*Matemática Contemporânea, Vol. 9, 91–110*
http://doi.org/10.21711/231766361995/rmc98


# ON METRICS OF CONSTANT SECTIONAL CURVATURE

**Walterson Pereira Ferreira**

## 1. Introduction

The Intrinsic Generalized Equation was introduced by Campos and Tenenblat in [7] as a class of systems of differential equations which contains the Intrinsic Generalized Wave and sine-Gordon Equations and the Intrinsic Generalized Laplace and Elliptic sinh-Gordon Equations.

The solutions of the Intrinsic Generalized Equation generically define Riemannian metrics with constant sectional curvature $K$, on open subsets $M \subset \mathbf{R}^n$. These Riemannian manifolds $M^n(K)$ of constant sectional curvature K have the extraordinary property of being isometrically immersed (locally) in a $(2n-1)$-dimensional simply connected pseudo-Riemannian manifold $\overline{M}_s^{2n-1}(\overline{K})$ of constant sectional curvature $\overline{K}$, with $K \neq \overline{K}$ and index $s$, $0 \leq s \leq n-1$ [2] (see also [1,5,6,8,10,12,16]).

Campos and Tenenblat in [7] give a Backlund Transformation for the Intrinsic Generalized Equation. Such transformation provides new solutions for the equation from a given one.

In this work, we determine the symmetry group of the Intrinsic Generalized Equation. The symmetry group of a system of differential equations consists of transformations which act on the space of independent and dependent variables for the system. These transformations transform solutions graph on solutions graph of the system. Moreover, the knowledge of such a group has other applications [11]. In this paper, we use this group to determine special solutions of the Intrinsic Generalized Equation, called invariant group solutions.

In section 2, we show that the symmetry group of the Intrinsic Generalized Equation consists only of translations if $K \neq 0$ and of translations and dilations if $K = 0$. In section 3, we determine all the solutions which are invariant under an $(n-1)$-dimensional translation subgroup of the symmetry group.

The submanifolds $M^n(K) \subset \overline{M}_s^{2n-1}(\overline{K})$ associated to such invariant solutions have interesting properties. For instance, when the solutions depend only on one variable many of these submanifolds are toroidal submanifolds [5,6,12]. In the general case, geometrical properties are given by Barbosa, Ferreira and Tenenblat in [2]. There it is shown, for instance, that such submanifolds are foliated by $(n-1)$-dimensional flat submanifolds which have constant mean curvature and each leaf of the foliation is itself foliated by curves of $\overline{M}$ which have constant curvatures.

This work is part of my doctoral thesis at the Universidade de Brasília. I would like to thank professor Keti Tenenblat for proposing the problem, for helpful conversations and encouragement during the preparation of the work.

## 2. The Intrinsic Generalized Equation

The Intrinsic Generalized Equation was defined by Campos and Tenenblat in [7] as the following system of differential equations on $\mathbf{R}^n$

$$vJv^t - 1 = 0 \tag{2.1}$$

$$v^i_{x_j} - v^j h^{ji} = 0\,, \quad i \neq j \tag{2.2}$$

$$v^i_{x_i} + J_{ii} \sum_{s \neq i} J_{ss} v^s h^{is} = 0 \tag{2.3}$$

$$
\begin{aligned}
h^{ij}_{x_i} &+ h^{ji}_{x_j} + \sum_{s\neq i,j} h^{si}h^{sj} + Kv^i v^j = 0\,, \quad i<j && (2.4)\\
h^{ij}_{x_s} &- h^{is}h^{sj} = 0\,, \quad i,j,s \ \ \text{distinct} && (2.5)\\
J_{jj}h^{ij}_{x_j} &+ J_{ii}h^{ji}_{x_i} + \sum_{s\neq i,j} J_{ss}h^{is}h^{js} = 0 \quad i<j && (2.6)
\end{aligned}
$$

where $J=(J_{ij})$ is the $n\times n$ diagonal matrix

$$J=\mathrm{diag}(\underbrace{1,\ldots,1}_{p\ times},\underbrace{-1,\ldots,-1}_{q\ times})$$

with $p+q=n$.

A solution of system (2.1)-(2.6), defined on an open subset $\Omega\subset\mathbf{R}^n$, is a pair $(v,h)$ where $v(x)=(v^1(x),\ldots,v^n(x))$ is a vector field in $\Omega$ and $h(x)=(h^{ij}(x))$ is an off-diagonal $n\times n$ matrix.

A solution of equations (2.1)-(2.6) such that $v^i(x)\neq 0$, $1\leq i\leq n$, on an open subset $\Omega\subset\mathbf{R}^n$, defines a metric on $\Omega$ given by $g_{ij}=\delta_{ij}(v^i)^2$ for which the sectional curvature is constant equal to $K$. In this case, the matrix $h$ is determined by $v$ and the equations (2.3) and (2.6) follow from equations (2.1) and (2.2).

When $J$ is the identity matrix the system (2.1)-(2.6) is called the Generalized Intrinsic sine-Gordon and Wave Equations [4]. Tenenblat and Winternitz in [15] computed the Symmetry Groups of these equations. When $p=1$ and $q=n-1$ the system (2.1)-(2.6) is called the Generalized Intrinsic sinh-Gordon and Laplace Equations [16]. For $n=2\,,p=2$ and $n=2\,,p=1$, respectively, by taking $v=\left(cos\frac{u}{2},sin\frac{u}{2}\right)$ and $v=\left(cosh\frac{u}{2},sinh\frac{u}{2}\right)$, respectively, the equations (2.1)-(2.6) reduce to the classical equations

$$u_{x_1x_1}-u_{x_2x_2}=-Ksinu \quad \text{and} \quad u_{x_1x_1}+u_{x_2x_2}=-Ksinhu\,,$$

respectively.

