Matemática Contemporânea, Vol. 9, 111–116
http://doi.org/10.21711/231766361995/rmc99

LOCAL STABILITY OF THE FIRST EIGENVALUE OF THE LAPLACIAN

Luquésio P. de M. Jorge **Levi L. de Lima**

Abstract

It is proved a Morse index formula for the variation problem arising from smoothly deforming a bounded regular domain in Euclidean space and computing the variational formulae for the first Dirichlet eigenvalue of the Laplacian along the deformation. As a consequence, a local stability result, first proved by N. Shimakura, is retrieved.

Let $\Omega \subset R^n$ be a bounded regular domain, $\lambda_1 = \lambda_1(\Omega)$ the first eigenvalue for the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \Sigma = \partial\Omega \end{cases} \tag{1}$$

and u the (normalized) first eigenfunction, i.e., u satisfies (1) for $\lambda = \lambda_1$ and, furthermore, $u > 0$ in Ω and $\int_\Omega u^2 = 1$. In this work, we shall study the dependence of λ_1 on Ω. More precisely, let $t \in (-\epsilon, \epsilon) \mapsto \varphi_t : R^n \to R^n$, $\varphi_0 = Id_{R^n}$, be a smooth variation by diffeomorphisms. We shall put $\Omega_t = \varphi_t(\Omega)$ and denote by $V : \Sigma \to R$ the *variational vector field* associated to φ_t, i.e.,

$$V(x) = \frac{d}{dt}\varphi_t(x)|_{t=0}, x \in \Sigma.$$

We shall suppose that φ_t preserves volume, i.e., $\text{vol}(\Omega_t) = \text{vol}(\Omega)$. This means that $\int_\Sigma f = 0$ where $f = \langle V, \nu\rangle$ and ν is the interior unit normal field to Ω. Conversely, if $f : \Sigma \mapsto R$ is given satisfying $\int_\Sigma f = 0$, it is possible to construct a volume preserving variation whose variational field V satisfies $f = \langle V, \nu\rangle$.

A first result concerning the functional $t \mapsto \lambda(t) = \lambda_1(\Omega_t)$ is the following classical *first variational formula* due to Hadamard ([S]):

$$\lambda'(0) = \int_\Sigma f \left(\frac{\partial u}{\partial \nu}\right)^2, \tag{2}$$

Let $\Sigma' \subset \Sigma$ be open and connected. Following Shimakura ([S]), we say that Ω is *Σ'-critical* if $\lambda'(0) = 0$ for any variation supported on Σ' (i.e. such that $f(x) = 0$ for $x \in \Sigma \backslash \Sigma'$). From (2), this is the case if and only if $\partial u/\partial \nu = k =$ const. on Σ'. A theorem of Serrin ([Se]) then implies that the only Σ-critical domains are the spheres $S_r(x_0) = \{x \in R^n; |x - x_0| = r\}, x_0 \in R^n, r > 0$. Furthermore, the annulus

$$S_{r_1,r_2} = \{x \in R^n; r_1 < |x - x_0| < r_2\}$$

is Σ_i-critical, where $\Sigma_i = S_{r_i}(x_0), i = 1, 2$.

From now on we suppose that Ω is Σ'-critical. Then we have the *second variation formula* ([S]):

$$\lambda''(0) = 2k^2 \int_\Sigma \left\{-f\frac{\partial f}{\partial \nu} + Hf^2\right\}. \tag{3}$$

In the formula, H denotes the mean curvature of Σ and $\partial f/\partial \nu$ is defined as follows. Let $W_f : \Omega \to R$ be the unique solution to the problem

$$\begin{cases} \Delta W_f + \lambda_1 W_f = 0 & \text{in } \Omega \\ W_f = f & \text{in } \Sigma \\ \int_\Omega W_f u = 0 & \end{cases} \tag{4}$$

Then $\partial f/\partial \nu = \partial W_f/\partial \nu$.

Again following Shimakura, we say that Ω is *Σ'-stable* if $\lambda''(0) \geq 0$ for any variation supported on Σ'. In ([S]) it is proved the following result.

