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# Semiglobal solvability for a class of first order operators

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**Abstract.** In this work, we deal with solvability near the characteristic set of equations in the form $Lu = pu + f$, where the operator $L = \partial_t + (x^n a(x) + ixb(x))\partial_x$ is defined on $\Omega = \mathbb{R} \times S^1$, $a$ and $b$ are real-valued smooth functions on $\mathbb{R}$, with $b(0) \neq 0$, $n \in \mathbb{N}$, and $p, f \in C^\infty(\Omega)$. We show that, for fixed $k \geq 1$, and given $p$ and $f$ belonging to convinient subspaces of $C^\infty(\Omega)$ of finite codimension (depending on $k$), there is $u \in C^k(\Omega)$ solution to the equation $Lu = pu + f$ in a neighborhood of the characteristic set.

**Keywords:** condition (P), characteristic set, periodic solutions.

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# 1 Introduction

Let $\mathcal{L}$ be a non singular smooth complex vector field defined on a smooth $m$-dimensional manifold $\Omega$. For a fixed function $p \in C^\infty(\Omega)$ consider the equation

$$\mathcal{L}u = pu + f, \tag{1.1}$$

where $f \in C^\infty(\Omega)$.

It is well-known that the *Nirenberg-Treves* condition $(\mathcal{P})$ characterizes the local solvability of (1.1), see [1] and [21]. Recalling that local solvability for $\mathcal{L}$ in $\Omega$ means that: for every $x \in \Omega$ and $f \in C^\infty(\Omega)$, there is $u \in \mathcal{D}'(\Omega)$ such that $\mathcal{L}u = pu + f$ in some neighborhood of $x$ in $\Omega$.

The solvability of the equation (1.1) in a neighborhood of a compact subset $K$ of $\Omega$ remains an interesting problem with non-intuitive answers, as evidenced by references [5, 6, 8, 9, 11, 12, 13, 14], and many others. For others related papers see [2, 3, 4, 7, 19, 20] and references therein.

Let $K$ be a compact set of $\Omega$, we say that $P = \mathcal{L} - p$ is solvable at the compact $K$ if for each $f$ in a subspace of finite codimension of $C^\infty(\Omega)$ there is $u \in \mathcal{D}'(\Omega)$ such that $Pu = f$ in a neighborhood of $K$.

In [17], or [15], Hörmander proved that condition $(\mathcal{P})$ is necessary for solvability of $P$ at $K$. If besides condition $(\mathcal{P})$, the following geometric condition is satisfied:

(GC) every characteristic point of $P$ over $K$ lies on a compact interval of a bicharacteristic of $\Re(\ell q)$, on which $q \neq 0$, with no characteristic endpoint over $K$, where $\ell(x, \xi)$ denotes the principal symbol of $P$ and $q$ is a smooth function on $T^*\Omega \setminus \{0\}$

then $\mathcal{L} - p$ is solvable at $K$ and the solutions to (1.1) can be found in $C^\infty(K)$.

Recall that a bicharacteristic of $\ell q$ is an integral curve of the Hamilton field

$$H_{\ell q} = \frac{\partial(\ell q)}{\partial \tau}\frac{\partial}{\partial t} - \frac{\partial(\ell q)}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial(\ell q)}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial(\ell q)}{\partial x}\frac{\partial}{\partial \xi},$$

where $\ell q = 0$ (see [17]).

Let us describe the problem that we plan to investigate in this manuscript. First, consider $\Omega = \mathbb{R} \times S^1$, and let

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0, \tag{1.2}$$

be a complex vector field defined on $\Omega$, where $a$ and $b$ are real-valued smooth functions in $\mathbb{R}$.

Assume that $(a+ib)(0) = 0$ and $b(x) \neq 0$ if $x \neq 0$. Hence, $\Sigma = \{0\} \times S^1$ is the set where $L$ fails to be elliptic and, moreover, the operator $L$ is of infinity type along $\Sigma$.

We recall that a point $(x_0, t_0) \in \Sigma$ is said to be of finite type $\nu$ ($\nu \in \mathbb{Z}_+$) if there exists a Lie bracket of $L$ and $\overline{L}$ of length $\nu$ which is nonzero at $(x_0, t_0)$. When $(x_0, t_0) \in \Sigma$ is not of finite type, we say that this point is of infinite type.

