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SOME RESULTS ON CONFORMALLY FLAT SUBMANIFOLDS

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An immersed submanifold $f\colon M^n \to \mathbf{R}^{n+p}$ into Euclidean space, endowed with the induced metric, is said to be *conformally flat* if each point has a neighborhood conformal to $\mathbf{R}^n$. Around 1919, nonflat conformally flat hypersurfaces ($p = 1$, $n \geq 4$) were completely described by E. Cartan ([**Ca**$_2$]) as being any envelope of a 1–parameter family of spheres. In this case, the geometric parametric description (see [**CDM**], [**AD**$_2$] or [**Da**]) is an immediate consequence of the existence, at any point, of a principal curvature of multiplicity at least $n-1$.

For higher but still low codimension, namely, $p \leq n-3$, from the work of Moore ([**Mo**]) we know that at each point x there is an *umbilical* subspace $\mathcal{U}(x) \subset T_xM$ such that $\dim \mathcal{U}(x) \geq n-p$. This means that there is $\eta \in T^{\perp}_{f(x)}M$ unitary and $\lambda \in \mathbf{R}$, $\lambda \geq 0$, so that the second fundamental form satisfies

$$\alpha_f(Z,X) = \lambda\langle Z,X\rangle\eta, \quad \forall Z \in \mathcal{U}(x),\ \forall X \in T_xM. \tag{1}$$

It is a well known fact that, on any open subset where the *index of conformal nullity* $\nu^c_f(x) := \dim \mathcal{U}(x)$ is constant, the umbilical distribution $\mathcal{U}$ is integrable its leaves being extrinsic spheres in M^n and part of round spheres in $\mathbf{R}^{n+p}$. Recall that an *extrinsic sphere* Σ of a Riemannian manifold M^n is an umbilical submanifold with parallel mean curvature vector such that the sectional curvature of M^n is constant along planes tangent to Σ.

On the other hand, we know ([**AD**$_1$], [**Da**]) that a simply connected Riemannian manifold $M^n, n \geq 3$, is conformally flat if and only if it can be realized as a hypersurface of the light cone $\mathbf{V}^{n+1}$ of the standard flat Lorentzian space

$\mathbf{L}^{n+2}$. Recall that

$$\mathbf{V}^{n+1} = \{X \in \mathbf{L}^{n+2} : \langle X, X \rangle = 0, X \neq 0\}.$$

Hence, in order to obtain an example of a conformally flat submanifold M^n of $\mathbf{R}^{n+p}$, it suffices to produce a Riemannian manifold N^{n+1} which admits isometric immersions $F: N^{n+1} \to \mathbf{R}^{n+p}$ and $G: N^{n+1} \to \mathbf{L}^{n+2}$ and then take M^n as the intersection $G(N^{n+1}) \cap \mathbf{V}^{n+1}$.

Our first main result is that, in fact, the above procedure for $p \leq n - 3$ generates all simply connected examples. In particular, Moore's spherical foliation is nothing else but the intersection with $\mathbf{V}^{n+1}$ of the (at least $(n-p+1)$–dimensional) relative nullity foliation common to F and G.

We say that an isometric immersion $F: N^{n+1} \to \widetilde{N}^{n+p}$ *extends* an isometric immersion $f: M^n \to \widetilde{N}^{n+p}$ if there exists an isometric embedding of M^n into N^{n+1} such that $F|_M = f$.

Theorem 1. *Let $f: M^n \to \mathbf{R}^{n+p}$, $n \geq 5$, $p \leq n-3$, be a simply connected conformally flat submanifold without flat points. If f has constant index of conformal nullity $\nu_f^c = \ell$, then there exist an extension $F: N^{n+1} \to \mathbf{R}^{n+p}$ of f and an isometric immersion $G: N^{n+1} \to \mathbf{L}^{n+2}$ so that $M^n = G(N^{n+1}) \cap \mathbf{V}^{n+1}$. Moreover, F and G carry a common $(\ell+1)$–dimensional relative nullity foliation.*

The maps F and G in the above are produced by "replacing" the extrinsic spheres by the affine subspaces of one dimension higher which contain them.

The rest of the paper is devoted to the classification of all local conformally flat submanifolds in codimension $p = 2$. In fact, this goal is achieved by two different means. Our first approach consists in putting together the above result with a description of all Riemannian manifolds N^m which can be realized, simultaneously, as hypersurfaces in $\mathbf{R}^{m+1}$ and $\mathbf{L}^{m+1}$.