**Theorem 1.** *The symmetry group of the system (2.1)-(2.6), for $n \geq 3$, is the group of transformations*

$$(x, v, h) \mapsto (e^{\lambda}x + b, v, e^{-\lambda}h),\ \lambda \in \mathbf{R} \quad and \quad b \in \mathbf{R}^n \quad if \quad K = 0$$

*or*

$$(x, v, h) \mapsto (x + b, v, h),\ b \in \mathbf{R}^n \quad if \quad K \neq 0.$$

Initially we show that system (2.1)-(2.6) has maximal rank through the following lemma [11].

**Lemma 2.** *The system (2.1)-(2.6) has maximal rank.*

**Proof.** The jet space for the system (2.1)-(2.6) is the space

$$J^{(1)} = \{(x_i, v^i, h^{ij}, v^i_{x_s}, h^{ij}_{x_s}) : 1 \leq i, j, s \leq n, i \neq j\} \cong \mathbf{R}^{n^3 + n(n+1)}.$$

The function $F(x_i, v^i, h^{ij}, v^i_{x_s}, h^{ij}_{x_s})$ which defines the system as level submanifold of $J^{(1)}$ gets value into $\mathbf{R}^m$ where $m = 1 + n - n^2 + n^3$ is the number of the equations of the system. We observe that the $m$ vectors

$$\frac{\partial F}{\partial v^1}, \frac{\partial F}{\partial v^i_{x_s}}, \frac{\partial F}{\partial h^{ij}_{x_i}}, \quad \text{with} \quad i < j, \frac{\partial F}{\partial h^{ij}_{x_s}}, \quad \text{with}$$

$i, j, s$ distinct and $\dfrac{\partial F}{\partial h^{ij}_{x_j}}$ with $i < j$, are linearly independent in all the points of the $F^{-1}(0)$. So, the system has maximal rank.

**Proof of the Theorem.** Let

$$V = \sum_{i=1}^{n} \xi^i(x, v, h)\frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \phi^i(x, v, h)\frac{\partial}{\partial v^i} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \phi^{ij}(x, v, h)\frac{\partial}{\partial h^{ij}}$$

be a vector field into the space of the independent and dependent variables. The first prolongation of $V$ is the vector field on $J^{(1)}$

$$pr^{(1)}V = V + \sum_{i,s=1}^{n} \phi_i^{(s)}\frac{\partial}{\partial v^i_{x_s}} + \sum_{\substack{i,j,s \\ i \neq j}}^{n} \phi_{ij}^{(s)}\frac{\partial}{\partial h^{ij}_{x_s}}$$

where:

$$\begin{aligned}
\phi_i^{(s)} &= D_s(\phi^i) - \sum_{\ell=1}^{n} D_s(\xi^\ell) v^i_{x_\ell} \\
\phi_{ij}^{(s)} &= D_s(\phi^{ij}) - \sum_{\ell=1}^{n} D_s(\xi^\ell) h^{ij}_{x_\ell} \quad \text{and} \\
D_s &= \frac{\partial}{\partial x_s} + \sum_{\ell=1}^{n} v^\ell_{x_s} \frac{\partial}{\partial v^\ell} + \sum_{\substack{r,t=1 \\ r\neq t}}^{n} h^{rt}_{x_s} \frac{\partial}{\partial h^{rt}}
\end{aligned}$$

We apply $pr^{(1)}V$ to each equation of the system (2.1)-(2.6) and we will make the following substitutions

$$\begin{aligned}
vJv^t &= 1 && (2.7)\\
v^i_{x_j} &= v^j h^{ji}, \quad i \neq j && (2.8)\\
v^i_{x_i} &= -J_{ii} \sum_s J_{ss} v^s h^{is} && (2.9)\\
h^{ij}_{x_i} &= -h^{ji}_{x_j} - \sum_{s\neq i,j} h^{si} h^{sj} - K v^i v^j, \quad i<j && (2.10)\\
h^{ij}_{x_s} &= h^{is} h^{sj}, \quad i,j,s \quad \text{distinct} && (2.11)\\
h^{ij}_{x_j} &= -J_{jj} J_{ii} h^{ji}_{x_i} - \sum_{s\neq i,j} J_{jj} J_{ss} h^{is} h^{js}, \quad i<j && (2.12)
\end{aligned}$$

After this, we obtain a system of polynomial equations on the variables $h^{st}_{x_\ell}$, $\ell = s$ or $\ell = t$ and $s > t$. Equating to zero the coefficients of independent monomials we will obtain a linear system, of partial differential equations, called the determining system, which defines the functions $\xi^i$, $\phi^i$ and $\phi^{ij}$.

By fixing distinct indices $i, j, s$ and by applying $pr^{(1)}V$ to equation (2.5) we obtain

$$\phi^{(s)}_{ij} - h^{is}\phi^{sj} - h^{sj}\phi^{is} = 0 \qquad (2.13)$$

Substituting $\phi^{(s)}_{ij}$ into equation (2.13) and then the relations (2.10)-(2.12) we get a polynomial equation of degree two on the variables $h^{rt}_{x_\ell}$, $\ell = r$ or $\ell = t$ and $r > t$.