Theorem 1. *Suppose that Ω is Σ'-critical then Ω is locally stable in the sense that for each $x \in \Sigma'$ we can find a neighborhood $W \subset \Sigma'$ such that $x \in W$ and Ω is W-stable.*

In this work we give a new proof of this result. In fact, we shall put Shimakura's result in a more conceptual framework by proving a Morse index formula (see our theorem below) for the underlying variational problem, as explained in the sequel. We shall use standard facts on Sobolev spaces which can be found in [LM].

Let $m \geq 1/2$ and $\gamma : H^m(\Omega) \to H^{m-1/2}(\Sigma)$ be the trace map. We know that γ is linear continuous, surjective and $\ker \gamma = H_0^m(\Omega)$. Hence, γ induces an isomorphism $\gamma^* : H^m(\Omega)/H_0^m(\Omega) \to H^{m-1/2}(\Sigma)$. Furthermore, the map $f \in H^{m-1/2}(\Sigma) \mapsto \xi(f) = W_f \in H^{m-3/2}(\Omega) \subset H^m(\Omega)$, defined by solving the elliptic problem (4), is also linear and continuous and, since for $f \equiv 0$ we get, up to a constant, $\xi(f) = u \in H_0^m(\Omega)$, Fredholm alternative implies that ξ is an inverse for γ^*. We use this for $m = 1/2$ and $m = 1$ to obtain the estimates

$$\begin{aligned} |W_f|_{H^0(\Omega)} &\leq |W_f|_{H^{1/2}(\Omega)} \leq c_1 |f|_{H^0(\Sigma)}, \\ |f|_{H^{1/2}(\Sigma)} &\leq c_2 |W_f|_{H^1(\Omega)}. \end{aligned} \tag{5}$$

We shall view $\partial/\partial\nu$ as an operator from $H^{1/2}(\Sigma)$ to $H^1(\Sigma) \subset H^0(\Sigma)$ so that the righthandside of (3) can be written as

$$\int_\Sigma f \mathcal{L} f, \tag{6}$$

where $\mathcal{L} = 2k^2(-\partial/\partial\nu + h) : H^{1/2}(\Sigma) \to H^0(\Sigma)$. We see easily that $\partial/\partial\nu$ is symmetric. In fact, by Green's formula,

$$\begin{aligned} \int_\Sigma g \frac{\partial f}{\partial \nu} - \int_\Sigma f \frac{\partial g}{\partial \nu} &= \int_\Omega W_f \Delta W_2 - \int_\Omega W_g \Delta W_f \\ &= \int_\Omega W_f(-\lambda_1 W_g) - \int_\Omega W_g(-\lambda_1 W_f) \\ &= 0 \end{aligned}$$

Hence, $\mathcal{L}$ is also symmetric and (6) defines a quadratic form in f, denoted $Q(f)$. We shall prove below that Q satisfies an inequality of Garding type

$$Q(f) \geq c_3 |f|^2_{H^1/2(\Sigma)} - c_4 |f|^2_{H^0(\Sigma)} \tag{7}$$

This, together with the symmetry of $\mathcal{L}$ and standard spectral theory, implies that $\mathcal{L}$ has a discrete real spectrum accumulating at $+\infty$. Furthermore, if

$\Sigma'' \subset \Sigma'$ is open and $H_0^{1/2}(\Sigma'') = \{f \in H^{1/2}(\Sigma''); f \text{ is supported on } \Sigma''\}$, then the *index* and *nullity* of Σ'', defined by

$$\operatorname{ind}(\Sigma'') = \dim\{f \in H_0^{1/2}(\Sigma''); Q(f) < 0\},$$
$$\operatorname{nul}(\Sigma'') = \dim\{f \in H_0^{1/2}(\Sigma''); Q(f) = 0\}$$

are both finite. Now consider a smooth deformation $t \in [0,1] \mapsto \Sigma''_t \subset \Sigma$ such that $\Sigma''_0 = \Sigma''$ and $\Sigma''_1 = \{x\} \subset \Sigma''$. With this notation, our result is the following index formula.

Theorem 2. $\operatorname{ind}(\Sigma'') = \sum_{0<t<1} \operatorname{nul}(\Sigma''_t)$.

