d^{\mathrm{mon}}$ to be the connected components of the preimages $\mathcal{L}^{-1}(W_b)$ of the strata W_b of the W-stratification of $\mathbb{C}[u]_{d-1}^{\mathrm{mon}}$.

Using Lemma 3.2 again, it is clear that the graph Γ_f does not change its isomorphism class as f varies within one such component of $\mathcal{L}^{-1}(W_b)$. Thus, we have proved:

Corollary 3.12. *The W'-stratification of $\mathbb{C}[z]_d^{\mathrm{mon}}$ is a refinement of the Γ-stratification.*

Looking more closely at the W'-strata, we find the following.

Proposition 3.13. *Every connected component of $\mathcal{L}^{-1}(W_b)$ is contractible.*

Proof. Fix one component $C \subset W_b \subset \mathscr{S}_{k,a}$ and choose once and for all a valid reference configuration for the roots of some element $h \in C$. For instance, one can choose $r+1$ different real rays emanating from the origin and passing through the $r+1$-st roots of unity. On the k-th such ray one now places the $b_k + j$-th root of h with that argument in distance j from the origin.

Usually, there is no continuous map taking a polynomial h to an enumerated configuration of its roots, but restricted to C, we are –by construction– provided with natural choices of enumerating the roots of h (for instance by ordering them by increasing argument and, subsequently, their distance to the origin) and this enumeration can not change as we vary h in C. Thus we do have a continuous lift from C to the set of root configurations and, similar to the Gram-Schmidt procedure for matrices, we can easily construct a continuous retraction bringing any h' to the standard form h. These retractions can then be lifted to the strata of the W'-stratification of $\mathbb{C}[z]_d^{\mathrm{mon}}$. $\square$

Remark 3.14. The W'-stratification is strictly finer than the Γ-stratification, i.e. there are pairs of adjacent strata M and N, $N \subset \overline{M}$, in W' such that for all $f \in M$ and $g \in N$, the graphs Γ_f and Γ_g are isomorphic. This will be illustrated in the following two examples.

Example 3.15. Let $f = z^4 - z^2 = z^2(z-1)(z+1)$. The real graph of f is shaped like a W with three critical points at $C = \{0, \pm 1/\sqrt{2}\}$, see Figure 3.2. While f posses three distinct critical points, it has only two critical values $V = \{0, -1/4\}$, the second one of which has multiplicity 2.

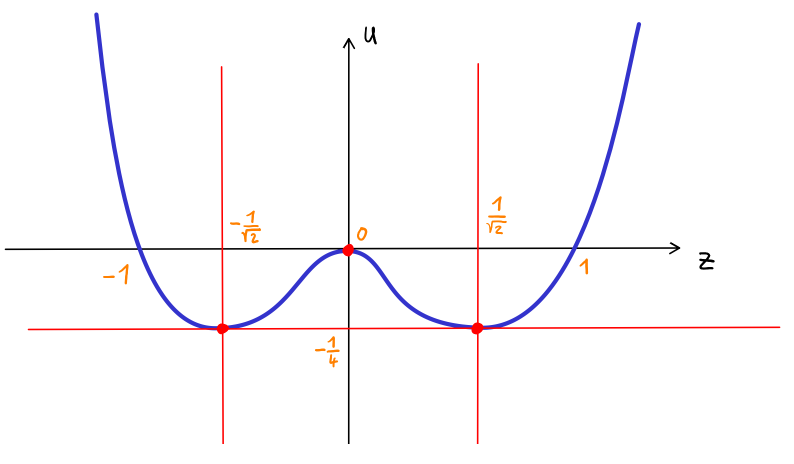

Figure 3.2: The real graph of $f = z^4 - z^2$

Indeed, we find

$$\mathcal{L}(f) = u^3 - \frac{1}{2}u^2 + \frac{1}{16}u = u \cdot \left(u - \frac{1}{4}\right)^2.$$

We infer that this polynomial belongs to the set $\mathscr{S}'_{1,(0,1,0,0,\dots)}$. Its flow graph is rather simple and shown in Figure 3.3.