We suppose $i < j$. Equating to zero the coefficients of $h^{ji}_{x_j} h^{st}_{x_s}$ with $t < s$ and $h^{ji}_{x_j} h^{ts}_{x_t}$ with $t > s$ we obtain

$$\xi^i_{h^{st}} = 0 \,, \quad \forall t \quad , t \neq s \,. \tag{2.14}$$

The coefficients of $h^{ji}_{x_j} h^{st}_{x_t}$ with $t < s$ and $h^{ji}_{x_j} h^{ts}_{x_s}$ with $t > s$ provide

$$\xi^i_{h^{ts}} = 0 \,, \quad \forall t \quad , t \neq s \,. \tag{2.15}$$

From equations (2.14) and (2.15) we conclude that $\xi^i$ depends only on $x$ and $v$, i.e., $\xi^i = \xi^i(x, v)$.

Next, we consider the first order terms of equation (2.13).
We suppose $i < j$. the coefficients of $h^{st}_{x_s}$ with $t < s$ and $h^{ts}_{x_t}$ with $t > s$ provide

$$\phi^{ij}_{h^{st}} = 0 \quad \forall\, t, \; t \neq s \,. \tag{2.16}$$

Equating to zero the coefficients of $h^{st}_{x_t}$ with $t < s$ and $h^{ts}_{x_s}$ with $t > s$ we obtain

$$\phi^{ij}_{h^{ts}} = 0 \quad \forall t, \quad t \neq s \tag{2.17}$$

From the coefficient of $h^{ji}_{x_j}$ it follows that

$$\xi^i_{x_s} + \sum_\ell v^\ell_{x_s} \xi^i_{v^\ell} = 0 \quad , \qquad i \neq s. \tag{2.18}$$

If $i > j$ we obtain the same equations (2.16)-(2.18). From equations (2.16) and (2.17) it follows that $\phi^{ij}$ depends only on $x, v, h^{ij}$ and $h^{ji}$, i.e.,

$$\phi^{ij} = \phi^{ij}(x, v, h^{ij}, h^{ji}) \quad , \quad i \neq j. \tag{2.19}$$

Substituting (2.8) and (2.9) into equation (2.18) and using the fact that $\xi^i$ depends only on $x$ and $v$ we obtain

$$\begin{aligned} \xi^i_{x_s} &= 0 \quad \forall\, s, \; s \neq i \,. & (2.20) \\ -J_{ss} J_{\ell\ell} v^\ell \xi^i_{v^s} &+ \; v^s \xi^i_{v^\ell} = 0, \; \forall\, s, \; s \neq i & (2.21) \end{aligned}$$

From equation (2.21) we conclude that $\xi^i$ depends on $\sum_{s=1}^n J_{ss}(v^s)^2$ which is equal to 1. So $\xi^i$ does not depend on $v$. Therefore, $\xi^i$ depends only on $x_i$, i.e.,

$$\xi_i = \xi^i(x_i)$$

We fix $i \neq j$ and apply $pr^{(1)}V$ to equation (2.2). Then substituting the relations (2.8)-(2.12) and using information (2.19) we get

$$\phi^i_{h^{jt}} = \phi^i_{h^{tj}} = 0 \quad \forall\, t,\ t \neq j. \tag{2.22}$$

$$-J_{jj}J_{\ell\ell}v^\ell \phi^i_{v^j} + v^j \phi^i_{v^\ell} = 0,\ \forall\, \ell,\ \ell \neq i, j. \tag{2.23}$$

$$\phi^i_{x_j} + (-J_{jj}J_{ii}v^i\phi^i_{v^j} + v^j\phi^i_{v^i} - \xi^j_{x_j}v^j - \phi^j)h^{ji} - v^j\phi^{ji} = 0. \tag{2.24}$$

We conclude from equation (2.23) that $\phi^i$ depends on $\sum\limits_{s\neq i}^{n} J_{ss}(v^s)^2$ which is equal to $1 - J_{ii}(v^i)^2$. So, $\phi^i$ depends only on $x$ and $v^i$, i.e.,

$$\phi^i = \phi^i(x, v^i)\,. \tag{2.25}$$

Taking the derivative of equation (2.24) with respect to $h^{ij}$ and then twice with respect to $h^{ji}$ we obtain

$$\phi^{ji}_{h^{ij}} = 0 \qquad \text{and} \qquad \phi^{ji}_{h^{ji}h^{ji}} = 0.$$

Therefore, $\phi^{ij}$ does not depend on $h^{ji}$ and is linear on $h^{ij}$, i.e.,

$$\phi^{ij} = A^{ij}(x,v)h^{ij} + B^{ij}(x,v),\ \forall\, i,j,\ i \neq j.$$

Using the information above about $\phi^{ij}$, the equation (2.13) provides

$$B^{ij} \ = \ 0,\ \forall\, i,j,\ i \neq j. \tag{2.26}$$

$$A^{ij} \ - \ A^{sj} - A^{is} = \xi^s_{x_s},\ i,j,s \quad \text{distinct}. \tag{2.27}$$

Substituting $\phi^{ij}$ and using (2.25) into equation (2.24) we get

$$\phi^i_{x_j} \ = \ 0,\ \forall\, i,j\,,\ i \neq j. \tag{2.28}$$

$$v^j\phi^i_{v^i} \ - \ \xi^j_{x_j}v^j - \phi^j - v^jA^{ji} = 0, \quad \forall\, i,j,\ i \neq j. \tag{2.29}$$

Fixing indices $i < j$ and applying $pr^{(1)}V$ to equation (2.4) we obtain

$$\phi^{(i)}_{ij} + \phi^{(j)}_{ji} + \sum_{s\neq i,j} \phi^{sj}h^{si} + \sum_{s\neq i,j} \phi^{si}h^{sj} + Kv^j\phi^i + Kv^i\phi^j = 0 \tag{2.30}$$