For fixed $p \in C^\infty(\Omega)$, we are interested in the solvability at $\Sigma$ of

$$P = L - p,$$

where $L$ is given by (1.2).

Under our assumptions the operator $P$ satisfies condition $(\mathcal{P})$ (since the function $b$ does not change sign on any integral curve of $\partial/\partial t + a(x)\partial/\partial x$ (see [16])). On the other hand, the geometric condition (GC) is not satisfied for $K = \Sigma$. As a consequence, the results from [17] cannot be used to determine its solvability at $\Sigma$. Indeed, the characteristic set of $P$ is

$$C(P) = \{(x, t, \xi, \tau) \in \Omega \times (\mathbb{R}^2 \setminus \{(0, 0)\}) \,:\, \ell(x, t, \xi, \tau) = 0\},$$

where $\ell(x, t, \xi, \tau) = \tau + a(x)\xi + ib(x)\xi$; hence,

$$C(P) = \{(0, t, \xi, 0) \,:\, t \in S^1 \text{ and } \xi \in \mathbb{R} \setminus \{0\}\}.$$

It is easy to verify that a bicharacteristic of $\ell q$ passing through a point $(0, t_0, \xi_0, 0) \in C(P)$ has the form $\gamma(s) = (0, t(s), \xi(s), 0)$, $s \in \mathbb{R}$; hence, $\gamma(s) \in C(P)$ for all $s \in \mathbb{R}$.

In this paper, we will assume that $b$ vanishes of order 1 at $x = 0$. Therefore, in some neighborhood $(-\epsilon, \epsilon) \times S^1$ of $\Sigma$ we can write

$$(a + ib)(x) = x^n a_0(x) + ixb_0(x),$$

where $n \geq 1$, $a_0$ and $b_0$ are real-valued smooth functions in $(-\epsilon, \epsilon)$, and $b_0(x) \neq 0$ for all $x \in (-\epsilon, \epsilon)$.

It follows from [18] that

$$\lambda = b_0(0) - ia_0(0) \tag{1.3}$$

is related to an invariant of $L$.

Let us assume that $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. For each fixed $k \in \mathbb{Z}_+$, it follows from [10] and [18] that in new coordinates, via a $C^k$-diffeomorphism, $L$ is a multiple of the operator

$$T_\lambda = \frac{\partial}{\partial t} - i\lambda x \frac{\partial}{\partial x};$$

moreover, as showed in [18], for all $f$ belonging to a subspace of finite codimension of $C^\infty(\Omega)$ the equation $T_\lambda u = pu + f$ has a $C^k$ solution in a neighborhood of $\Sigma$.

It is worth mentioning that in [5] was showed that there is $f \in C^\infty(\Omega)$ such that the equation $T_\lambda u = f$ does not have $C^\infty$ solution.

The remainder case to be considered is $\lambda \in \mathbb{Q}$. In this case we will present sufficient conditions on $f$ and $p$ to obtain $C^k$ solutions to the equation $Lu = pu + f$ in a neighborhood of $\Sigma$.

We stressed that in the case $\lambda \in \mathbb{Q}$ our class of vector fields can not be normalized to $T_\lambda$ by any $C^k$ diffeomorphism, $k \geq 1$.

Our arguments are motived by those given in [18]. Where one of the key points in finding $C^k$ solutions to $T_\lambda u = pu + f$ is that for $\lambda \in \mathbb{C} \setminus \mathbb{Q}$ the equation $T_\lambda u = p - p(0,t)$ has $C^k$ solutions, for any $p \in C^\infty(\Omega)$. In our situation where $\lambda \in \mathbb{Q}$ we need compatibility conditions on $p - p(0,t)$.

## 2 Results

Let $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, and let

$$L = \partial/\partial t + (x^n a_0(x) + ixb_0(x))\partial/\partial x, \quad n \geq 2, \tag{2.1}$$

be a complex vector field defined on $\Omega_\epsilon$, where $a_0$ and $b_0$ are real-valued smooth functions in $(-\epsilon, \epsilon)$. Assume that $b_0(x) \neq 0$, for all $x \in (-\epsilon, \epsilon)$, and $b_0(0) \in \mathbb{Q}$. Without loss of generality we may assume that $b_0(0) > 0$.