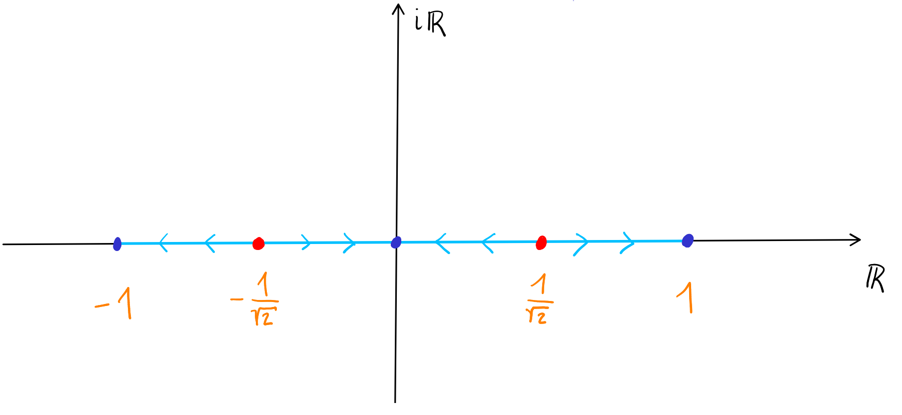

Figure 3.3: The flow graph of $f = z^4 - z^2$

Let us perturb f slightly by some complex parameter ε as

$$f_\varepsilon = z^2 \cdot (z - 1 + \varepsilon) \cdot (z + 1 + \varepsilon).$$

We have

$$h_\varepsilon := \mathcal{L}(f_\varepsilon) = u^3 + \left(\frac{1}{2} + \frac{5}{4}\varepsilon^2 - \frac{1}{16}\varepsilon^4\right) u^2 + \frac{1}{16}\left(1 - \varepsilon^2\right)^3 u,$$

a calculation that is certainly possible, but rather tedious to carry out by hand. Given the polynomial description of $\mathcal{L}$ from Proposition 2.1, we can merely substitute the expressions for the coefficients of f_ε to arrive at h_ε.

The roots of h_ε can be computed using the $p - q$-formula:

$$V_\varepsilon = \{v_0, v_{\pm\varepsilon}\} = \left\{0, -\frac{1}{4} - \frac{5}{8}\varepsilon^2 + \frac{1}{32}\varepsilon^4 \pm \frac{\varepsilon}{\sqrt{2}}\sqrt{\left(1 + \frac{\varepsilon}{4}\right)^3}\right\}.$$

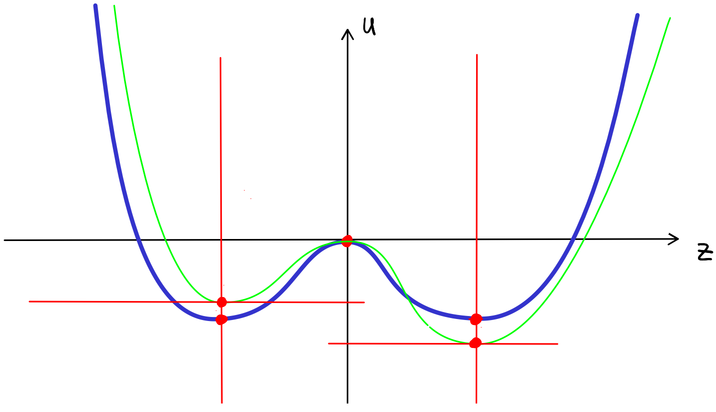

Figure 3.4: The real graph of f and its perturbation f_ε

Since the expression for the root is locally solvable around $\varepsilon = 0$, the dominating term in the expansion of the non-zero critical values of f_ε around $\varepsilon = 0$ is linear in $\pm\varepsilon$.

Note that for $\varepsilon \neq 0$ the polynomial f_ε has *three* distinct critical values and therefore belongs to $\mathscr{S}'_{0,(2,0,0,\dots)}$, i.e. we change strata in the $\mathscr{S}'$-stratification as we pass from $\varepsilon = 0$ to $\varepsilon \neq 0$.

Necessarily, this implies that doing so, we also change the W'-stratum. While $\varepsilon \in \mathbb{R}$ is real, also the two non-zero critical values are real and have the same argument. Thus, the root configuration of h_ε is special and for $\varepsilon \neq 0$ we find ourselves – at least locally around f – in two different components of the W'-stratification: one for $\varepsilon > 0$ and one for $\varepsilon < 0$.

For $\varepsilon \in \mathbb{C}\backslash\mathbb{R}$, on the other hand, it is easy to see from the above expressions that the non-zero critical values of f_ε have different arguments and, hence, f_ε comes to lie in yet two other components of the W'-stratification, depending on the half space of $\mathbb{C} \setminus \mathbb{R}$ in which ε is located.