Substituting $\phi_{ij}^{(i)}, \phi_{ji}^{(j)}$ and the relations (2.8)-(2.12) into the equation (2.30) we obtain from the coefficient of $h_{x_j}^{ji}$ that

$$-A^{ij} + \xi_{x_i}^{i} + A^{ji} - \xi_{x_j}^{j} = 0 \quad , \quad i < j. \tag{2.31}$$

Maintaining $i < j$ and applying $pr^{(1)}V$ to equation (2.6) and using the informations above we get from the coefficient of $h_{x_i}^{ji}$ that

$$-A^{ij} + \xi_{x_j}^{j} + A^{ji} - \xi_{x_i}^{i} = 0, \; i < j. \tag{2.32}$$

Adding the equations (2.31) and (2.32) we obtain

$$A^{ij} = A^{ji} \quad , \quad i \neq j \tag{2.33}$$

and

$$\xi_{x_i}^{i} = \xi_{x_j}^{j}. \tag{2.34}$$

From equation (2.34) and using the fact that $\xi^i$ depends only on $x_i$ we conclude that

$$\xi^i = a x_i + \alpha_i \quad , \quad \forall \;\; 1 \leq i \leq n \tag{2.35}$$

where a and $\alpha_i$ are real constants.

Interchanging $s$ with $i$ on equation (2.27) and considering the equations (2.33) and (2.35) we get

$$A^{sj} - A^{ij} - A^{is} = a \quad , \quad i, j, s \quad \text{distinct.} \tag{2.36}$$

From the equations (2.27) and (2.36) we obtain

$$A^{is} = -a \quad \forall \, s, \quad s \neq i. \tag{2.37}$$

Hence, it follows that

$$\phi^{ij} = -a h^{ij} \quad , \quad i \neq j. \tag{2.38}$$

Substituting the equations (2.35) and (2.37) into equation (2.29), it reduces to

$$\phi^j = v^j \phi_{v^i}^{i} \quad , \quad i \neq j. \tag{2.39}$$

Since $\phi^j$ depends only on $x_j$ and $v^j$ taking the derivatives of equation (2.37) with respect to these variables, we obtain

$$\phi^j = cv^j \quad , \quad c \quad \text{constant.} \tag{2.40}$$

Applying $pr^{(1)}V$ to equation (2.1), we find $c = 0$ and so

$$\phi^i = 0, \quad \forall \quad 1 \leq i \leq n. \tag{2.41}$$

Substituting the equations (2.38) and (2.41) into equation (2.30) we get $a = 0$ if $K \neq 0$.

Therefore, the infinitesimal generators of the symmetry group is given by

$$V = \sum_{i=1}^{n}(ax_i + \alpha_i)\frac{\partial}{\partial x_i} - a\sum_{\substack{i,j \\ i\neq j}} h^{ij}\frac{\partial}{\partial h^{ij}} ,$$

where $a = 0$ if $K \neq 0$.

## 3. Invariant Solutions

Now we compute solutions of the Intrinsic Generalized Equation which are invariant under an $(n-1)$-dimensional translation subgroup of the symmetry group. In all this section, we consider, $p = 1$ and $q = n - 1$. Moreover, also consider without loss of generality only the cases $K = -1, 0, 1$. We observe that similar solutions for $p = n$ were given in [15].

Let

$$V_j = \sum_{i=1}^{n} b_i^j \frac{\partial}{\partial x_i}, \ 1 \leq j \leq n-1, \ b_i^j \in \mathbf{R},$$

be $n - 1$ linearly independent vector fields of the Lie algebra of the symmetry group.

We suppose that these vector fields are invariant by the following function

$$\xi = \sum_{i=\ell}^{m} \alpha_i x_i, \ \alpha_i \in \mathbf{R}\backslash\{0\} \ \forall\, \ell \leq i \leq m \leq n, \quad \ell = 1 \text{ or } 2.$$

In this case, solutions of the form $v^i(\xi)$, $h^{ij}(\xi)$ will be invariant under the translation subgroup associated by $\{V_j\}_{j=1}^{n-1}$.

Moreover, we are only interested in solutions, $v(\xi) = (v^1(\xi), \ldots, v^n(\xi))$ such that $v^i(\xi) \neq 0$, $\forall\ 1 \leq i \leq n$, on an open subset $\Omega \subset \mathbf{R}^n$. In this case, the equations (2.3) and (2.6) are consequence of the remaining equations. So, the system (2.1)-(2.6) is reduced to

$$(v^1)^2 - \sum_{j\geq 2}(v^j)^2 = 1 \tag{3.1}$$

$$v^j_{x_i} = v^i h^{ij}, \quad i \neq j \tag{3.2}$$

$$h^{ij}_{x_i} + h^{ji}_{x_j} + \sum_{s\neq i,j} h^{si}h^{sj} = -Kv^iv^j, \quad i \neq j \tag{3.3}$$

$$h^{ij}_{x_s} = h^{is}h^{sj}, \quad i, j, s \text{ distinct}\,. \tag{3.4}$$

The main result of this section is the theorem below.

**Theorem 3.**

*a) If $n \geq 4$ and $K = -1$ the system (3.1) - (3.4) has solutions of the form $v(\xi)$ if and only if $\ell \leq m \leq \ell + 1$.*

*b) If $n \geq 4$ and $K = 0$ then the system (3.1) - (3.4) has nonconstant solutions of the form $v(\xi)$ if and only if $\ell \leq m \leq \ell + 2$.*

*c) If $n \geq 4$ and $K = 1$ then the system (3.1) - (3.4) has no solutions of the form $v(\xi)$ $\forall\ \ \ell \leq m \leq n$.*

*d) If $n = 2$ or $3$ then the system (3.1) - (3.4) has solutions of the form $v(\xi)$ for any $K$.*

Theorem 3 will be proved by first establishing the cases for which the system (3.1)-(3.4) has no solution. After this, we determine the solutions that exist.

