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On the Existence of Hyperspheres with Prescribed Anisotropic Mean Curvature

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. We prove the existence of hyperspheres with prescribed anisotropic mean curvature in the Euclidean space, extending a classical result of Treibergs and Wei [10] to the anisotropic setting.

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1 Introduction

The notion of anisotropic mean curvature arises naturally in the study of variational problems as a generalization of the usual mean curvature. Moreover, in the Euclidean space this notion of curvature has also a natural geometric interpretation. In fact, consider the parametric functional of the form

$$\mathcal{F}(X) = \int_M F(N)\,\mathrm{d}M,$$

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where the integrand $F \in C^\infty(\mathbb{R}^{n+1}\backslash\{0\})$ is a positive Lagrangian satisfying the homogeneity condition

$$F(tz) = tF(z), \qquad \text{for all } z \in \mathbb{S}^n,\ t > 0,$$

and $X : M \longmapsto \mathbb{R}^{n+1}$ is an immersed closed and oriented hypersurface with Gauss mapping N and induced volume element $\mathrm{d}M$. Moreover, F is always assumed to be elliptic, i.e.,

$$D^2F(z) = \Big(\frac{\partial^2 F}{\partial z^i \partial z^j}(z)\Big)_{i,j=1,\dots,n+1} : z^\perp \longmapsto z^\perp \tag{1.1}$$

is a positive definite endomorphism for all $z \in \mathbb{S}^n$, or equivalently

$$\lambda = \lambda(F) = \inf_{z\in\mathbb{S}^n,\, v\in z^\perp,\, |v|=1} \langle D^2F(z)\cdot v, v\rangle > 0.$$

Geometrically, the ellipticity condition (1.1) implies that F is the support function of some convex body

$$\bigcap_{z\in\mathbb{S}^n} \{y \in \mathbb{R}^{n+1} \,:\, \langle y, z\rangle \leq F(z)\},$$

the boundary $\mathcal{W}_F$ of which is the convex hypersurfaces parametrized by

$$\Phi : \mathbb{S}^n \longmapsto \mathcal{W}_F, \qquad \Phi(z) = DF(z).$$

In the terminology of Taylor [9], $\mathcal{W}_F = \Phi(\mathbb{S}^n)$ is called the *Wulff shape.* Finally, notice that $\mathcal{F}$ generalizes the area functional

$$\mathcal{A}(X) = \int_M \mathrm{d}M,$$

which is obtained when $F(N) = \|N\| = 1$ is the *area integrand.*

Let us now consider an arbitrary variation X_ε of $X = X_0$ with variation vector field $Y = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(X_\varepsilon)|_{\varepsilon=0}$. Decomposing $Y = \varphi N +$ tangential terms, it is well known (see [8], [11] and [4]) that the first variation of $\mathcal{F}$ is given by

$$\delta\mathcal{F}(X,Y) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(X_\varepsilon)|_{\varepsilon=0} = -\int_M H_F\varphi \mathrm{d}M,$$

where H_F is the anisotropic mean curvature of X which is defined as follows. Let

$$N_F : M \longmapsto \mathcal{W}_F, \qquad N_F = \Phi \circ N,$$

denotes the generalized Gauss mapping into the Wulff shape. The operator $S_F = -\mathrm{d}X^{-1} \circ \mathrm{d}N_F$ is named the anisotropic Weingarten operator. We note that

$$S_F = A_F \circ A,$$

where $A = -\mathrm{d}X^{-1} \circ \mathrm{d}N$ is the classical Weingarten operator of X and A_F is the symmetric positive definite $(1,1)$-tensor given by

$$A_F = \mathrm{d}X^{-1} \circ \mathrm{d}\Phi \circ \mathrm{d}X = -\mathrm{d}X^{-1} \circ D^2F(N) \circ \mathrm{d}X.$$

Finally, the anisotropic mean curvature of X is defined by

$$H_F = \mathrm{tr}(S_F).$$

For instance, the anisotropic mean curvature of the sphere $\mathbb{S}^n(r)$ of radius r is

$$H_F = \Delta F(-z), \qquad z \in \mathbb{S}^n(r). \tag{1.2}$$

In fact, the unit normal vector of $\mathbb{S}^n(r)$ at a point z is $N = -\frac{1}{r}z$ and its Weingarten operator is $A = \frac{1}{r}I$. Hence,

$$H_F(z) = \frac{1}{r}\mathrm{tr}\Big(D^2F\Big(-\frac{1}{r}z\Big)\Big) = \Delta F(-z),$$

since D^2F is homogeneous of degree -1 and $D^2F|_N(N,N) = 0$.

Although the anisotropic Weingarten operator is not necessarily symmetric, it has n real eigenvalues (see e.g. [6]). In fact, to see this we define the abstract metric

$$g_F(v,w) = \langle A_F^{-1}v, w\rangle, \qquad v, w \in TM.$$

Note that the operator A_F is positive definite, hence it is invertible and its inverse is also positive. We have

$$g_F(S_Fv, w) = \langle A_F^{-1}(A_FA)v, w\rangle = \langle Av, w\rangle = \langle v, Aw\rangle = g_F(v, S_Fw)$$

for all $v, w \in TM$, which gives that S_F is symmetric with respect to this inner product. Thus there exists an orthonormal basis (with respect to the metric g_F) that diagonalize S_F. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of S_F are called the anisotropic principal curvatures of X and H_F is the sum of these curvatures.