While there are two $\mathscr{S}'$-strata and five W'-strata involved in the local variation of ε, the flow graphs of the f_ε are all the same. One way to see this is to observe that the pairs of roots of $f_\varepsilon - *$ that are being permuted as we follow the prescribed paths encircling either one of the two points

$v_{\pm\varepsilon}$, are disjoint – for the different critical values in case $\varepsilon \neq 0$, as well as for the single critical value for $\varepsilon = 0$.

Example 3.16. Let $f = z^4 - \frac{4}{3}z^3 - 16$. The critical points of f are $C = \{0, 1\}$ and the associated critical values are $V = \{v_0, v_1\} = \{-16, -16\frac{1}{3}\}$. Indeed, we have

$$h = \mathcal{L}(f) = (u+16)^2 \cdot \left(u + 16\frac{1}{3}\right) = u^3 + \frac{145}{3}u^2 + \frac{2336}{3}u + \frac{12544}{3}.$$

Clearly, $f \in \mathscr{S}'_{0,(1,1,0,0,0\dots)}$. As in the previous example, let us consider f as a special element in a family with a complex parameter ε,

$$f_\varepsilon = z^4 - \frac{4\varepsilon}{3}z^3 - 16,$$

only that this time the default value is $\varepsilon = 1$.

On one hand, for $\varepsilon \to 0$ we approach the deeper stratum $\mathscr{S}'_{3,(0,0,\dots)}$. On the other hand, if we set $\varepsilon = \exp(i\varphi)$ for $\varphi \in [0, 2\pi)$, we are roaming within one and the same $\mathscr{S}'$-stratum.

Consider the movement of the critical values: $v_0 = -16$ is independent of the value of ε, but

$$v_1 = v_1(\varepsilon) = -16 - \frac{1}{3}\varepsilon^4.$$

That means for one move of ε around the origin, the configuration of critical values repeats itself four times!

In this example we do encounter relevant wall-crossings in the W'-stratification where the flow graph of f_ε changes. These occur around the parameters $\varepsilon = \exp\left(\frac{2\pi i k}{8}\right)$, $k \in \mathbb{Z}$, when $\varepsilon = \exp(i\varphi)$ is an eighth root of unity and are illustrated in Figure 3.5.

4 Connections with Galois theory

The motivation for this last section comes from the following observation already made in the proof of Proposition 3.7: The monodromy action of the fundamental group on the complements of the critical values

$$\sigma\colon \pi_1(\mathbb{C} \setminus \{0, v_1, \dots, v_n\}, *) \to \mathfrak{S}_d$$

on the roots of a given *monic* polynomial $f \in \mathbb{C}[z]_d^{\text{mon}}$ is completely determined by the connected component C of the stratum of the $\mathscr{S}'$-stratification of $\mathbb{C}[z]_d^{\text{mon}}$ in which f comes to lie.

This holds, because according to Proposition 3.7 the restriction $\mathcal{L}\colon C \to \mathcal{L}(C)$ is a topological covering map. For $h := \mathcal{L}(f)$ we have chosen a particular set of generators for the fundamental group in the beginning of Section 3 (Note that γ_0 might be trivial, depending on whether or not 0 is a critical value of f). Moving h within its $\mathscr{S}$-stratum requires to drag the representatives of our generators for the fundamental group along. For our preferred choice of generators, we extract the following observation:

Lemma 4.1. *Let $f \in \mathbb{C}[z]_d^{\text{mon}}$ be arbitrary, C its component of the $\mathscr{S}'$-stratification, and $\mathcal{L}(C)$ the stratum of its image $h = \mathcal{L}(f) \in \mathbb{C}[u]_{d-1}^{\text{mon}}$. Then for the choice of generators for the fundamental group*

$$\pi_1(\mathbb{C} \setminus \{0, v_1, \dots, v_n\}, *)$$

of the complement of the roots of h in $\mathbb{C}^*$ from Section 3, a real wall crossing in the W-refinement of C corresponds to a change of generators by conjugation; i.e. for every such wall crossing there exists a finite set of generators γ_j which is replaced by their conjugation $\tau_j \cdot \gamma_j \cdot \tau_j^{-1}$ with τ_j the product of other generators $\neq \gamma_j$.

Instead of giving a formal proof of the above statement, which would require us to introduce even more technical notations, we shall simply illustrate the lemma in a simple example and leave the general case to the reader.