Initially, we observe that there exist constant solutions $v$ of the system (3.1)-(3.4) if and only if $K = 0$. If $v$ is a nonconstant solution, then there are two distinct indices $j$ such that $v^j$ is nonconstant. In this case, without loss of generality, we can suppose that $v^1$ and $v^2$ are nonconstant or $v^2$ and $v^3$ are nonconstant.

The proposition below establishes those cases for which there are no solutions.

**Proposition 4.**

*a) If $n \geq 4$, $m \geq \ell+2$ and $K \neq 0$ then the system (3.1)-(3.4) has no solution of the form $v(\xi)$.*

*b) If $n \geq \ell+3$, $m \geq \ell+3$ and $K = 0$ then the system (3.1)-(3.4) has no nonconstant solution of the form $v(\xi)$.*

*c) If $n \geq 4$ and $K = 1$ then the system (3.1)-(3.4) has no solution of the form $v(\xi)$, $\forall \ell \leq m \leq n$.*

**Proof.** The equations (3.2) provides

$$\begin{aligned} h^{ij} &= \alpha_i \frac{v^j_\xi}{v^i} \quad \text{if} \quad \ell \leq i \leq m. && (3.5) \\ h^{ij} &= 0 \quad \text{if} \quad i > m \quad \text{or} \quad 1 \leq i < \ell. && (3.6) \end{aligned}$$

If $n \geq 3$, $m \geq \ell+1$ and $i, j, s$ are three distinct indices with $\ell \leq i, s \leq m$, then from equations (3.5) and (3.4) we obtain

$$\left(\frac{v^j_\xi}{v^i}\right)_\xi = \frac{v^s_\xi}{v^s}\frac{v^j_\xi}{v^i}. \tag{3.7}$$

From equation (3.7) we conclude that

$$v^j_\xi = p_{jis} v^i v^s, \quad \text{where} \quad p_{jis} \in \mathbf{R}. \tag{3.8}$$

We suppose that $v$ is a nonconstant solution of the system (3.1)-(3.4). We will consider separately the cases $\ell = 1$ or $\ell = 2$.

**Case A.** $\ell = 1$. If $m \geq 4$ it follows from equation (3.8) that there are constants $\lambda_j \neq 0, 1 \leq j \leq m$, and a nonconstant function $\varphi(\xi)$ such that

$$v^j = \lambda_j \varphi(\xi)\,, \quad 1 \leq j \leq m. \tag{3.9}$$

Substituting equation (3.9) into equation (3.7) we obtain

$$\left(\frac{\varphi_\xi}{\varphi}\right)_\xi = \left(\frac{\varphi_\xi}{\varphi}\right)^2. \tag{3.10}$$

From this last equation it follows that

$$\varphi = \frac{a}{\xi + c}\,,$$

where $a$, and $c$ are real constants.

Hence,

$$v^j = \frac{\lambda_j a}{\xi + c}\,, \quad 1 \leq j \leq m. \tag{3.11}$$

and

$$v^j = -\frac{\beta_j a^2}{\xi + c} + \gamma_j\,, \quad m < j \leq n\,, \tag{3.12}$$

where $\beta_j$ and $\gamma_j$ are real constants.

Substituting the equations (3.11) and (3.12) into equation (3.1) we obtain a contradiction.

If $m = 3$ and $v^j_\xi \equiv 0 \;\forall\, j \geq 4$, then equation (3.3), for indices $1 \leq i \leq n$ and $j \geq 4$, reduces to $Kv^iv^j = 0,\; i \neq j$. Hence, if $K \neq 0$ no solutions exist.

Now, we suppose that $m = 3$ and there is an index $j \geq 4$ such that $v^j_\xi \neq 0$. In this case, it follows from equation (3.8) that there are $\alpha$ and $\beta \in \mathbf{R}\backslash\{0\}$ such that

$$v^2 = \beta v^1 \quad \text{and} \quad v^3 = \alpha v^1.$$

Then equation (3.7) provides

$$\left(\frac{v^1_\xi}{v^1}\right)_\xi = \left(\frac{v^1_\xi}{v^1}\right)^2.$$

Therefore,

$$v^1 = \frac{\lambda}{\xi + d}, \quad \lambda \text{ and } d \in \mathbf{R}\backslash\{0\} \text{ with } \lambda \neq 0,$$
$$v^2 = \frac{\beta\lambda}{\xi + d}$$
$$v^3 = \frac{\alpha\lambda}{\xi + d} \quad \text{and} \quad v^j = -\frac{c_j\lambda^2}{\xi + d} + q_j,\ j \geq 4,\ c_j \text{ and } q_j \in \mathbf{R}.$$

Equation (3.1) leads again to a contradiction.

To conclude the proof of Proposition 4, in the case $l = 1$, we observe that Campos [5] showed that the system (3.1)-(3.4) has no solution for $n \geq 4,\ m = 1$ and $K = 1$. Hence, the case $m = 2$ remains to be proved.

Equation (3.3) with $i = 1$ and $j \geq 3$ provides

$$p_{j12}\left(\alpha_1^2\frac{v_\xi^2}{v^1} + \alpha_2^2\frac{v_\xi^1}{v^2}\right) = -v^j \tag{3.13}$$

Considering indices $i, j \geq 3,\ i \neq j$ in equation (3.3), and making use of equation (3.13), we get

$$\left(\alpha_1^2\frac{v_\xi^2}{v^1} + \alpha_2^2\frac{v_\xi^1}{v^2}\right)^2 = -\left(\alpha_1^2(v^2)^2 + \alpha_2^2(v^1)^2\right)$$

which is a contradiction.