For completeness, we shall indicate the well-known argument that shows how Shimakura's result cited above follows from our theorem. Suppose that Ω is Σ'-critical and let $x \in \Sigma'$. Let $\Sigma'' \subset \Sigma'$ be a small neighbordhood containing x and Σ''_t a deformation as above. By our theorem, there are $t_1, > \ldots > t_n$ such that

$$\operatorname{ind}(\Sigma'') = \sum_{i=1}^{n} \operatorname{nul}(\Sigma''_{t_i}). \tag{8}$$

Let $t^* > t_n$. Now, if ind $(\Sigma''_{t^*}) > 0$, we can find, again by the index formula, $t^{**} > t^*$ such $\operatorname{nul}(\Sigma''_{t^{**}}) > 0$ and this contradicts (8). Hence, $\operatorname{ind}(\Sigma''_{t^*}) = 0$, i.e., Ω is W-stable with $W = \Sigma''_{t^*}$.

Now we give the proof of our theorem. We shall follow the recipe of ([FT]), so that we have to prove two facts about the operator $\mathcal{L}$, namely, that:

- the quadratic form Q associated to $\mathcal{L}$ satisfies Garding enequality (7);

- $\mathcal{L}$ satisfies the *unique continuation property*, i.e., if $\mathcal{L}f = \mu f$ on Σ, μ real, and $f \equiv 0$ in $U \subset \Sigma$ then $f \equiv 0$ in Σ.

Clearly, it suffices to prove Garding inequality for $-\partial/\partial\nu$ (since $H = \mathcal{L} + \partial/\partial\nu$ is a zero order operator and a zero order perturbation of an operator, satisfying Garding inequality also satisfies the same inequality) and this is an easy consequence of Green's formula and estimates (5). In fact,

$$
\begin{aligned}
-\int_{\Sigma} f\frac{\partial f}{\partial \nu} &= \int_{\Omega} W_f \Delta W_f + \int_{\Omega} |\Delta W_f|^2 \\
&= -\lambda_1 \int_{\Omega} W_f^2 + \int_{\Omega} |\Delta W_f|^2 \\
&= -(\lambda_1 + 1)|W_f|^2_{H^0(\Omega)} + |W_f|^2_{H^1(\Omega)} \\
&= c_3 |f|^2_{H^{1/2}(\Sigma)} - c_4 |f|^2_{H^0(\Sigma)}
\end{aligned}
$$

where $c_3 = 1/c_2^2$ and $c_4 = c_1^2(\lambda_1 + 1)$, as desired. As for the unique continuation property, notice that, under the stated conditions, we also have $\frac{\partial f}{\partial \nu} \equiv 0$ on U. Since the operator $L = \Delta + \lambda_1$ is elliptic, U is *non-characteristic* for L, and we are in a position to apply Holmgren's uniqueness theorem ([H]) to conclude that $W_f \equiv 0$ in a neighborhood of Ω adjacent to U. Now recall that L, as a second order elliptic operator, satisfies the unique continuation property ([H]). Hence, $W_f \equiv 0$ in Ω and from this we get $f \equiv 0$ in Σ. The theorem is proved.

References

[FT] Fried, H. and Thayer, F. J., *An abstract version of the Morese index formula and its application to hypersurfaces of constant mean curvature*, Bol. da Soc. Bras. de Mat., 20, (1979).

[H] Hormander, L., *Linear Partial differential operators*, Springer-Verlag, Berlin, (1969).

[LM] Lions, J. L. and Magenes, E., *Non-homogeneous boundary value problems and applications* I, Springer-Verlag, Belin.

[Se] Serrin, J., *A symmetry problem in potential theory*, Arch. Rat. Mech. and Anal., 43 (1973), 304-318.

[S] Shimakura, N., *La premiere valeur propre du Laplacien pour le probleme de Dirichlet*, J. des Math. Pures et Ap., 62 (1983), 129-152.

Luquésio P. de M. Jorge
Levi L. de Lima
Departamento de Matemática
Universidade Federal do Ceará
Campus do Pici
60455-760 Fortaleza-Ce, Brazil
e-mail: ljorge@lia.ufc.br
e-mail: levi@lia.ufc.br