Example 4.2. Let us consider again the polynomial $f = z^3 - 2z - 4$ from Example 3.1. Its critical values are

$$v_1(f) = -4 - 2\sqrt{\frac{2}{3}}^3, \quad v_2(f) = -4 + 2\sqrt{\frac{2}{3}}^3.$$

and its image $h = \mathcal{L}(f) \in \mathbb{C}[u]_2^{\text{mon}}$ comes to lie in the stratum $\mathscr{S}_{0,(2,0,\dots)}$ of the $\mathscr{S}$-stratification. If we label the roots of f as

$$p_1 = 2, \quad p_2 = -1 + i, \quad p_3 = -1 - i,$$

then it is easy to see that the monodromy representation for f takes the generators γ_0, γ_1 and γ_2 (i.e. the loops around the origin, v_1, and v_2, respectively) to the permutations

$$\sigma(\gamma_0) = (\,), \quad \sigma(\gamma_1) = (1,2), \quad \sigma(\gamma_2) = (2,3)$$

in the usual cycle notation in $\mathfrak{S}_3$.

On the W-stratification, we are situated in a lower dimensional substratum of $\mathscr{S}_{0,(2,0,\dots)}$, namely in $W_{(0,2)}$. Now, using the considerations from Example 2.6, we set

$$h_t = u^2 + 8u + 16 - \frac{32}{27}e^{2\pi i t}$$

and its lift

$$f_t = z^3 - 2e^{\frac{2\pi i}{3}t}z - 4$$

for *real valued* $t \in \mathbb{R}$, $1 \gg |t|$. Since the roots of h_t are

$$v_{1,2}(t) = -4 \mp \sqrt{\frac{32}{27}}e^{\frac{2\pi i}{2}t},$$

which have different arguments for $t \neq 0$, we find h_t to be in a component of $W_{(0,1,1)}$ in these cases. Thus, $t \to 0$ is a real wall crossing in the W-refinement of $\mathscr{S}_{0,(2,0,\dots)}$.

Let us fix small values $t^+ > 0$ and $t^- < 0$ close to 0. Then we obtain new paths $\gamma_j^{\pm}$ for each case and $j = 0,1,2$ from the discussion in Section 3. It is easy to see that, by construction,

$$\sigma(\gamma_0^-) = (\,), \quad \sigma(\gamma_1^-) = \sigma(\gamma_1) = (1,2), \quad \sigma(\gamma_2^-) = \sigma(\gamma_2) = (2,3)$$

coincides with the monodromy representation for $t = 0$. But for t^+ we find a different situation. Here

$$\sigma(\gamma_0^+) = (\,), \quad \sigma(\gamma_1^+) = (1,3), \quad \sigma(\gamma_2^+) = \sigma(\gamma_2) = (2,3).$$

Why this is the case, can be infered from Figure 4.1: If we had been using some parallel transport to drag along γ_2^- as we were moving with t from

t^- to t^+, we would necessarily have ended up with a path in the homotopy class of $\tilde{\gamma}_1^-$ in the lower right picture. But the homotopy class of the latter is equal to

$$[\tilde{\gamma}_1^-] = [\gamma_2^+ \cdot \gamma_1^+ \cdot \left(\gamma_2^+\right)^{-1}],$$

the conjugation of γ_1^+ by γ_2^+. Therefore, also

$$\begin{aligned} \sigma(\gamma_1^+) &= \sigma\left(\left(\gamma_2^+\right)^{-1} \cdot \tilde{\gamma}_1^- \cdot \gamma_2^+\right) \\ &= \sigma\left(\left(\gamma_2^-\right)^{-1} \cdot \gamma_1^- \cdot \gamma_2^-\right) \\ &= (1,2) \circ (2,3) \circ (1,2) = (1,3). \end{aligned}$$

Note that for $t \neq 0$, the vertices involved in the transpositions for a critical point c are always precisely the vertices adjacent to c in the flow graph Γ_f.