The following result was proved in [13]:

**Theorem 2.1.** *Let $L$ be given by (2.1). Let $\mathsf{p}$ and $\mathsf{q}$ be positive integer numbers such that $b_0(0) = \mathsf{p}/\mathsf{q}$ and $\gcd(\mathsf{p}, \mathsf{q}) = 1$. For each fixed $k \in \mathbb{Z}_+$ there exists $N = N(k) \in \mathbb{Z}_+$ such that given $f \in C^\infty(\Omega_\epsilon)$, satisfying*

$$\int_0^{2\pi} f(0,t)dt = 0 \tag{2.2}$$

*and conditions bearing on the derivatives of $f$ of order up to $j_0\mathsf{q}$, where $j_0 = \max\{j \in \mathbb{Z} : j\mathsf{q} \leq N(k)\}$, there exists $u \in C^k(\Omega_\epsilon)$ solution of the equation $Lu = f$, in a neighborhood of $\Sigma$.*

**Remark 2.2.** Since the compatibility conditions above involve only a finite number of derivatives of $f$, it is possible to obtain $C^k$ solutions to equation $Lu = f$ in a neighborhood of $\Sigma$ assuming that $f \in C^\ell(\Omega_\epsilon)$, for $\ell \in \mathbb{Z}_+$ sufficiently larger depending on $k$.

**Example 2.3.** Consider

$$L = \frac{\partial}{\partial t} + \left(a(x) + i\,\frac{\mathsf{p}}{\mathsf{q}}\,x\right)\frac{\partial}{\partial x}\,, \tag{2.3}$$

defined on $\Omega$, where $\mathsf{p}, \mathsf{q} \in \mathbb{Z}_+$, $\gcd(\mathsf{p}, \mathsf{q}) = 1$, $a(x) \in C^\infty(\mathbb{R}; \mathbb{R})$ and, $a$ is flat at $x = 0$.

Given $f \in C^\infty(\Omega)$, the compatibility conditions given by Theorem 2.1 are

$$\int_0^{2\pi} \partial_x^{(m\mathsf{q})} f(0,t) e^{im\mathsf{p}t} dt = 0, \tag{2.4}$$

for $m = 0, 1, \ldots, j_0$. See [13] for more details.

□

**Example 2.4.** Consider the following operator defined on $\Omega$

$$T_{\frac{1}{2}} = \frac{\partial}{\partial t} + i\,\frac{x}{2}\,\frac{\partial}{\partial x}. \tag{2.5}$$

We claim that the equation

$$T_{\frac{1}{2}} u = x^2 e^{-it} \tag{2.6}$$

does not have solution $u \in C^k(\Omega)$, for $k \geq 4$, in any neighborhood of $\Sigma$.

Indeed, for $f(x,t) = x^2 e^{-it}$ we have

$$\int_0^{2\pi} \partial_x^2 f(t,0) e^{it} dt = \int_0^{2\pi} 2\, e^{-it} e^{it} dt = 4\pi \neq 0,$$

that is, $f$ does not satisfy (2.4).

Let us assume by contradiction that (2.6) has a solution $u \in C^4(\Omega)$. Then, by using Taylor's formula, we can write

$$u(x,t) = \sum_{j=0}^{3} u_j(t) x^j + R(x,t)\,,$$

where $R \in O(|x|^4)$. From (2.6) we have

$$u_2'(t) + i u_2(t) = e^{-it}. \tag{2.7}$$

Thus $u_2(t) = (t + u_2(0))e^{-it}$, which lead us to a contradiction, since the function $t \mapsto t\, e^{-it}$ is not $2\pi$-periodic. $\square$

For $k \in \mathbb{Z}_+$, define

$$\mathcal{F}_k(\Sigma) = \{f \in C^\infty(\Omega);\ Lu = f \text{ in a neighborhood of } \Sigma, \text{ for some } u \in C^k(\Omega)\}. \tag{2.8}$$

Note that Theorem 2.1 gives sufficient conditions for $f \in C^\infty(\Omega)$ to be in $\mathcal{F}_k(\Sigma)$.