**Case B.** $\ell = 2$. We suppose that $v_\xi^1 \not\equiv 0$. If $n \geq 4$ and $m \geq 4$, it follows from equation (3.8) that there are constants $c_j \neq 0,\ 2 \leq j \leq m$, such that

$$v^j = c_j v^2,\ 2 \leq j \leq m.$$

Substituting this last equation into equation (3.7) we obtain

$$v^2 = \frac{b}{\xi + p},$$

where $b$ and $p$ are real constants.

Therefore,

$$v^j = \frac{bc_j}{\xi + p}, \quad 2 \leq j \leq m \quad \text{and}$$
$$v^j = -\frac{q_j b^2}{\xi + p} + r_j,\ m < j \leq n \quad \text{or } j = 1,$$

where $q_j$ and $r_j$ are real constants. Substituting these relations into the system (3.1)-(3.4) we obtain a contradiction.

Now we suppose that $v^1_\xi \equiv 0$. If $n \geq 4$, $m \geq 4$ and $K \neq 0$, then equation (3.3), with the indices $i = 1$ and $j \geq 2$, leads to the following contradiction $Kv^1v^j = 0$.

If $K = 0$, the system (3.1)-(3.4) reduces to the following

$$\begin{aligned} &\sum_{j\geq 2}(v^j)^2 = A^2 \quad , \quad A^2 = (v^1)^2 - 1 \\ &v^j_{x_i} = v^i h^{ij} \quad , \quad i,j \geq 2 \quad \text{and} \quad i \neq j \\ &h^{ij}_{x_i} + h^{ji}_{x_j} + \sum_{s\neq i,j} h^{si}h^{sj} = 0, \quad i \neq j, \quad i,j \geq 2 \\ &h^{ij}_{x_s} = h^{is}h^{js} \quad , \quad i,j,s \text{ distinct, } i,j,s \geq 2 \end{aligned} \tag{3.14}$$

Tenenblat and Winternitz showed in [15] that system (3.14) has no solution if $n \geq 5$ and $m \geq 5$.

To complete the proof of Proposition 4, in the case $l = 2$, the cases $K = 1$ and $2 \leq m \leq 3$ remain to be proved. The equation (3.3) with $i = 1$, $j \geq 4$ provides

$$p_{j23}p_{123}(\alpha_2^2(v^3)^2 + \alpha_3^2(v^2)^2) = -v^1v^j \ , \ j \geq 4 \tag{3.15}$$

Now we get $i = 2$, $j = 1$ or $j \geq 4$ in equation (3.3). Making use of equation (3.4), we get

$$p_{j23}\left(\alpha_2^2\frac{v^3_\xi}{v^2} + \alpha_3^2\frac{v^2_\xi}{v^3}\right) = -v^j \ , \ j = 1 \text{ or } j \geq 4 \tag{3.16}$$

Substituting equation (3.16) into equation (3.15) we obtain a contradiction. The case $m = 2$ is proved in Campos [5].

Now, we will determine those solutions of the form $v(\xi)$ for the system (3.1)-(3.4) that exist. In order to simplify the statements we will indicate henceforth by

$$\xi = \sum_{i=1}^{m} \alpha_i x_i, \ \alpha_i \in \mathbf{R}\backslash\{0\} \ \forall\, 1 \leq i \leq m \leq n$$

and by

$$\eta = \sum_{i=2}^{m} \beta_i x_i, \ \beta_i \in \mathbf{R}\backslash\{0\} \ \forall\, 2 \leq i \leq m \leq n.$$

**Proposition 5.** *If $n \geq 4$, $m = 3$ and $K = 0$, then the nonconstant solutions of the system (3.1)-(3.4), of the form $v(\xi)$, are given by*

$$\begin{aligned}
(v^1_\xi)^2 &= b(a-b)\left((v^1)^2 - \frac{c}{b}\right)\left((v^1)^2 - \frac{a(1+\lambda^2)-c}{a-b}\right) \\
(v^2)^2 &= \frac{b}{a}\left((v^1)^2 - \frac{c}{b}\right) \\
(v^3)^2 &= \frac{a-b}{a}\left((v^1)^2 - \frac{a(1+\lambda^2)-c}{a-b}\right) \\
v^j &= \lambda_j \quad \forall \quad j \geq 4
\end{aligned} \tag{3.17}$$

*where $a, b, \lambda_j \in \mathbf{R}\backslash\{0\}$, $c \in \mathbf{R}$, $\lambda^2 = \sum_{j=4}^{n} \lambda_j^2$, $a \neq b$ and $a, b$ satisfy the equation*

$$(a-b)(b\alpha_1^2 + a\alpha_2^2) + ab\alpha_3^2 = 0.$$

We obtaim real solutions of equations (3.17) in terms of elementary functions for special values of the constants involved in those equations. For instance, for $c = a(1+\lambda^2)$ and $b(a-b) > 0$ we have

$$\begin{aligned}
v^1 &= \epsilon_1 \sqrt{\frac{a}{b}(1+\lambda^2)} \sec\left[\sqrt{a(a-b)(1+\lambda^2)}(\xi - \xi_0)\right] \\
v^2 &= \epsilon_2 \sqrt{1+\lambda^2} \ \mathrm{tg}\ \left[\sqrt{a(a-b)(1+\lambda^2)}(\xi - \xi_0)\right] \\
v^3 &= \epsilon_3 \sqrt{\frac{a-b}{b}(1+\lambda^2)} \left[\sqrt{a(a-b)(1+\lambda^2)}(\xi - \xi_0)\right] \\
v^j &= \lambda_j \, , \ j \geq 4,
\end{aligned}$$

where $\xi_0 \in \mathbf{R}$ and $\epsilon_i = \pm 1$, $i = 1, 2, 3$.