This last result is of independent interest and has other consequences. For example, it reveals a completely unexpected strong relation with the classical

Sbrana–Cartan theory ([**Sb**], [**Ca**$_1$]) of isometrically deformable Euclidean hypersurfaces. It turns out that in order to admit an isometric immersion into $\mathbf{L}^{m+1}$, a nonflat hypersurface N^m of $\mathbf{R}^{m+1}$ is either in one of three (out of five) classes of deformable Sbrana–Cartan hypersurfaces or has a similar structure as the elements of a fourth class. As a consequence, we will see that in codimension 2 any 'generic' conformally flat submanifold either has as many isometric deformations as a certain surface in $\mathbf{R}^3$ or in the sphere $\mathbf{S}^3$, admits precisely a 1–parameter family of deformations or is isometrically rigid.

Our classification of Riemannian manifolds which can be realized as a hypersurface in Euclidean and Lorentzian spaces simultaneously, makes use of a special class of spherical and hyperbolic surfaces which we describe next.

Associated to a coordinate system (u,v) for a surface $h\colon V^2 \to \mathbf{S}^m(1)$, $m \geq 2$, we denote by Γ^1, Γ^2, the connection functions of the Riemannian connection ∇' of V^2 determined by

$$\nabla'_{\partial_u}\partial_v = \Gamma^1\partial_u + \Gamma^2\partial_v,$$

where ∂_u, ∂_v stand for the coordinate vector fields. Notice that h is just a coordinate system for $\mathbf{S}^m(1)$ when $m = 2$. System (u,v) is called *conjugate* whenever the second fundamental form of h verifies

$$\alpha_h(\partial_u, \partial_v) = 0. \tag{2}$$

Equation (2), in terms of the coordinate functions $h = (h^1, \ldots, h^{m+1})$ in Euclidean space, takes the form

$$\mathrm{Hess}_{h^j}(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle h^j = 0, \quad 1 \leq j \leq m+1. \tag{3}$$

Given a spherical surface with conjugate coordinates $\{h, (u,v)\}$, we call *associated function* any negative solution τ of the system of equations

$$\begin{cases} \tau_u = 2\Gamma^2\tau(1-\tau) \\ \tau_v = 2\Gamma^1(1-\tau). \end{cases} \tag{4}$$

System (4) has the integrability condition

$$(\Gamma^2_v - 2\Gamma^1\Gamma^2)\tau - \Gamma^1_u + 2\Gamma^1\Gamma^2 = 0.$$

We then say that $\{h,(u,v)\}$ a surface of *first type* when its metric verifies

$$\Gamma^1_u = \Gamma^2_v = 2\Gamma^1\Gamma^2. \tag{5}$$

If not of first type, we call it of *second type* when

$$\tau = \frac{\Gamma^1_u - 2\Gamma^1\Gamma^2}{\Gamma^2_v - 2\Gamma^1\Gamma^2} \tag{6}$$

is a (necessarily unique) associated function.

Remark 2. In their classification of isometrically deformable Euclidean hypersurfaces without flat points, Sbrana ([**Sb**]) and Cartan ([$\mathbf{Ca_1}$]) considered two classes of spherical surfaces (called by Sbrana of first and second species) which carry either real or complex conjugate coordinates. While surfaces of first type are nothing else but surfaces of first species for real conjugate coordinates, surfaces of second type are not of second species but of a similar kind. Namely, τ given by (6) is still a solution of (4) but it is a positive one.

The notion of spherical surfaces of first or second type extends to surfaces k: $V^2 \to \mathbf{H}^m(-1) \subset \mathbf{L}^{m+1}$ in hyperbolic space. In this case, equation (3) takes the form

$$\mathrm{Hess}_{k^j}(\partial_u, \partial_v) - \langle \partial_u, \partial_v \rangle k^j = 0, \quad 1 \le j \le m+1.$$

Now, consider a hypersurface F: $N^m \to \mathbf{R}^{m+1}$ with constant index of relative nullity $\nu_F = \ell$, $0 \le \ell \le m-1$. In this situation, we may locally parametrize F by means of the *Gauss parametrization* which we briefly describe next for later use and refer to [**DG**] for further details.