Here we are interested on the existence of closed hypersurfaces with prescribed anisotropic mean curvature. Treibergs and Wei considered this problem for the mean curvature in [10]. More precisely, they considered the following problem raised by Yau: Is there an embedding $Y : \mathbb{S}^n \longmapsto \mathbb{R}^{n+1}$ of the n-dimensional sphere into Euclidean $(n+1)$-space, whose mean curvature is a preassigned sufficiently smooth function H defined on $\mathbb{R}^{n+1}$? A theorem of Bakelman and Kantor [1] together with the results obtained in [10] asserts the existence of such hypersurfaces assuming only the natural condition that H decay faster than the mean curvature of concentric spheres. Specifically, they proved that, if H is a C^1 positive function defined on the closure of the annular region $U = \{z \in \mathbb{R}^{n+1} : r_1 < |z| < r_2\}$, where $0 < r_1 \leq 1 \leq r_2$, that satisfies

$$\frac{\partial}{\partial \rho} \rho H(\rho z) \leq 0 \quad \text{for all } \rho z \in U \tag{1.3}$$

and

$$\begin{aligned} H(z) &> |z|^{-1} \quad \text{for } |z| = r_1 \\ H(z) &< |z|^{-1} \quad \text{for } |z| = r_2 \end{aligned} \tag{1.4}$$

then, for some $0 < \alpha < 1$, there exists an embedded hypersphere $Y \in C^{2,\alpha}(\mathbb{S}^n)$ with mean curvature H which is also a graph over the unit sphere and also satisfies $r_1 \leq |Y| \leq r_2$. It is worth to point out that the Dirichlet problem associated to the anisotropic mean curvature equation was investigated recently in [3], where the authors were able to establish a existence theorem for vertical graphs with prescribed anisotropic curvature similar to the classical Serrin's Theorem.

Our main result is an extension of Treibergs-Wei theorem for the anisotropic mean curvature under natural hypothesis. More precisely,

Theorem 1.1. *Assume $H \in C^1(\bar{U})$ satisfies condition (1.3) and*

$$H(z) > \Delta F(-z) \text{ if } |z| \leq r_1 \text{ and } H(z) < \Delta F(-z) \text{ if } |z| \geq r_2. \tag{1.5}$$

Then there exists $u \in C^2(\mathbb{S}^n)$ whose radial graph is contained in U and has prescribed anisotropic curvature H. Moreover, if there is a second function $v \in C^2(\mathbb{S}^n)$ whose radial graph also lives in U and has prescribed anisotropic curvature H, then

$$v = (1 + t_0)u$$

for some $t_0 > -1$, and, for all $0 \leq \theta \leq 1$, the radial graph of $v_\theta = (1+\theta t_0)u$ has anisotropic mean curvature H.

We reduce the proof of Theorem 1.1 to the existence of solutions for a quasi-nonlinear elliptic equation over $\mathbb{S}^n$. The existence of solution is proved by applying the method of continuity and a degree theory argument once the *a priori* gradiente estimates for the solutions have been established.

The article is organized as follows. In Section 2 we list some basic formulae which are needed later and describe an appropriate analytical formulation for the problem. In Section 3 we deal with the *a priori* gradiente estimates for prospective solutions. Finally in Section 4 we complete the proof of Theorem 1.1 using the continuity method and a degree theory argument with the aid of previously established estimates.

2 The Anisotropic Mean Curvature

In this section we will derive a suitable expression for the anisotropic mean curvature of a radial graph. First we calculate the second fundamental form of the graph using moving frames. In this section we adopt the convention that lower case indices $i, j, k, \ldots$ are summed from 1 to n and $a, b, c, \ldots$ from 1 to $n+1$.

Let $\{e_1, \cdots, e_{n+1}\}$ be a local orthonormal frame field defined in $\mathbb{R}^{n+1}$ such that e_{n+1} is the outward radial direction.

Let u be a smooth function defined on the sphere $\mathbb{S}^n$. We denote by ∇ the connection of $\mathbb{S}^n$. The graph Y of u is conveniently represented by $Y = e^u e_{n+1}$ and we extend u to $\mathbb{R}^{n+1} \setminus \{0\}$ as a constant along radii.

The vector fields $E_i = e_i + e^u u_i e_{n+1}$ form a basis to the tangent space at Y and, in terms of this basis, the induced metric of Y has components

$$g_{ij} = \langle E_i, E_j \rangle = \delta_{ij} + e^{2u} u_i u_j.$$

Hence its inverse matrix is given by $g^{ij} = \delta_{ij} - W^{-2} e^{2u} u_i u_j$, where

$$W = \sqrt{(1 + e^{2u} |\nabla u|^2)}$$

The unit normal vector to Y is

$$N = \frac{1}{W}(e^u u_i e_i - e_{n+1}).$$

Therefore, the components of the second fundamental form b of Y are

$$b_{ij} = -\langle dN(E_i), E_j \rangle = \frac{e^{-u}}{W}(\delta_{ij} + e^{2u} u_i u_j - e^{2u} u_{ij}).$$

By the homogeneity of the derivatives of u, we can equate their values on Y and $\mathbb{S}^n$. Pulling back, we conclude that on $\mathbb{S}^n$

$$b_{ij} = (1 + |\nabla u|^2)^{-1/2} e^{-u} (\delta_{ij} + u_i u_j - u_{ij}). \tag{2.1}$$

On the other hand, the components of the bilinear form $\mathcal{A}_F$ metrically equivalent to the operator A_F are

$$\begin{aligned}(\mathcal{A}_F)_{ij} =& \mathcal{A}_F(E_i, E_j) = \langle A_F(E_i), E_j \rangle \\ =& F_{ab} E_i^a E_j^b \\ =& e^{2u} F_{n+1 n+1} u_i u_j + e^u F_{n+1 i} u_i + e^u F_{n+1 j} u_j + F_{ij},\end{aligned}$$

where F_{ab} denote the components of the Hessian of F in terms of the frame field $\{e_a\}$. Note that the above derivatives of F are calculated in N. In terms of matrices,

$$S_F = A_F A = (g^{-1} \mathcal{A}_F) g^{-1} b = (g^{-1} \mathcal{A}_F g^{-1}) b.$$