The homogeneous equation $Lu = pu$ is the focus of our first result. For this, given $p = p(x,t)$ in $C^\infty(\Omega)$, let $p_0 : C^\infty(S^1) \to C^\infty(S^1)$ be defined by

$$p_0(t) = p(0,t). \tag{2.9}$$

**Theorem 2.5.** *Let $L$ be given by (2.1) and let $k \in \mathbb{Z}_+$. If $p \in C^\infty(\Omega)$ satisfies*

$$\frac{1}{2\pi i}\int_0^{2\pi} p(0,t)dt \in \mathbb{Z}, \tag{2.10}$$

*and $p - p_0 \in \mathcal{F}_k(\Sigma)$, then there exists $u \in C^k(\Omega)$ solution of the equation $Lu = pu$, in a neighborhood of $\Sigma$, with $u \neq 0$ on $\Sigma$.*

*Proof.* Let $p_0$ be given by (2.9). Since $p$ satisfies (2.10), it follows that

$$u_0(t) = e^{\int_0^t p_0(\tau)d\tau}$$

is a function in $C^\infty(S^1)$; moreover, $u_0$ satisfies

$$u_0'(t) = p_0(t)u_0(t).$$

Let $\tilde{p}(x,t) = p(x,t) - p_0(t)$. By hypothesis, there is $w \in C^k(\Omega)$ solution to $Lw = \tilde{p}$, in a neighborhood of $\Sigma$.

Define

$$u(x,t) = u_0(t)e^{w(x,t)}.$$

We have $u \in C^k(\Omega)$ and

$$Lu = u_0'e^w + u_0e^wLw = p_0u_0e^w + u_0e^w\tilde{p} = (p_0 + \tilde{p})u = pu,$$

in a neighborhood of $\Sigma$. $\square$

Since Theorem 2.1 gives sufficient conditions for a smooth function f to belong to $\mathcal{F}_k$ we can rewrite Theorem 2.5 as follows:

**Corollary 2.6.** *Let $L$ be given by (2.1). Let $\mathsf{p}$ and $\mathsf{q}$ be positive integer numbers such that $b_0(0) = \mathsf{p}/\mathsf{q}$ and $\gcd(\mathsf{p},\mathsf{q}) = 1$. For each fixed $k \in \mathbb{Z}_+$ there exists $N = N(k) \in \mathbb{Z}_+$ such that given $p \in C^\infty(\Omega_\epsilon)$, satisfying*

$$\frac{1}{2\pi i}\int_0^{2\pi} p(0,t)dt \in \mathbb{Z}, \tag{2.11}$$

*and conditions given by Theorem 2.1 bearing on the derivatives of $p$ of order up to $j_0\mathsf{q}$, where $j_0 = \max\{j \in \mathbb{Z} : j\mathsf{q} \leq N(k)\}$, there is $u \in C^k(\Omega_\epsilon)$ solution of $Lu = pu$ in a neighborhood of $\Sigma$, with $u \neq 0$ on $\Sigma$.*

**Example 2.7.** Let $L$ be given by (2.3). For each fixed $k \in \mathbb{Z}_+$ there exists $N = N(k) \in \mathbb{Z}_+$ such that if $p \in C^\infty(\Omega_\epsilon)$, satisfies

$$\int_0^{2\pi} \partial_x^{(m\mathsf{q})} p(0,s) e^{imp\mathsf{s}} ds = 0,$$

for $m = 0, 1, \ldots, j_0$, where $j_0 = \max\{j \in \mathbb{Z} : j\mathsf{q} \leq N(k)\}$, then there is $u \in C^k(\Omega_\epsilon)$ solution of $Lu = pu$ in a neighborhood of $\Sigma$, with $u \neq 0$ on $\Sigma$.

Comparing with the situation where $p - p_0 \notin \mathcal{F}_k(\Sigma)$ we have following:

**Example 2.8.** If $u \in C^4(\Omega)$ is a solution to $T_{\frac{1}{2}} u = x^2 e^{-it} u$, where $T_{\frac{1}{2}}$ is given by (2.5), then $u_{|_\Sigma} \equiv 0$. Indeed, by using Taylor's formula we can write $u(x,t) = u_0(t) + u_1(t)x + u_2(t)x^2 + u_3(t)x^3 + R(x,t)$, where $R \in O(|x|^4)$; hence, we are lead to

$$u_0' = 0 \quad \Rightarrow \quad u_0 \equiv c, \quad \text{for some } c \in \mathbb{C},$$

and

$$u_2' + iu_2 = ce^{-it} \Rightarrow \left(u_2 e^{it}\right)' = c \Rightarrow u_2(t) = (ct + u_2(0))\, e^{-it}.$$

Therefore, since $u_2$ is $2\pi$-periodic, we have $c = 0$; consequently, $u_{|_\Sigma} \equiv 0$.