The particular interest in the monodromy action for $f \in \mathbb{C}[z]_d^{\mathrm{mon}}$ stems from the fact that its image in $\mathfrak{S}_d$ coincides with the *Galois group* of the field extension $\mathbb{C}(z)/\mathbb{C}(f)$. Here, we denote by $\mathbb{C}(z) = \mathrm{Quot}(\mathbb{C}[z])$ the fraction field of the polynomial ring in z. Since $f \in \mathbb{C}[z]$ satisfies no algebraic relation over $\mathbb{C}$, the subring $\mathbb{C}[f] \cong \mathbb{C}[u]$ is again isomorphic to a free polynomial ring in a new variable u and has a field of fractions for which we will write $\mathbb{C}(u) \cong \mathbb{C}(f) \subset \mathbb{C}(z)$. Note that z itself is indeed algebraic over $\mathbb{C}(u)$ with minimum polynomial

$$P(X) = f(X) - u \in \mathbb{C}(u)[X]$$

and, hence, the whole field extension $\mathbb{C}(z)/\mathbb{C}(u)$ is algebraic. For a more detailed discussion and proof of the introductory statement, see e.g. [10]. Now the preceeding discussion yields the following very concrete application of our topological considerations:

Corollary 4.3. *For every degree $d \in \mathbb{N}$ there exists a finite field extension $K/\mathbb{Q}$ and a finite number of polynomials*

$$P_j(a_0, a_1, \ldots, a_{d-1}) \in K[a_0, a_2, \ldots, a_{d-1}], \quad j = 1, \ldots, N$$

in the coefficients of $f \in \mathbb{C}[z]_d^{\mathrm{mon}}$*, a finite set* S_d *of subgroups of* $\mathfrak{S}_d$*, and a surjective map*

$$G : \{0,1\}^N \to S_d$$

such that for an arbitrary $f \in \mathbb{C}[z]_d^{\mathrm{mon}}$ *and its vector*

$$c = (c_1, \ldots, c_N), \quad c_j = \begin{cases} 0 & \text{if } P_j(f) = 0; \\ 1 & \text{otherwise} \end{cases}$$

the output $G(c) \in \mathfrak{S}_d$ *is the Galois group of the field extension* $\mathbb{C}(z)/\mathbb{C}(f)$*.*

This result is similar in spirit to the "parametric factorization" as, for instance, in [2]. Roughly speaking, for a polynomial

$$F \in \mathbb{Q}[\lambda_1, \ldots, \lambda_r][z_1, \ldots, z_n]$$

in z with coefficients c_i parametrized by the λ_k, this asks for the loci in the λ-plane where F has admits a non-trivial factorization. In the rather special case where

$$F = F(z, u) = z^d + c_{d-1}(\lambda) z^{d-1} + \cdots + c_1(\lambda) z + c_0(\lambda) - u$$

we could read off this locus as the preimage of the strata with non-transitive monodromy representation under the map given by the c_i. However, since the flow graph is always connected, we do not expect such strata to occur.

Remark 4.4. One word on how the field extension $K/\mathbb{Q}$ comes into play: Obviously, the P_j are the defining equations for the strata of the $\mathscr{S}'$ stratification. Since the Galois group of f depends on the *topological* component of the stratum in which f lies, it is a priori not sufficient to simply consider $\mathcal{L}^{-1}(\mathscr{S})$, but we really need to compute a decomposition of the latter. This can be done algebraically via primary decomposition, but we might need to extend the field of coefficients for that. We will see in Example 4.5 below, that this is not necessarily the case, though.

Note that, whenever f is defined over a finite extension of $\mathbb{Q}$, a precomputed table of the polynomials P_j and the map G in Corollary 4.3 can be used to quickly determine the Galois group of f by simply evaluating the P_j on the coefficients of f. To finish this article, we will illustrate this in one last simple example.

Example 4.5. Let $d = 4$. The open stratum $\mathscr{S}_{0,(3,0,\dots)} = \mathcal{L}^{-1}(\mathbb{C}[u]_3^{\mathrm{mon}} \setminus \Xi)$ of the $\mathscr{S}$-stratification in $\mathbb{C}[u]_3^{\mathrm{mon}}$ consists of polynomials h with pairwise distinct roots, such as, for instance

$$h = u^3 + \frac{18731}{4096} \cdot u^2 + \frac{8603}{2048} \cdot u - \frac{2089}{1024}.$$

One of its preimages under $\mathcal{L}$ is

$$f = z^4 - \frac{3}{2} \cdot z^3 - \frac{5}{2} \cdot z^2 + 3 \cdot z = \left(z + \frac{3}{2}\right) \cdot z \cdot (z-1) \cdot (z-2)$$

which has four rational roots

$$p_1 = -\frac{3}{2}, \quad p_2 = 0, \quad p_3 = 1, \quad p_4 = 2$$

and three real critical values c_1, c_2, c_3 in between either two consecutive roots. Hence, the flow graph Γ_f is particularly simple and it is easy to see that the monodromy representation is given by

$$\sigma(\gamma_1) = (1,2), \quad \sigma(\gamma_2) = (2,3), \quad \sigma(\gamma_3) = (3,4).$$

In particular, these elements generate the full symmetric group $\mathfrak{S}_4$.