On the other hand, decomposing the Hessian matrix of F as

$$D^2F = \begin{pmatrix} \hat{F} & F_{in+1} \\ F_{in+1} & F_{n+1n+1} \end{pmatrix},$$

we get from the Euler relation $F_{ab}(z)z^b = 0$ that

$$\begin{aligned}(g\hat{F}g)_{ij} =& g_{ik}F_{kl}g_{jl}\\ =&(\delta_{ik} + e^{2u}u_iu_k)(F_{kl})(\delta_{jl} + e^{2u}u_ju_l)\\ =&F_{ij} + e^{2u}u_ju_kF_{ik} + e^{2u}u_iu_kF_{kj} + e^{4u}u_iu_ju_ku_lF_{kl}\\ =&F_{ij} + e^{u}u_iF_{n+1j} + e^{u}u_iF_{n+1i} + e^{2u}F_{n+1n+1}u_iu_j\\ =&(\mathcal{A}_F)_{ij}.\end{aligned}$$

Then, in terms of matrices,

$$S_F = \hat{F}b.$$

We denote $S_F(E_i) = \sum_j s_{ij}E_j$. So

$$s_{ij} = \sum_k F_{ik}(N)b_{kj}, \tag{2.2}$$

which implies that

$$H_F = \sum_{i,j} F_{ij}(N)b_{ij}. \tag{2.3}$$

Hence the anisotropic mean curvature of the graph of u is given by

$$e^uWH_F = F_{ij}(N)(\delta_{ij} + u_iu_j - u_{ij}). \tag{2.4}$$

Thus, the radial graph of a function u has prescribed anisotropic mean curvature H if and only if u is a solution of the quasilinear elliptic equation

$$Q[x, u, u_i, u_{ij}] - H = 0,$$

where

$$Q[x, u, u_i, u_{ij}] = e^{-u}W^{-1}F_{ij}(N)(\delta_{ij} + u_iu_j - u_{ij}).$$

The second fundamental form of a Euclidean graph $(x, v(x)) \in \mathbb{R}^{n+1}$, of a smooth function v defined in a domain $\Omega \subset \mathbb{R}^n$, has components

$$b_{ij} = -\frac{v_{ij}}{\sqrt{1+|Dv|^2}}.$$

Hence, as it was done above, we conclude that the anisotropic mean curvature of the graph of v is

$$\sqrt{1+|Dv|^2} H_F = -F_{ij}(N) v_{ij}.$$

We finalize this section with a maximum principle for graphs with prescribed anisotropic mean curvature.

Proposition 2.1. *Suppose the radial graph Y has prescribed anisotropic mean curvature H and the function $H \in C^1(\mathbb{R}^{n+1} \setminus \{0\})$ satisfies the conditions (1.3) and (1.5). Then $r_1 < |Y| < r_2$.*

Proof. Let u be the function whose radial graph is Σ. By contradiction assume that $R = \sup e^u = e^u(x_0) \geq r_2$. Let $\mathcal{S}$ be the sphere of radius R centered at the origin. Observe that Σ and $\mathcal{S}$ are tangent at the point $Y(x_0) = e^{u(x_0)} x_0$. Furthermore, with respect to the inwards normal vector common to both hypersurfaces at this point, Σ lies above $\mathcal{S}$. Then the principal curvatures κ_i of Σ at this point are greater than or equal to $\frac{1}{R}$. Since the unit normal of Σ at $Y(x_0)$ is

$$N = \frac{1}{\sqrt{1+e^{2u}|\nabla u|^2}} (\nabla u - Y) = -\frac{1}{R} Y,$$

we conclude that

$$\begin{aligned} H =& \operatorname{tr}\Big(S_F\Big) = \sum_i \langle A_F A(e_i), e_i \rangle \\ =& \sum_i \kappa_i \langle A_F(e_i), e_i \rangle \geq \frac{1}{R} \sum_i \langle A_F(e_i), e_i \rangle \\ =& \frac{1}{R} \Delta F\Big(-\frac{1}{R} Y(x_0)\Big) = \Delta F\big(-Y(x_0)\big), \end{aligned}$$

where $\{e_i\}$ is an orthonormal basis of $(T_{x_0}\Sigma, \langle \cdot, \cdot \rangle)$ formed by eigenvectors of A. But the above inequality contradicts (1.5). Hence $u \leq r_2$. Proceeding in a similar way with the minimum of e^u we conclude that $e^u \geq r_1$. $\square$

3 Gradient Estimates

In this section, we prove a priori global estimate for gradient of prospective solutions of equation (2.4). To prove this estimate we follow the technique presented in [2].

Let $u \in C^3(\mathbb{S}^n)$ be a solution of the anisotropic mean curvature equation $H_F = H$. To estimate $|\nabla u|$ we will obtain a uniform positive constant $a = a(n, H, F, \sup |u|)$ that satisfies

$$\langle Y, N\rangle^2 \geq a > 0, \tag{3.1}$$

where N denotes the unit normal vector along $\Sigma = \mathrm{graph}|_u$ and $Y(x) = e^{u(x)}x$, is the position vector. This inequality implies the estimate of the gradient of u. In fact, since

$$N(Y(x)) = \frac{1}{e^u\sqrt{1+|\nabla u|^2}}\big(\nabla u - e^u x\big), \qquad x \in \mathbb{S}^n,$$

we have

$$\langle N, Y\rangle^2 = \frac{e^{2u}}{1+|\nabla u|^2},$$

which implies

$$\langle N, Y\rangle^2 \geq a \qquad \Leftrightarrow \qquad |\nabla u|^2 \leq \frac{e^{2u}}{a} - 1.$$