**Theorem 2.9.** *Let $L$ be given by (2.1)and let $k \in \mathbb{Z}_+$. If $p \in C^\infty(\Omega)$ satisfies*

$$\frac{1}{2\pi i} \int_0^{2\pi} p(0,t) dt \notin \mathbb{Z}, \tag{2.12}$$

*and $p - p_0 \in \mathcal{F}_k(\Sigma)$, then there exist $w \in C^k(\Omega_\epsilon)$ with $w \neq 0$ on $\Sigma$, and $g \in C^k((-\epsilon, \epsilon))$ vanishing of finite order at $x = 0$, such that $u(x,t) = g(x)w(x,t)$ is a solution of the equation $Lu = pu$, in a neighborhood of $\Sigma$.*

*Proof.* Let $\alpha, \beta$ be real numbers given by

$$\alpha + i\beta = \frac{1}{2\pi i} \int_0^{2\pi} p(0,t) dt;$$

by (2.12) we have $\alpha + i\beta \notin \mathbb{Z}$.

Let $\ell = m - \lfloor\alpha\rfloor$, where $m \in \mathbb{Z}_+$ will be chosen later and we use $\lfloor\alpha\rfloor = \max\{n \in \mathbb{Z}\,; n \leq \alpha\}$. Define

$$\mu = (\ell + \alpha) + i\beta;$$

note that $\Re(\mu) = m - \lfloor\alpha\rfloor + \alpha \geq m$.

Now, set the function

$$\tilde{p}(x,t) = p(x,t) - i\mu.$$

We have

$$\begin{aligned}\frac{1}{2\pi i}\int_0^{2\pi} \tilde{p}(0,t)dt &= \frac{1}{2\pi i}\int_0^{2\pi} p(0,t)dt - \mu \\ &= \alpha + i\beta - (\ell+\alpha) - i\beta = -\ell \in \mathbb{Z};\end{aligned}$$

hence, by Theorem 2.5, there is $w \in C^k(\Omega)$, $w \neq 0$ on $\Sigma$, solution to $Lw = \tilde{p}w$ in a neighborhood of $\Sigma$.

Let $g : (-\epsilon, \epsilon) \to \mathbb{C}$ be the function defined by

$$g(x) = \begin{cases} e^{-i\mu\int_x^\epsilon \frac{1}{a+ib}dy}, & x > 0 \\ 0, & x = 0 \\ e^{i\mu\int_{-\epsilon}^x \frac{1}{a+ib}dy}, & x < 0 \end{cases}.$$

We will show that $g$ vanishes of finite order at $x = 0$. Note that

$$\frac{-i}{a+ib} = -i\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2};$$

hence,

$$\begin{aligned}-i\mu\int_x^\epsilon \frac{1}{a+ib}dy &= (\Re(\mu) + i\Im(\mu))\left(-i\int_x^\epsilon \frac{a}{a^2+b^2}dy - \int_x^\epsilon \frac{b}{a^2+b^2}dy\right) \\ &= \left(-\Re(\mu)\int_x^\epsilon \frac{b}{a^2+b^2}dy + \Im(\mu)\int_x^\epsilon \frac{a}{a^2+b^2}dy\right) \\ &\quad + i\left(-\Re(\mu)\int_x^\epsilon \frac{a}{a^2+b^2}dy - \Im(\mu)\int_x^\epsilon \frac{b}{a^2+b^2}dy\right).\end{aligned}$$

Recall that $a(x) = x^n a_0(x)$, $b(x) = x\, b_0(x)$, and $b_0(0) > 0$. Then,

$$\frac{b}{a^2 + b^2} = \frac{1}{x}\left(\frac{1}{b_0(0)} + 0(|x|)\right).$$

Therefore, for $x > 0$,

$$g(x) = e^{-i\mu \int_x^\epsilon \frac{1}{a+ib} dy} = \left(\frac{x}{\epsilon}\right)^{\frac{\Re(\mu)}{b_0(0)}} e^{-i\frac{\Im(\mu)}{b_0(0)} \ln\left(\frac{x}{\epsilon}\right)} g_+(x),$$

where $g_+ \in C^\infty((0, \epsilon))$, the derivative $g_+^{(j)}$ is bounded in $(0, \epsilon)$, for all $j \in \mathbb{Z}_+$, and $|g_+(x)| \geq c$, for some $c > 0$ and for all $x \in (0, \epsilon)$.