The equation defining $\Xi \subset \mathbb{C}[u]_3^{\mathrm{mon}}$ in terms of the coefficients of $h = u^3 + b_2 u^2 + b_1 u + b_0$ is

$$H = b_0 \cdot \left(b_2^2 \cdot b_1^2 - 4b_1^3 - 4b_2^3 \cdot b_0 - 27b_0^2 + 18b_0 \cdot b_1 \cdot b_2\right)$$

which pulls back along $\mathcal{L}$ to

$$\begin{aligned}\tilde{H} = \mathcal{L}^*(H) &= -\frac{1}{256} \cdot (256a_0^3 - 192a_0^2a_1a_3 - 128a_0^2a_2^2 + 144a_0^2a_2a_3^2 \\ &\quad -27a_0^2a_3^4 + 144a_0a_1^2a_2 - 6a_0a_1^2a_3^2 - 80a_0a_1a_2^2a_3 \\ &\quad +18a_0a_1a_2a_3^3 + 16a_0a_2^4 - 4a_0a_2^3a_3^2 - 27a_1^4 + 18a_1^3a_2a_3 \\ &\quad -4a_1^3a_3^3 - 4a_1^2a_2^3 + a_1^2a_2^2a_3^2) \cdot \\ &\quad \frac{-1}{268435456} \cdot \left(108a_1^2 - 108a_1a_2a_3 + 27a_1a_3^3 + 32a_2^3 - 9a_2^2a_3^2\right)^3 \\ &\quad \cdot \left(8a_1 - 4a_2a_3 + a_3^3\right)^2 \\ &=: -\frac{1}{256} \cdot \tilde{H}_0 \cdot \frac{-1}{268435456} \cdot \tilde{H}_1^3 \cdot \tilde{H}_2^2\end{aligned}$$

The reader may verify that indeed

$$\tilde{H}(f) = \frac{2268173829278781703125}{288230376151711744} \neq 0$$

so that f does belong to $\mathscr{S}'_{0,(3,0,\dots)}$. Note that $\tilde{H}_0 = \mathcal{L}^*(b_0)$ is again merely $\frac{-1}{256}$ times the discriminant polynomial in degree 4. The remaining factorization involving $\tilde{H}_1^3$ and $\tilde{H}_2^2$ is more surprising and will play an important role below.

The next lower dimensional stratum of interest is $\mathscr{S}_{0,(1,1,0,\dots)}$ where we find polynomials

$$h = (u - v_1)^2 \cdot (u - v_2)$$

with one double root v_1 and one single root v_2, with $v_1 \neq v_2$ and $v_1, v_2 \neq 0$. The coefficients of such polynomials

$$h = u^3 + b_2u^2 + b_1u + b_0 = u^3 + (-2v_1 - v_2)u^2 + (v_1^2 + 2v_1v_2)u - v_1^2v_2$$

satisfy a relation

$$-4b_2^3b_0 + b_2^2b_1^2 + 18b_2b_1b_0 - 4b_1^3 - 27b_0^2$$

which is nothing but the usual discriminant equation. Those h for which $v_1 = v_2$ are annihilated by

$$b_1^2 - 3b_2b_0, \quad b_2b_1 - 9b_0, \quad b_2^2 - 3b_1$$

and thus for h to be in $\mathscr{S}_{0,(1,1,0,\dots)}$ at least one of the above polynomials and b_0 must be nonzero.