The estimate (3.1) will be obtained by estimating the maximum of the function φ defined on $\mathbb{S}^n$ by

$$\varphi(x) = \frac{1}{|Y|^2}\exp\left(\frac{1}{A\langle Y, N\rangle^2}\right) =: s\exp(t),$$

where A is a positive constant to be chosen later. Clearly, an upper bound for φ implies the estimate (3.1). We may assume (unless a rotation in the $\mathbb{R}^{n+1}$) that φ achieves its maximum at the north pole $q = (0, \cdots, 0, 1) \in \mathbb{S}^n$. In a small neighborhood of $Y(q)$ in Σ we may then use a local Cartesian representation for Σ, i.e., there exists a function $v \in C^3(U)$, such that $Y = (z, v(z)) \in \mathbb{R}^{n+1}$, $z \in U$, where $U \subset \mathbb{R}^n \times \{0\} \equiv \mathbb{R}^n \subset \mathbb{R}^{n+1}$ contains the origin and $(0, v(0)) = Y(q)$. In terms of v, the unit normal vector and the second fundamental form of Σ are given by

$$N = \left(\frac{Dv}{W}, -\frac{1}{W}\right), \qquad b_{ij} = -\frac{v_{ij}}{W},$$

where $W^2 = 1 + |Dv|^2$. Near q we may write φ as

$$\varphi(z) = \frac{1}{|z|^2 + v^2} \exp\left(\frac{1 + |Dv|^2}{A(z^k v_k - v)^2}\right), \qquad z \in U.$$

In particular,

$$\varphi(0) = \frac{1}{v^2} \exp\left(\frac{1 + |Dv|^2}{Av^2}\right).$$

Hence, the maximum value of φ, which is $\varphi(0)$, is controlled by $|Dv(0)|$. Therefore, it is sufficient to obtain a uniform constant $C = C(n, H, F, \sup |u|)$ that satisfies $|Dv(0)| \leq C$.

We may assume that $|Dv(0)| > 1$, otherwise we are done. After a rotation of the coordinates of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, if necessary, we have

$$Dv(0) = (v_1, 0, \ldots, 0) \in \mathbb{R}^n.$$

Since $z = 0$ is a maximum point of φ, we have $D\varphi(0) = 0$ and also $\big(\varphi_{ij}(0)\big)$ is a negative definite matrix.

We compute

$$D\varphi = e^t(Ds + sDt),$$

so $D\varphi(0) = 0$ implies

$$s_i(0) = -st_i(0), \quad i = 1, \ldots, n.$$

It follows that the expression

$$\varphi_{ij}(0) = e^t(s_{ij} + s_i t_j + s_j s_i + st_i t_j + t_{ij})(0)$$

takes the form

$$\varphi_{ij}(0) = (s_{ij} + st_{ij} - st_i t_j)e^t(0). \tag{3.2}$$

Now we compute the derivatives of the functions

$$s(z) = \frac{1}{|z|^2 + v^2} \quad \text{and} \quad t(z) = \frac{1 + |\nabla v|^2}{A(z^k v_k - v)^2}, \quad z \in U.$$

We have

$$\begin{aligned} s_i(z) =& -2\frac{z^i + v v_i}{(|z|^2 + v^2)^2}, \\ s_{ij}(z) =& 8\frac{(z^i + v v_i)(z^j + v v_j)}{(|z|^2 + v^2)^3} - 2\frac{\delta_{ij} + v_i v_j + v v_{ij}}{(|z|^2 + v^2)^2} \end{aligned}$$

and

$$\begin{aligned} t_i(z) =& \frac{2}{A}\left\{\frac{v^k v_{ki}}{(z^k v_k - v)^2} + \frac{z^k v_{ki}(1 + |\nabla v|^2)}{(z^k v_k - v)^3}\right\}, \\ t_{ij}(z) =& \frac{2}{A}\frac{v_i^k v_{kj} + v^k v_{kij}}{(z^k v_k - v)^2} + \frac{8}{A}\frac{v^k z^l v_{ki} v_{lj}}{(x^k v_k - v)^3} + \frac{2}{A}\frac{v_{ij} + z^k v_{kij}(1 + |\nabla v|^2)}{(z^k v_k - v)^3} \\ &+ \frac{6}{A}\frac{z^k z^l v_{ki} v_{lj}(1 + |\nabla v|^2)}{(z^k v_k - v)^4}. \end{aligned}$$

In particular, at the origin we have

$$s_i = -\frac{2}{v^3} v_i, \quad s_{ij} = \frac{8}{v^4} v_i v_j - \frac{2}{v^4}(\delta_{ij} + v_i v_j + v v_{ij}) \tag{3.3}$$

and

$$t_i = \frac{2}{Av^2} v^k v_{ki}, \quad t_{ij} = \frac{2}{Av^2}(v_i^k v_{kj} + v^k v_{kij}) + \frac{2}{Av^3} W^2 v_{ij}. \tag{3.4}$$

As we showed above, the anisotropic mean curvature of a Euclidean graph is given by

$$WH_F = -F^{ij}(N) v_{ij},$$

where, for sake of convenience, we use the notation $F^{ij} = \frac{\partial^2 F}{\partial z^i \partial z^j}$, with $(z^1, \ldots, z^n)$ being the Cartesian coordinates of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. We derive the equation $H_F = H$ with respect to z^k to obtain

$$\frac{v^l v_{lk}}{W^3} F^{ij} v_{ij} - F_\alpha^{ij} N_k^\alpha v_{ij} - F^{ij} v_{ijk} = H_k + H_{n+1} v_k. \tag{3.5}$$

Since

$$N_k^l = \frac{v_k^l}{W} - \frac{v^l v^p v_{pk}}{W^3}, \qquad N_k^{n+1} = -\frac{v^l v_{lk}}{W^3},$$

for $1 \le l \le n$, we have

$$\begin{aligned} H_k + H_{n+1} v_k =& \frac{v^l v_{lk}}{W^3} F^{ij} v_{ij} - \frac{1}{W^2} F_l^{ij} v_k^l v_{ij} \\ &+ \frac{v^p v_{pk}}{W^3} \left(\frac{v^l}{W} F_l^{ij} - \frac{1}{W} F_{n+1}^{ij} \right) v_{ij} - \frac{1}{W} F^{ij} v_{ij}. \end{aligned}$$