**Proposition 6.** *For $n \geq 4$, $m = 4$ and $K = 0$ the nonconstant solutions for* *the system (3.1)-(3.4), of the form $v(\eta)$, are given by*

$$\begin{aligned}
(v^2_\eta)^2 &= b(b+a)((v^2)^2 - c)\left(\frac{a\lambda^2+bc}{b+a} - (v^2)^2\right) \\
(v^3)^2 &= \frac{b}{a}((v^2)^2 - c) \\
(v^4)^2 &= \frac{b+a}{a}\left(\frac{a\lambda^2+bc}{b+a} - (v^2)^2\right) \\
v^j &= a_j \ , \ j=1 \quad \text{or} \quad 4 \leq j \leq n \ ,
\end{aligned}$$

*where* $a, b, a_j \in \mathbf{R}\backslash\{0\}$, $c \in \mathbf{R}$, $a_1^2 - \sum_{j=5}^{n} a_j^2 > 1$, $\lambda^2 = a_1^2 - \sum_{j=5}^{n} a_j^2 - 1, a \neq -b$ *and* $a, b$ *satisfy the equation*

$$(a+b)(b\alpha_2^2 + a\alpha_3^2) - ab\alpha_4^2 = 0.$$

**Proposition 7.** *The nonconstant solutions of the system (3.1)-(3.4), of the form* $v(\xi)$, *with* $n = m = 3$ *are given by*

$$\begin{aligned}
(v^1_\xi)^2 &= b(a-b)\left((v^1)^2 - \frac{c}{b}\right)\left((v^1)^2 - \frac{a-c}{a-b}\right) \\
(v^2)^2 &= \frac{b}{a}\left((v^1)^2 - \frac{c}{b}\right) \\
(v^3)^2 &= \frac{a-b}{a}\left((v^1)^2 - \frac{a-c}{a-b}\right) \ ,
\end{aligned}$$

*where* $a, b \in \mathbf{R}\backslash\{0\}$, $c \in \mathbf{R}$, $a \neq b$ *and* $a, b$ *satisfy the equation*

$$(a-b)(b\alpha_1^2 + a\alpha_2^2) + ab\alpha_3^2 + K = 0$$

*Moreover, we have additional solutions.*

*For* $K = -1$;

$$\begin{aligned}
v^1 &= A\cosh\left(\frac{\lambda}{\alpha_3}\xi - \xi_0\right) \\
v^2 &= \pm A senh\left(\frac{\lambda}{\alpha_3}\xi - \xi_0\right) \\
v^3 &= \lambda \ ,
\end{aligned}$$

*where* $\lambda \in \mathbf{R}\backslash\{0\}$, $\xi_0 \in \mathbf{R}$ *and* $A = \pm\sqrt{1+\lambda^2}$.

*For $K = 1$;*

$$\begin{aligned} v^1 &= \lambda \\ v^2 &= A\cos\left(\frac{\lambda}{\alpha_1}\xi - \xi_0\right) \\ v^3 &= \pm A \,\mathrm{sen}\left(\frac{\lambda}{\alpha_1}\xi - \xi_0\right), \end{aligned}$$

*where $\lambda \in (1,\infty)$, $\xi_0 \in \mathbf{R}$ and $A = \pm\sqrt{\lambda^2 - 1}$.*

**Proposition 8.** *If $n \geq 3$, $m = 2$ and $K = -1$, then the solutions of system (3.1)-(3.4) of the form $v(\xi)$ are given by*

$$\begin{aligned} (v^1_\xi)^2 &= \frac{1}{\alpha_1^2 + \alpha_2^2}((1 - \lambda^2\alpha_2^2)(v^1)^2 - c)((v^1)^2 - c_1) \\ (v^2)^2 &= \frac{1}{1 + \lambda^2\alpha_2^2}((1 - \lambda^2\alpha_2^2)(v^1)^2 - c) \\ (v^j)^2 &= \frac{b_j^2(\alpha_1^2 + \alpha_2^2)}{1 + \lambda^2\alpha_1^2}((v^1)^2 - A), \end{aligned}$$

*where $\lambda^2\alpha_2^2 - 1 \neq 0$, $\lambda^2 = \sum_{j=3}^{n} b_j^2$ and $b_j \in \mathbf{R}\backslash\{0\}$.*

*If $n = 3$, $c_1 = \dfrac{1 + \lambda^2\alpha_1^2 - c}{\lambda^2(\alpha_1^2 + \alpha_2^2)}$, $A = c_1$ and $c \in \mathbf{R}$.*

*If $n \geq 4$, $c = 1$, $A = \dfrac{\alpha_1^2}{\alpha_1^2 + \alpha_2^2}$ and $c_1 \in \mathbf{R}$.*

**Proposition 9.** *For $K = -1$, $n \geq 3$ and $m = 3$ the solutions of the system (3.1)-(3.4), of the form $v(\eta)$, are given by:*

*(i) If $\alpha_2^2 \neq \alpha_3^2$;*

$$\begin{aligned} (v^3_\eta)^2 &= \frac{1}{\alpha_3^2 - \alpha_2^2}\left(c - (\lambda\alpha_2^2 - 1)(v^3)^2\right)\left((v^3)^2 + c_1\right) \\ (v^2)^2 &= \frac{1}{\lambda\alpha_3^2 - 1}\left(c - (\lambda\alpha_2^2 - 1)(v^3)^2\right) \\ (v^i)^2 &= \frac{\beta_i^2(\alpha_3^2 - \alpha_2^2)}{\lambda\alpha_3^2 - 1}(\gamma + (v^3)^2),\ i = 1 \text{ or } i \geq 4, \end{aligned}$$

*when $\lambda\alpha_3^2 - 1 \neq 0$ where $\lambda = \beta_1^2 - \sum_{j\geq 4}\beta_j^2$ and $\beta_i \in \mathbf{R}\backslash\{0\}$, $i = 1$ or $i \geq 4$.*

*If* $n=3\,,\quad c_1=\dfrac{\lambda\alpha_3^2-1+c}{\lambda(\alpha_3^2-\alpha_2^2)}\,,\quad \gamma=c_1$ *and* $c \in \mathbf{R}$.