Similarly, for $x < 0$,

$$g(x) = e^{i\mu \int_{-\epsilon}^x \frac{1}{a+ib} dy} = \left(\frac{|x|}{\epsilon}\right)^{\frac{\Re(\mu)}{b_0(0)}} e^{-i\frac{\Im(\mu)}{b_0(0)} \ln\left(\frac{|x|}{\epsilon}\right)} g_-(x),$$

where $g_- \in C^\infty((-\epsilon, 0))$, the derivative $g_-^{(j)}$ is bounded in $(-\epsilon, 0)$, for all $j \in \mathbb{Z}_+$, and $|g_-(x)| \geq c$, for some $c > 0$ and for all $x \in (-\epsilon, 0)$.

Finally, define $u : \Omega \to \mathbb{C}$ by

$$u(x, t) = g(x) w(x, t);$$

for a suitable choice of $m$ (given in $\Re(\mu)$) we obtain that $u$ has the desired properties.

$\square$

**Remark 2.10.** The function $g$ defined in $(-\epsilon, \epsilon)$ and obtained in the proof of the Theorem above can be rewrite in the form $g(x) = x^N G(x)$, where $|G(x)| \geq C$, for some $C > 0$ and for all $x \neq 0$; moreover, the derivative $G^{(j)}$ is bounded for $x \neq 0$, for all $j$. These properties will be used in the proof of Theorem 2.12 below.

Since Theorem 2.1 gives sufficient conditions for a smooth function $f$ to belong to $\mathcal{F}_k$ we can rewrite Theorem 2.9 as follows:

**Corollary 2.11.** *Let $L$ be given by (2.1). Let $\mathsf{p}$ and $\mathsf{q}$ be positive integer numbers such that $b_0(0) = \dfrac{\mathsf{p}}{\mathsf{q}}$ and $\gcd(\mathsf{p}, \mathsf{q}) = 1$. For each fixed $k \in \mathbb{Z}_+$ there exists $N = N(k) \in \mathbb{Z}_+$ such that given $p \in C^\infty(\Omega_\epsilon)$, satisfying*

$$\frac{1}{2\pi i}\int_0^{2\pi} p(0,t)dt \notin \mathbb{Z}, \tag{2.13}$$

*and conditions given by Theorem 2.1 bearing on the derivatives of $p$ of order up to $j_0\mathsf{q}$, where $j_0 = \max\{j \in \mathbb{Z} : j\mathsf{q} \le N(k)\}$, there exists $w \in C^k(\Omega_\epsilon)$ with $w \neq 0$ on $\Sigma$, and $g \in C^k((-\epsilon, \epsilon))$ vanishing of finite order at $x = 0$, such that $u(x,t) = g(x)w(x,t)$ is a solution of the equation $Lu = pu$, in a neighborhood of $\Sigma$.*

The following two results are related to nonhomogeneous equations.

**Theorem 2.12.** *Let $\mathsf{p}$ and $\mathsf{q}$ be positive integer numbers such that $b_0(0) = \mathsf{p}/\mathsf{q}$ and $\gcd(\mathsf{p}, \mathsf{q}) = 1$. Assume that $p \in C^\infty(\Omega_\epsilon)$ satisfies*

$$\frac{1}{2\pi i}\int_0^{2\pi} p(0,t)dt - j\frac{\mathsf{p}}{\mathsf{q}} \notin \mathbb{Z}, \quad \textit{for all } j \in \mathbb{Z}_+. \tag{2.14}$$

*Then, for each fixed $k \in \mathbb{Z}_+$ there exists $\ell = \ell(k) \in \mathbb{Z}_+$ such that if, besides (2.14), $p - p_0 \in \mathcal{F}_\ell(\Sigma)$, where $p_0$ is given by (2.9), then given $f \in C^\infty(\Omega)$ there exists $u \in C^k(\Omega_\epsilon)$ solution to the equation $Lu = pu + f$, in a neighborhood of $\Sigma$ (recalling that $L$ is given by (2.1)).*