Applying the Euler relation

$$F_\alpha^{ij}(X) X^\alpha = -F^{ij}(X), \qquad \alpha = 1, \dots, n+1, \tag{3.6}$$

we get

$$\frac{v^l}{W} F_l^{ij} - \frac{1}{W} F_{n+1}^{ij} = -F^{ij}.$$

Replacing this into the above equation,

$$-\frac{1}{W^2} F_l^{ij} v_k^l v_{ij} - \frac{1}{W} F^{ij} v_{ijk} = H_k + H_{n+1} v_k.$$

As $g_i = -g f_i$ at the origin, it follows from (3.3) and (3.4) that $v^k v_{ki} = A v v_i$. In particular,

$$v_{11} = Av \quad \text{and} \quad v_{1i} = 0, \ (i > 1).$$

Thus, contracting equation (3.5) with v^k, we obtain (at the origin)

$$-\frac{v_1}{W^2} F_1^{ij} v_{11} v_{ij} - \frac{v_1}{W} F^{ij} v_{ij1} = H_1 v_1 + H_{n+1} v_1^2. \tag{3.7}$$

We use the Euler relation (3.6) again to get

$$-\frac{v_1}{W} F_1^{ij} = -\frac{1}{W} F_{n+1}^{ij} + F^{ij}.$$

Hence, equation (3.7) becomes

$$\frac{v_{11}}{W} F^{ij} v_{ij} - \frac{v_{11}}{W^2} F_{n+1}^{ij} v_{ij} - \frac{v_1}{W} F^{ij} v_{ij1} = H_1 v_1 + H_{n+1} v_1^2. \tag{3.8}$$

Using again that $WH_F = -F^{ij}v_{ij} = WH$,

$$-\frac{v_{11}}{W^2}F^{ij}_{n+1}v_{ij} - \frac{v_1}{W}F^{ij}v_{ij1} = Hv_{11} + H_1v_1 + H_{n+1}v_1^2. \tag{3.9}$$

Now we will eliminate from equation (3.9) the first and the second derivatives of v. To proceed, we note that $F^{ij}\varphi_{ij} \le 0$, since the matrix $\big(F^{ij}\big)$ is positive definite and $\big(\varphi_{ij}\big)$ is negative. Thus, it follows from (3.2) that

$$F^{ij}s_{ij} + sF^{ij}t_{ij} - sF^{ij}t_it_j \le 0.$$

Using (3.3) and (3.4) the above inequality becomes

$$\begin{aligned} 0 \ge &\frac{8}{v^4}F^{ij}v_iv_j - \frac{2}{v^4}F^{ij}(\delta_{ij} + v_iv_j + vv_{ij}) - \frac{4}{A^2v^6}F^{ij}v^kv^lv_{ki}v_{lj} \\ &+ \frac{2}{Av^4}F^{ij}(v_i^kv_{kj} + v^kv_{kij}) + \frac{2W^2}{Av^5}F^{ij}v_{ij}. \end{aligned}$$

Dividing this inequality by $\frac{v^4}{2}$ we get

$$\begin{aligned} -v^kF^{ij}v_{kij} \ge &4AF^{ij}v_iv_j - AF^{ij}(\delta_{ij} + v_iv_j + vv_{ij}) - \frac{2}{Av^2}F^{ij}v^kv^lv_{ki}v_{lj} \\ &+ F^{ij}v_i^kv_{kj} + \frac{W^2}{v}F^{ij}v_{ij}. \end{aligned}$$

Since $WH_F = -F^{ij}v_{ij} = WH$ and $v_i = 0$, $(i > 1)$, $v_{11} = Av$, we have

$$-v_1F^{ij}v_{1ij} \ge AF^{11}v_1^2 + F^{ij}v_i^kv_{kj} - AF^{ij}\delta_{ij} + AWvH - \frac{W^3}{v}H. \tag{3.10}$$

After rotation of the coordinates $(z^2, \dots, z^n)$ we may assume that $\big(v_{ij}(0)\big)$ is diagonal. Hence,

$$-\frac{v_1}{W}F^{ij}v_{1ij} \ge \frac{A}{W}F^{11}v_1^2 + \frac{1}{W}F^{ii}v_{ii}^2 - \frac{A}{W}F^{ij}\delta_{ij} + AvH - \frac{W^2}{v}H. \tag{3.11}$$

Since $v_{1ij} = v_{ij1}$, we may apply inequality (3.11) to obtain from (3.9) that

$$\begin{aligned} Hv_{11} + H_1v_1 + H_{n+1}v_1^2 \ge &-\frac{v_{11}}{W^2}F^{ii}_{n+1}v_{ii} + \frac{A}{W}F^{11}v_1^2 + \frac{1}{W}F^{ii}v_{ii}^2 \\ &-\frac{A}{W}F^{ij}\delta_{ij} + AvH - \frac{W^2}{v}H. \end{aligned} \tag{3.12}$$

Note that we eliminate the third derivatives of v on the last equation. To do the same with the second derivatives we first note that $F^{ii} > 0$, for any $i = 1, \cdots n$. In fact,

$$F^{ii} = \mathrm{Hess}(F)_{|_N}(e_i, e_i) = \mathrm{Hess}(F)_{|_N}(e_i^T, e_i^T) \geq \lambda |e_i^T|^2 > 0,$$

since the tangent component e_i^T of the vector e_i do not vanish whereas N is not multiple of e_i. Thus we may apply the Cauchy inequality with epsilon,

$$ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2,$$

with $a = |v_{ii}|$, $b = |F^{ii}_{n+1}|$ and $\varepsilon = \frac{WF^{ii}}{v_{11}} > 0$, for each $1 \leq i \leq n$ fixed. Then

$$v_{11}|F^{ii}_{n+1} v_{ii}| \leq WF^{ii} v_{ii}^2 + v_{11}^2 \frac{(F^{ii}_{n+1})^2}{WF^{ii}} \leq WF^{ii} v_{ii}^2 + A^2 v^2 \frac{(F^{ii}_{n+1})^2}{W\lambda}.$$