Now $\mathscr{S}'_{0,(1,1,0,\dots)} = \mathcal{L}^{-1}(\mathscr{S}_{0,(1,1,0,\dots)})$ decomposes into two components C_1 and C_2 corresponding to the factors $\tilde{H}_1$ and $\tilde{H}_2$ above. The factor $\tilde{H}_0$ does not play a role since it describes the strata in the preimages of $\mathscr{S}_{k,(\dots)}$ for $k > 0$. From what we have seen so far, it is easy to construct two candidates for polynomials in either one of the C_j. First consider

$$f_2 = z^4 - z^2 - 1,$$

a shifted version of the polynomial in Example 3.15. We immediately deduce from the flow graph of f_2 that the monodromy representation for f takes two relevant generators of the fundamental group to the permutations

$$(1,2)(3,4) \quad \text{and} \quad (2,3)$$

which generate a subgroup of $\mathfrak{S}_4$ of order 6. This subgroup is itself isomorphic to $\mathfrak{S}_3$. Indeed, we find that

$$\tilde{H}_1(f_2) = -32, \quad \tilde{H}_2(f_2) = 0$$

so that $f_2 \in C_2$. Hence C_2 is the stratum in the $\mathscr{S}'$-stratification in which the polynomials have precisely two distinct critical points with the *same* critical value $\neq 0$ and one other critical point with another critical value $\neq 0$.

As a candidate for the other component, consider

$$f_3 = z^4 - \frac{4}{3}z^3 - 16$$

and its perturbations from Example 3.16. For an appropriate enumeration of the roots, the two relevant generators of the fundamental group are taken to the cycles

$$(1,2,3) \quad \text{and} \quad (3,4)$$

in $\mathfrak{S}_4$. Note that these again generate the full symmetric group. Evaluating on f_3 yields

$$\tilde{H}_1(f_3) = 0, \quad \tilde{H}_2(f_3) = -\frac{64}{27}$$

so that indeed f_3 comes to lie in the other component C_1 which is hence characterized by the fact that their polynomials have only *two* critical points with distinct critical values $\neq 0$.

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References

[1] N. A'Campo, Flowbox decomposition for gradients of univariate polynomials, *preprint* (2021).

[2] A. Ayad, On factoring parametric multivariate polynomials, *Advances in Applied Mathematics* **45**(2010), 607–623.

[3] E. V. Brieskorn, Sur les groupes de tresses, *Séminaire N. Bourbaki* **401**(1973), 21–44.

[4] C. Chevalley, Invariants of finite groups generated by reflections *American Journal of Mathematics* **77**(1955), No. 4, 778–782.

[5] E. Looijenga, The Complement of the Bifurcation Variety of a Simple Singularity, *Inventiones math.* **23**(1974) 105–116.

[6] O. V. Lyashko, The geometry of bifurcation diagrams, *Uspekhi Mat. Nauk.* **34(3)**(1979), No. 207, 205–206.

[7] O. V. Lyashko, Geometry of bifurcation diagrams, *Itogi Nauki i Tekhniki* **22**(1983), 94–129.

[8] R. Miranda, *Algebraic curves and Riemann surfaces*, Graduate Studies in Mathematics, Vol. 5, American Mathematical Society, 1995.

[9] G. C. Shephard and J. A. Todd, Finite unitary reflection groups, *Canadian Journal of Mathematics* **6**(1954), 274–304.

[10] T. Szamuely, *Galois Groups and Fundamental Groups* Cambridge University Press, 2021.