*Proof.* By using Taylor's expansion, we can write

$$f(x,t) \simeq \sum_{j\ge 0} f_j(t)x^j,\ p(x,t) \simeq \sum_{j\ge 0} p_j(t)x^j,\ \text{and } (a+ib)(x) \simeq \sum_{j\ge 1} c_j x^j.$$

After substituting them into equation $Lv = pv + f$, where we write

$$v(x,t) \simeq \sum_{j\ge 0} v_j(t)x^j,$$

we have:

$$
\begin{aligned}
v_0' &= p_0 v_0 + f_0, && (2.15)\\
v_1' + (c_1 - p_0)v_1 &= p_1 v_0 + f_1, && (2.16)\\
v_2' + (2c_1 - p_0)v_2 &= (p_1 - c_2)v_1 + p_2 v_0 + f_2, && (2.17)\\
v_3' + (3c_1 - p_0)v_3 &= (p_2 - c_3)v_1 + (p_1 - 2c_2)v_2 + p_3 v_0 + f_3, && (2.18)\\
&\ \vdots\\
v_j' + (jc_1 - p_0)v_j &= \sum_{k=1}^{j-1}(p_{j-k} - kc_{j-k+1})v_k + p_j v_0 + f_j,\ \ j \geq 4. && (2.19)
\end{aligned}
$$

Since $c_1 = i\mathsf{p}/\mathsf{q}$ and $p_0$ satisfies (2.14), we obtain

$$
v_0(t) = e^{\int_0^t p_0(s)ds}\left[\int_0^t e^{-\int_0^s p_0(\sigma)d\sigma} f_0\, ds + K_0\right],
$$

where

$$
K_0 = \frac{e^{\int_0^{2\pi} p_0(t)dt}}{1 - e^{\int_0^{2\pi} p_0(t)dt}} \int_0^{2\pi} e^{-\int_0^t p_0(\sigma)d\sigma} f_0\, dt.
$$

Next

$$
v_1(t) = e^{-i\frac{\mathsf{p}}{\mathsf{q}}t + \int_0^t p_0(s)ds}\left[\int_0^t e^{i\frac{\mathsf{p}}{\mathsf{q}}s - \int_0^s p_0(\sigma)d\sigma}(p_1 v_0 + f_1)\, ds + K_1\right],
$$

where

$$
K_1 = \frac{e^{-i2\pi\frac{\mathsf{p}}{\mathsf{q}} + \int_0^{2\pi} p_0(t)dt}}{1 - e^{-i2\pi\frac{\mathsf{p}}{\mathsf{q}} + \int_0^{2\pi} p_0(t)dt}} \int_0^{2\pi} e^{i\frac{\mathsf{p}}{\mathsf{q}}t - \int_0^t p_0(\sigma)d\sigma}(p_1 v_0 + f_1)dt.
$$

And, for $j \geq 2$, we have

$$
\begin{aligned}
v_j(t) = e^{-ij\frac{\mathsf{p}}{\mathsf{q}}t + \int_0^t p_0(s)ds} &\int_0^t e^{ij\frac{\mathsf{p}}{\mathsf{q}}s - \int_0^s p_0(\sigma)d\sigma}\\
&\times\left(\sum_{k=1}^{j-1}(p_{j-k} - kc_{j-k+1})v_k + p_j v_0 + f_j\right) ds\\
&+ K_j\, e^{-ij\frac{\mathsf{p}}{\mathsf{q}}t + \int_0^t p_0(s)ds}
\end{aligned}
$$

where

$$K_j = C_j \int_0^{2\pi} e^{ij\frac{\mathtt{p}}{\mathtt{q}}t - \int_0^t p_0(\sigma)d\sigma} \left( \sum_{k=1}^{j-1} (p_{j-k} - kc_{j-k+1})v_k + p_j v_0 + f_j \right) dt\,,$$

and

$$C_j = \frac{e^{-i2\pi j\frac{\mathtt{p}}{\mathtt{q}} + \int_0^{2\pi} p_0(t)dt}}{1 - e^{-i2\pi j\frac{\mathtt{p}}{\mathtt{q}} + \int_0^{2\pi} p_0(t)dt}}.$$

Let $\mu_m : \Omega \to \mathbb{C}$ be the function defined by

$$\mu_m(x,t) = \sum_{j=0}^{m} v_j(t)x^j.$$

We have

$$L\mu_m - p\mu_m - f = x^{m+1}h,$$

where $h : \Omega \to \mathbb{C}$ is a smooth function.