Adding on i we get

$$v_{11}|F^{ii}_{n+1} v_{ii}| \leq WF^{ii} v_{ii}^2 + \frac{A^2 B}{W},$$

where

$$B = v(0)^2 \sup_{\mathbb{S}^n} \frac{(F^{ii}_{n+1})^2}{\lambda} > 0.$$

Hence,

$$\frac{v_{11}}{W^2} F^{ii}_{n+1} v_{ii} \geq -\frac{v_{11}}{W} |F^{ii}_{n+1} v_{ii}| \geq -\frac{1}{W} F^{ii} v_{ii}^2 - \frac{A^2 B}{W^3} \geq -\frac{1}{W} F^{ii} v_{ii}^2 - A^2 B.$$

Replacing the last inequality into (3.12) we obtain

$$\begin{aligned} Hv_{11} + H_1 v_1 + H_{n+1} v_1^2 \geq & - A^2 B + \frac{A}{W} F^{11} v_1^2 - \frac{A}{W} F^{ij} \delta_{ij} \\ & + AvH - \frac{W^2}{v} H. \end{aligned} \tag{3.13}$$

As we have $v_{11} = Av$ e $W^2 = 1 + v_1^2$ (at the origin), the above equation may be rewritten as

$$H_1 v_1 + v_1^2 (H_{n+1} + \frac{H}{v}) + \frac{H}{v} \geq -A^2 B + \frac{A}{W} F^{11} v_1^2 - \frac{A}{W} F^{ij} \delta_{ij}.$$

It follows from hypothesis (1.3) that

$$H_{n+1} + \frac{H}{v} \leq 0.$$

In fact,

$$0 \geq \frac{\partial}{\partial \rho} \left(\rho H(\rho(0, v(0))\right)_{|\rho=1} = H(0, v(0)) + v(0) H_{n+1}(0, v(0)).$$

Hence, we conclude from (3.13) that

$$H_1 v_1 + \frac{H}{v} \geq -A^2 B + \frac{A}{W} F^{11} v_1^2 - \frac{A}{W} F^{ij} \delta_{ij}. \tag{3.14}$$

Since $v_1 > 1$ we have $\frac{v_1^2}{W} \geq \frac{v_1}{\sqrt{2}}$, so

$$\frac{v_1^2}{W} F^{11} \geq \frac{v_1}{\sqrt{2}} F^{11} \geq \frac{v_1}{\sqrt{2}} \lambda.$$

Therefore,

$$H_1 v_1 + \frac{H}{v} \geq -A^2 B + A \frac{v_1}{\sqrt{2}} \lambda - \frac{A}{W} F^{ij} \delta_{ij}. \tag{3.15}$$

Since

$$\frac{1}{W} F^{ij} \delta_{ij} \leq n \Lambda,$$

where Λ is the largest eigenvalue of $D^2 F$, then it follows from (3.15) that

$$v_1 \Big(\frac{A\lambda}{\sqrt{2}} - H_1 \Big) \leq \frac{H}{v} + A^2 B + n A \Lambda.$$

Thus, if we choose the constant $A > 0$ large such that $A > \frac{\sqrt{2}}{\lambda} \sup |DH|$, we obtain

$$v_1 \leq \frac{H/v + A^2 B + n A \Lambda}{\frac{A\lambda}{\sqrt{2}} - H_1}.$$

So, denoting

$$\bar{C} = \frac{\frac{H}{v}(0) + A^2 B + n A \Lambda}{\frac{A\lambda}{\sqrt{2}} - H_1(0)},$$

we obtain $|Dv(0)| \leq \bar{C}$, with $\bar{C} = \bar{C}(n, H, F, \sup |u|)$, which proves the following theorem.

Theorem 3.1. *Under the conditions of Theorem 1.1, if $u \in C^3(\mathbb{S}^n)$ is a solution of the prescribed anisotropic mean curvature equation $H_F = H$, then there exists a uniform constant $C = C(n, H, F, \sup|u|)$ such that*

$$|\nabla u| \leq C.$$

4 Proof of Theorem 1.1

In order to prove Theorem 1.1 we use the degree theory for nonlinear elliptic partial differential equations developed by Yan Yan Li. We refer the reader to [7] for more details.

We consider for each t, $0 \leq t \leq 1$, the map

$$H_t(z) = tH(z) + (1-t)\phi(|z|)\Delta F(-z), \qquad z \in U, \tag{4.1}$$

where ϕ is a positive real function defined in $\mathbb{R}_+$ which satisfies the following conditions

$$\begin{aligned} \phi(t) > 1 \quad &\text{for} \quad t \leq r_1, \\ \phi(t) < 1 \quad &\text{for} \quad t \geq r_2 \end{aligned} \tag{4.2}$$

and $\phi' < 0$. Note that these conditions imply the existence of a unique point $r_0 \in (r_1, r_2)$ such that $\phi(r_0) = 1$. We point out that, with this choice of the function ϕ, H_t also satisfies the conditions in Theorem 1.1. In fact, it follows from (1.5) that

$$\begin{aligned} H_t(z) =& tH(z) + (1-t)\phi(|z|)\Delta F(-z) \\ >& (t + (1-t)\phi(|z|))\Delta F(-z) \geq \Delta F(-z) \end{aligned}$$

for $|z| \leq r_1$. Similarly, we verify that $H_t(z) < \Delta F(-z)$ for $|z| \geq r_2$. To prove condition (1.3) we compute

$$\begin{aligned} \frac{\partial}{\partial \rho}\Big(\rho H_t(\rho z)\Big) =& \frac{\partial}{\partial \rho}\Big(t\rho H(\rho z) + \rho(1-t)\phi(\rho|z|)\Delta F(-\rho z)\Big) \\ =& t\frac{\partial}{\partial \rho}\Big(\rho H(\rho z)\Big) + (1-t)|z|\phi'(|z|)\Delta F(-z) \\ \leq& t\frac{\partial}{\partial \rho}\Big(\rho H(\rho z)\Big) \leq 0, \end{aligned}$$

where we use that ΔF is homogeneous of degree -1 and is a positive function.