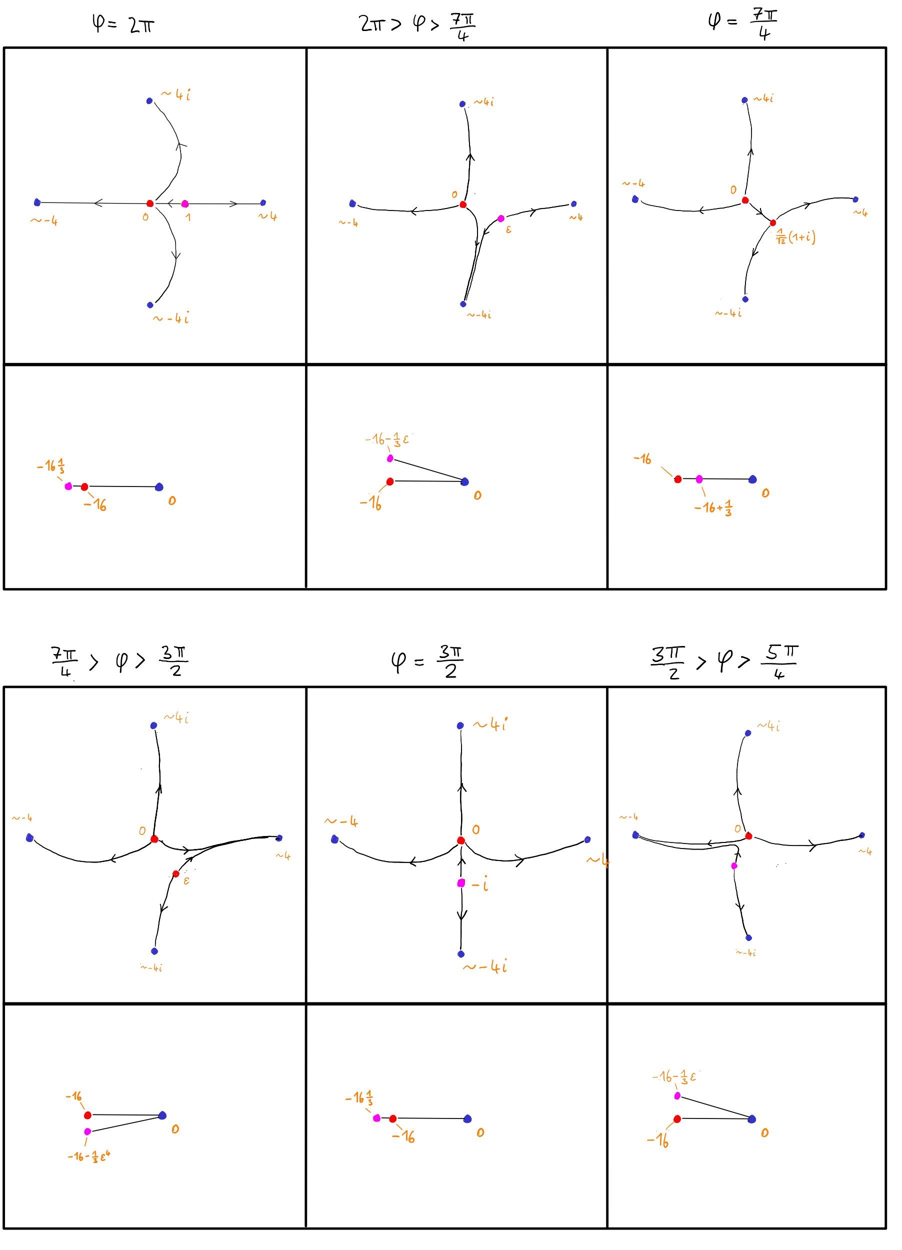

Figure 3.5: Wall crossings in the W'-stratification and associated graph transitions for varying φ; the top row shows the positions of the roots and critical points of f_ε together with its flow graph while the row below depicts the associated configuration of critical values.

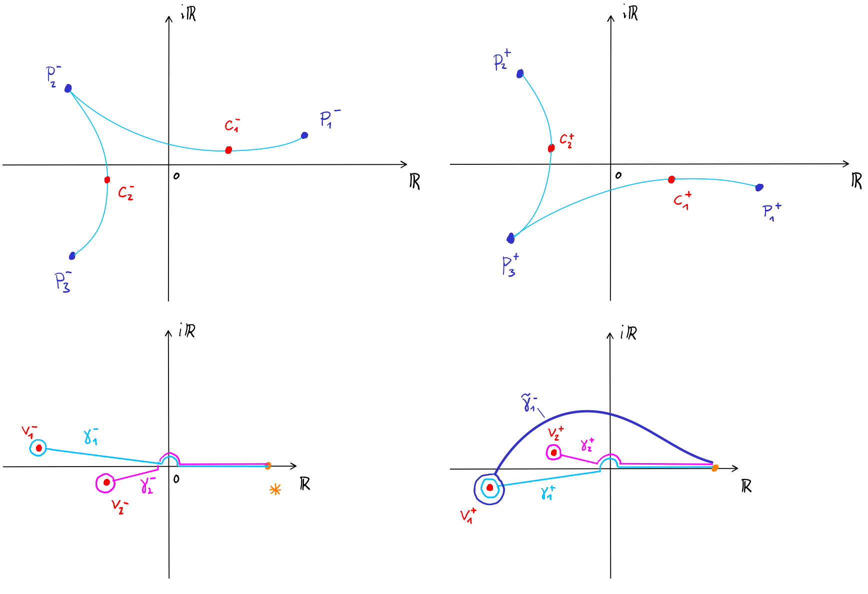

Figure 4.1: Real wall crossing for $t \to 0$; the upper row shows the positions of the roots and critical points of $f_\pm$ in the z-plane while the lower row depicts the corresponding critical values in the u-plane