Now, for $\ell \in \mathbb{Z}_+$ to be choosen later, assume that $p - p_0 \in \mathcal{F}_\ell(\Sigma)$. It follows from Theorem 2.9 that there is $w \in C^\ell(\Omega_\epsilon)$, with $w \neq 0$ on $\Sigma$, and $g \in C^\ell((-\epsilon,\epsilon))$ vanishing of finite order at $x = 0$, such that $\nu(x,t) = g(x)w(x,t)$ is a solution of the equation $L\nu = p\nu$, in a neighborhood of $\Sigma$.

Thanks the properties of $g$ (see remark 2.10), for a suitable choice of $\ell$ and $m$ we obtain that $F : \Omega \to \mathbb{C}$ given by

$$F(x,t) = -\frac{x^{m+1}h(x,t)}{\nu(x,t)}$$

belongs to $C^r(\Omega)$, with $r$ large enough as in remark 2.2. Hence, the equation $L\mathtt{v} = F$ has a $C^k$ solution $\mathtt{v}$.

Define $u = \mu_m + \nu\mathtt{v}$. Hence, $u \in C^k(\Omega)$ and

$$\begin{aligned} Lu &= L\mu_m + (L\nu)\mathtt{v} + \nu(L\mathtt{v}) \\ &= p\mu_m + f + x^{m+1}h + p\nu\mathtt{v} - \nu\frac{x^{m+1}h}{\nu} \\ &= pu + f, \end{aligned}$$

in a neighborhood of $\Sigma$. $\square$

**Theorem 2.13.** *Let $L$ be given by (2.1). Let $\mathsf{p}$ and $\mathsf{q}$ be positive integer numbers such that $b_0(0) = \mathsf{p}/\mathsf{q}$ and $\gcd(\mathsf{p},\mathsf{q}) = 1$. Assume that for some $j \in \mathbb{Z}_+$ the function $p \in C^\infty(\Omega_\epsilon)$ satisfies*

$$\frac{1}{2\pi i}\int_0^{2\pi} p(0,t)dt - j\frac{\mathsf{p}}{\mathsf{q}} \in \mathbb{Z}. \tag{2.20}$$

*Then, for each fixed $k \in \mathbb{Z}_+$ there exist positive integers $\ell = \ell(k)$ and $N = N(k)$ such that if, besides (2.14), $p - p_0 \in \mathcal{F}_\ell(\Sigma)$, where $p_0$ is given by (2.9), and $f$ satisfies certain conditions bearing on its derivatives of order up to $N$, then there exists $u \in C^k(\Omega)$ solution to the equation $Lu = pu + f$, in a neighborhood of $\Sigma$.*

*Proof.* The proof is analogous that of Theorem 2.12. Indeed, assume that for $N \in \mathbb{Z}_+$ to be choosen later, for each $j \leq N$ such that (2.20) holds, the function $f$ safisfies

$$\int_0^{2\pi} e^{ij\frac{\mathsf{p}}{\mathsf{q}}s - \int_0^s p_0(\sigma)d\sigma}\left(\sum_{k=1}^{j-1}(p_{j-k} - kc_{j-k+1})v_k + p_j v_0 + f_j\right)dt = 0,$$

where $v_0$ and $v_k$ are given by the equations (2.15)-(2.19).

Hence, we can find $\mu_N$ so that $L\mu_N - p\mu_N - f = x^{N+1}h(x,t)$, with $h$ smooth. Therefore, by applying conveniently Theorem 2.5, and proceeding as in the proof of Theorem 2.12 we can find $u \in C^k$ solution to $Lu = pu + f$ in a neighborhood of $\Sigma$. □

**Remark 2.14.** In the case where $p \equiv 0$, the hypotheses on $f$ in Theorem 2.13 is in agreement with those given in Theorem 2.1.


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