Now we consider the family of equations

$$\Upsilon(t,u) = H_F\big(Y\big) - H_t(Y) = 0, \qquad Y = e^{u(x)}x,\ x \in \mathbb{S}^n, \tag{4.3}$$

where H_F is the anisotropic mean curvature of the radial graph defined by $u \in C^2(\mathbb{S}^n)$. It follows from the expression obtained above to H_F that we may write (4.3) in the form

$$\Upsilon(t, x, u, \nabla u, \nabla^2 u) = 0, \qquad x \in \mathbb{S}^n. \tag{4.4}$$

Notice that the constant function $u = \ln r_0$ is a solution to the problem corresponding to $t = 0$. We denote it by u_0. The following result ensures the uniqueness of u_0.

Lemma 4.1. *Fixed $t = 0$ there exists a unique solution u_0 of the equation $\Upsilon(t, u(x)) = 0$, namely $u_0 = \ln r_0$, where r_0 satisfies $\phi(r_0) = 1$.*

Proof. That u_0 is a solution to the problem it follows from (1.2) and

$$\begin{aligned}\Upsilon(0, u_0) =& H_F\big(Y\big) - \phi\big(|Y|\big)\Delta F\big(-Y\big)\\ =& \Delta F\big(-Y\big) - \Delta F\big(-Y\big) = 0,\end{aligned}$$

where $Y(x) = e^{u_0}x = r_0 x,\ x \in \mathbb{S}^n$. Let $\bar{u}$ be a solution of $\Upsilon(0, u(x)) = 0$. This means that

$$H_F\big(\bar{Y}\big) - \phi\big(|\bar{Y}|\big)\Delta F\big(-\bar{Y}\big) = 0, \qquad \bar{Y}(x) = e^{\bar{u}(x)}x,\ x \in \mathbb{S}^n.$$

Now, let $x_0 \in \mathbb{S}^n$ be a minimum point of $\bar{u}$. At this point, we have $\nabla \bar{u} = 0$ and $\nabla^2 \bar{u}$ is positive-definite. We compute explicity at $\bar{Y}(x_0)$

$$b_{ij} = e^{-\bar{u}}(\delta_{ij} - \bar{u}_{ij}).$$

Therefore, if we consider a local frame $\{e_i\}$ around x_0 which is orthonormal at x_0 and which diagonalize $\nabla^2 \bar{u}$ at this point, we obtain

$$\kappa_i \leq e^{-\bar{u}},$$

where κ_i are the principal curvature of the radial graph defined by $\bar{u}$. Hence, since at $\bar{Y}(x_0)$ the unit normal of the graph $\bar{Y}$ is

$$\bar{N} = -\frac{1}{|\bar{Y}|}\bar{Y} = -e^{-\bar{u}}\bar{Y},$$

the anisotropic mean curvature of $\bar{Y}$ satisfies

$$H_F\big(\bar{Y}(x_0)\big) = \sum_i \kappa_i \langle A_F e_i, e_i\rangle \leq e^{-\bar{u}}\Delta F\big(N(x_0)\big) = \Delta F\big(-\bar{Y}(x_0)\big).$$

Therefore, at x_0,

$$\phi\big(|\bar{Y}|\big)\Delta F\big(-\bar{Y}\big) = H_F\big(\bar{Y}\big) \leq \Delta F\big(-\bar{Y}\big) = \phi\big(|Y|\big)\Delta F\big(-\bar{Y}\big).$$

Hence, since ϕ is a decreasing function we conclude from the choice of x_0 as a minimum point that

$$\bar{u}(x) \geq \bar{u}(x_0) \geq u_0,$$

for all $x \in \mathbb{S}^n$. In a similar way, we prove that

$$\bar{u}(x) \leq u_0$$

for all $x \in \mathbb{S}^n$. Thus, we get $\bar{u} = u_0$. This finishes the proof. □

In the two last sections we proved that a differentiable function u which solves the equations $\Upsilon(t,u) = 0$ for some $0 \leq t \leq 1$ satisfies the following bounds

$$r_1 \leq e^u \leq r_2 \tag{4.5}$$

and

$$|u|_1 \leq C, \tag{4.6}$$

for some positive constant C which depends on n, r_1, r_2, H and F. The standard elliptic regularity theory then provides $C^{2,\alpha}$ estimates. If we suppose that H is a $C^{2,\alpha}$ data, then the regularity of the solution may be improved for $C^{4,\alpha}$. Thus, we obtain a bound

$$|u|_{4,\alpha} < \hat{C} \tag{4.7}$$

for some constant $\hat{C} > 0$.

We then denote by $\mathcal{O}$ the open ball in $C^{4,\alpha}(\mathbb{S}^n)$ with radius $\hat{C}$. Thus, our reasoning above shows that any solution u of $\Upsilon(t,u) = 0$ for some $0 \leq t \leq 1$ is contained in $\mathcal{O}$. In particular, if we consider the restriction

$$\Upsilon : \bar{\mathcal{O}} \subset C^{4,\alpha}(\mathbb{S}^n) \longmapsto C^{2,\alpha}(\mathbb{S}^n),$$

then we conclude that

$$\Upsilon(t,\,\cdot\,)^{-1}(0) \cap \partial\mathcal{O} = \emptyset, \qquad 0 \leq t \leq 1.$$

Thus, according to Definition 2.2 in [7] the degree $\deg(\Upsilon(t,\,\cdot\,), \mathcal{O}, 0)$ is well-defined for all $0 \leq t \leq 1$.