*If* $n\geq 4\,,\quad c=1$ *and* $\gamma=\dfrac{\alpha_3^2}{\alpha_3^2-\alpha_2^2}$. .

*Moreover, we also have the solutions for* $\lambda\alpha_3^2-1=0$,

$$v^3=a$$
$$v^2=A senh\left(\frac{a}{\alpha_3}\eta-\eta_0\right)\,,\qquad \eta_0 \in \mathbf{R}.$$
$$v^i=\beta_i\alpha_3 A\cosh\left(\frac{a}{\alpha_3}\eta-\eta_0\right)\,,\ i=1\ or\ i\geq 4\,.$$

*If* $n=3\,,\quad a \in \mathbf{R}\backslash\{0\}\,,\quad \beta_1=\pm\dfrac{1}{\alpha_3}$ *and* $A=\pm\sqrt{1+a^2}$ .

*If* $n\geq 4\,,\quad \beta_i \in \mathbf{R}\backslash\{0\}\,,\quad i=1$ *or* $i\geq 4\,,\quad a^2=\dfrac{\alpha_3^2}{\alpha_2^2-\alpha_3^2}$ *and* $A=\pm\sqrt{1+a^2}$ .

*(ii) If* $\alpha_2^2=\alpha_3^2$ *the system (3.1)-(3.4) has no solution for* $n\geq 4$.

*When* $n=3$ *and* $\alpha_2^2=\alpha_3^2$ *the solutions are given by any three functions* $v^1, v^2$ *and* $v^3$ *satisfying*

$$(v^1)^2-(v^2)^2-(v^3)^2=1$$
$$v^1_\eta=\pm\frac{1}{\alpha_2}v^2v^3\,.$$

The proof of Propositions 5-9 follows from a straightforward computation. More details can be found in [9].

We conclude by observing that results analogous to Theorem 3 and Proposition 4 of this section and to Theorem 2 of [15] can be obtained for the Intrinsic Generalized Equation for $p\neq 1$ or $p\neq n$.

## References

[1] Aminov, Ju. A, *On the imersion of domains n-Lobaschevsky space* $(2n-1)$-*dimensional euclidian space*, Soviet Math. Dokl. 18 (1977), 1210-1213.

[2] Barbosa, J.L., Ferreira, W., Tenenblat, K., *Submanifolds of Constant Sectional Curvature in Pseudo-Riemannian Manifolds*, preprint UnB.

[3] Bianchi, L., *Lezione di Geometria Differenciale*, Bologna, Nila Zanihell ed., (1927).

[4] Beals, R., Tenenblat, K., *An Instrinsic Generalization for the Wave and sine-Gordon Equations*, Differential Geometry, B. Lawson et all eds., Pitman Monographs and Surveys in Pure an Applied Math. 52 (1991), 25-46.

[5] Campos, P.T.A., *Sobre as Generalizações Geométricas das Equações de Laplace e sinh-Gordon Elíptica*, PhD. Thesis, Universidade de Brasília, (1992).

[6] Campos, P.T.A., *Submanifolds of Constant Non Negative Curvature*, Mat. Contemp. 4 (1993), 55-73.

[7] Campos, P.T.A., Tenenblat, K., *Backlund Transformations for a Class of Systems of Differential Equations*, Geom. and Funct. Analysis 4 (1994), 270-287.

[8] Dajczer, M., Tojeiro, R., *Isometric Immersions and the Generalized Laplace and sinh-Gordon Equations*, Preprint IMPA.

[9] Ferreira, W., *Soluções Invariantes pelos Grupos de Simetria de Lie das Equações Generalizadas Intrínsecas de Laplace e sinh-Gordon Elíptica e Propriedades Geométricas das Subvariedades Associadas*, PhD. Thesis, Universidade de Brasília, (1994).

[10 ] Moore, J.D., *Submanifolds of Constant Positive Curvature*, Duke Math. J. 44 (1977), 449-484.

[11] Olver, P. J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, (1986).

[12] Rabelo, M.L., Tenenblat, K., *Submanifolds of Constant Nonpositive Curvature*, Mat. Contemp. 1 (1991), 71-81.

[13] Tenenblat, K., *Backlund's theorem for submanifolds of space forms and a generalized wave equation*, Bol. Soc. Bras. Mat. 16 (1985), 69-94.

[14] Tenenblat, K., Terng, C.L., *Backlund's theorem of n-dimensional submanifolds of* $\mathbf{R}^{2n-1}$. Ann. of Math. 111 (1980), 447-490.

[15] Tenenblat, K., Winternitz, P., *On the Symmetry Groups of the Intrinsic Generalized Wave and sine-Gordon Equations*, J. Math. Phys. 34 (1993), 3527-3542.

[16] Tojeiro, R., *Imersões isométricas entre espaços de curvatura constante*, PhD. Thesis, IMPA (1991).

IMF-Departamento de Matemática
Universidade Federal de Goiás
Campus II - Samambaia
74001-970 - Goiânia, GO
Brazil