Since Lemma 4.1 assures that $u_0 = \ln r_0$ is the unique solution to $\Upsilon(0,u) = 0$ in $C^{4,\alpha}(\mathbb{S}^n)$, we must prove that the Frechét derivative $\Upsilon_u(0,u_0)$ calculated around u_0 is an invertible operator from $C^{4,\alpha}(\mathbb{S}^n)$ to $C^{2,\alpha}(\mathbb{S}^n)$. We compute

$$\begin{aligned}\Upsilon(0,\rho u_0) =& H_F\big(Y_\rho\big) - \phi\big(|Y_\rho|\big)\Delta F\big(-Y_\rho\big)\\ =& \Delta F\big(-Y_\rho\big) - \phi\big(|Y_\rho|\big)\Delta F\big(-Y_\rho\big),\end{aligned}$$

where $Y_\rho(x) = e^{\rho u_0}x,\ x \in \mathbb{S}^n$. Using the fact that $\phi(r_0) = 1$ and that $\phi'(r_0) < 0$ we get

$$\Upsilon_u(0,u_0)\cdot u_0 = \frac{\mathrm{d}}{\mathrm{d}\rho}\Upsilon(0,\rho u_0)|_{\rho=1} = -\phi'(r_0)\Delta F\big(-Y_1\big) > 0.$$

On the other hand, since obviously $\nabla u_0 = 0$ and $\nabla^2 u_0 = 0$, then $\Upsilon_u(0,u_0)\cdot u_0$ is just a multiple of the zeroth order term in $\Upsilon_u(0,u_0)$. We conclude that $\Upsilon_u(0,u_0)$ is an invertible elliptic operator.

We finally calculate $\deg(\Upsilon(1,\,\cdot\,), \mathcal{O}, 0)$. From Proposition 2.2 in [7], it follows that $\deg(\Upsilon(t,\,\cdot\,), \mathcal{O}, 0)$ does not depend on t. In particular,

$$\deg(\Upsilon(1,\,\cdot\,), \mathcal{O}, 0) = \deg(\Upsilon(0,\,\cdot\,), \mathcal{O}, 0).$$

On the other hand, we had just proved that the equation $\Upsilon(0,u) = 0$ has a unique solution u_0 and that the linearized operator $\Upsilon_u(0,u_0)$ is invertible.

Thus, by Proposition 2.3 in [7] we get

$$\deg(\Upsilon(0,\,\cdot\,),\mathcal{O},0)=\deg(\Upsilon_u(0,u_0),\mathcal{O},0)=\pm1,$$

and, therefore,

$$\deg(\Upsilon(1,\,\cdot\,),\mathcal{O},0)=\pm1.$$

Thus, the equation $\Upsilon(1,u)=0$ has at least one solution $u\in\mathcal{O}$. This completes the proof of the existence in Theorem 1.1. To obtain the uniqueness result we follow the idea presented in [10]. First we extend the prescribed function H to $\mathbb{R}^{n+1}\setminus\{0\}$ on such a way that (1.3) remains true. Let $Y^i(x)=e^{u^i}x$, $i=1,2$, solutions of the prescribed anisotropic mean curvature equation. It follows from Proposition 2.1 that $r_1<|Y^i|<r_2$. Suppose that $u^1>u^2$ at some point. Let $t>1$ such that the radial graph

$$\tilde{Y}^2:=te^{u^2}=e^{\tilde{u}^2}$$

satisfies $|Y^2|\geq|Y^1|$ and $\tilde{Y}^2(x_0)=Y^1(x_0)$ for some point $x_0\in\mathbb{S}^n$. Let H_F^i and $\tilde{H}_F^2$ be the anisotropic mean curvature of Y^i and $\tilde{Y}^2$, respectively. We have

$$\tilde{H}_F^2(\tilde{Y}^2)=\frac{1}{t}H_F^2(Y^2)=\frac{1}{t}H(Y^2).$$

On the other hand, since the function $\psi(\rho)=\rho H(\rho z)$ is decreasing we have

$$\frac{1}{t}H(Y^2)=\frac{1}{t}H(\frac{1}{t}\tilde{Y}^2)\geq H(\tilde{Y}^2). \tag{4.8}$$

Hence

$$\tilde{H}_F^2(\tilde{Y}^2)\geq H(\tilde{Y}^2),$$

which implies that

$$-Q[\tilde{u}^2]+H(\tilde{Y}^2)\leq 0.$$

As

$$-Q[u^1]+H(Y^1)=0,$$

$u^1\leq\tilde{u}^2$, and $u_1(x_0)=\tilde{u}^2(x_0)$, we may apply the maximum principle to obtain (see e.g., [2]) that $\tilde{u}^2=u^1$. In particular, $\tilde{Y}^2=Y^1$ is a solution of the anisotropic mean curvature equation, hence equality (4.8) holds. Using condition (1.3) we may conclude from (4.8) that

$$\frac{1}{s}H(\frac{1}{s}Y^1) = H(Y^1), \quad 1 \leq s \leq t.$$

Thus, since $H_F(sY) = \frac{1}{s}H_F(Y)$, each radial graph $\tilde{Y} = sY^1$, $1 \leq s \leq t$, is a solution. In fact,

$$H_F(\tilde{Y}) = \frac{1}{s}H_F(Y^1) = \frac{1}{s}H(Y^1) = \frac{1}{s}H(\frac{1}{s}sY^1) = H(sY^1) = H(\tilde{Y}).$$

This completes the proof of Theorem 1.1.

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