Let $V^{m-\ell}$ be the quotient space of relative nullity leaves in an open subset $U \subset N^m$ with projection π: $U \to V^{m-\ell}$. The Gauss map ξ: $U \to \mathbf{S}^m(1)$ induces an isometric (with the induced metric) immersion h: $V^{m-\ell} \to \mathbf{S}^m(1)$ so that $h \circ \pi = \xi$. Let $\mathcal{N}$ denote the normal bundle of h in $\mathbf{S}^m(1) \subset \mathbf{R}^{m+1}$ and let γ be the "support function" defined by $\gamma \circ \pi = \langle F, \xi \rangle$. The Gauss parametrization Ψ: $\mathcal{N} \to \mathbf{R}^{m+1}$ is given by

$$\Psi(\vartheta) = \gamma(x)h(x) + \mathrm{grad}\,\gamma(x) + \vartheta, \quad x = \pi(\vartheta),$$

where we fiberwise identify the affine relative nullity bundle over a cross section with the vector bundle $\mathcal{N}$ by parallel transport in Euclidean space.

Our first approach to the classification of conformally flat submanifolds in codimension 2 concludes with the following result.

Theorem 3. *Let N^m, $m \geq 2$, be a Riemannian manifold without flat points and let $F\colon N^m \to \mathbf{R}^{m+1}$ and $G\colon N^m \to \mathbf{L}^{m+1}$ be isometric immersions. Then, F is locally given by the Gauss parametrization $\Psi\colon \mathcal{N} \to \mathbf{R}^{m+1}$,*

$$\Psi(\vartheta) = \gamma h + grad\,\gamma + \vartheta,$$

in terms of a surface of first or second type $\{h, (u, v)\}$ and a solution γ of the differential equation

$$Hess_\gamma(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle \gamma = 0. \tag{7}$$

Conversely, any parametrized hypersurface in $\mathbf{R}^{m+1}$ as above can be locally isometrically immersed in $\mathbf{L}^{m+1}$. A similar description holds for G.

Remarks 4. 1) The intersection of $G(N^{m+1})$ with a foliation of $\mathbf{L}^{m+2}$ by light cones provides a local foliation of N^{m+1} by m-dimensional conformally flat submanifolds of $\mathbf{R}^{m+2}$. See also Theorem 1.10 of [**AD**$_1$].
2) Satisfying the assumptions of the above result we have the following special examples:

i) $N^m = N^2 \times \mathbf{R}^{m-2}$, where N^2 admits isometric immersions $F'\colon N^2 \to \mathbf{R}^3$ and $G'\colon N^2 \to \mathbf{L}^3$, with $F = F' \times \mathrm{Id}$ and $G = G' \times \mathrm{Id}$;

ii) $N^m = CN^2 \times \mathbf{R}^{m-3}$, where $CN^2 \cong N^2 \times \mathbf{R}_+$ is a cone over a surface N^2 which admits isometric immersions $F'\colon N^2 \to \mathbf{S}^3(1) \subset \mathbf{R}^4$ and $G'\colon N^2 \to \mathbf{S}^3_1(1) \subset \mathbf{L}^4$ into Lorentzian unit sphere, where $F = CF' \times \mathrm{Id}$ and $G = CG' \times \mathrm{Id}$.

We introduce next two new definitions in order to deal with the rigidity of conformally flat submanifolds in codimension 2. We call a conformally flat submanifold $f\colon M^n \to \mathbf{R}^{n+2}$, $n \geq 5$, *generic* when its umbilical direction $\eta \in T_f^\perp M$

(recall (1)) possesses everywhere a nonzero principal curvature λ of multiplicity $n-2$. We say that a generic f is *surface-like* if its isometric extension $F\colon N^{n+1} \to \mathbf{R}^{n+2}$ is as either one of the examples in Remark 4.2.

Using the theory of deformable Euclidean hypersurfaces of Sbrana and Cartan, we conclude from Theorem 3 the following.

Theorem 5. *Any local isometric deformation of a generic conformally flat submanifold $f\colon M^n \to \mathbf{R}^{n+2}$, $n \geq 5$, is the restriction to M^n of an isometric deformation of its isometric extension $F\colon N^{n+1} \to \mathbf{R}^{n+2}$. Moreover, if nowhere surface-like, f admits, precisely, a 1-parameter family of isometric deformations when F is generated by a surface of first type and is isometrically rigid otherwise. In the surface-like situation, all deformations of f are determined by isometric deformations of the surface in the first factor of F.*

Making use of our previous results, we are now able to describe all nonflat codimension 2 conformally flat submanifolds in a parametric form. This second approach turns out to be much more involved than Cartan's description because, aside from hypersurfaces, the existence alone of a spherical foliation is far from being sufficient to conclude conformal flatness.

From now on, by $f\colon M^n \to \mathbf{R}^{n+2}$ being a *composition* we mean that there exist $U \subset \mathbf{R}^{n+1}$ open and isometric immersions $\tilde{f}\colon M^n \to U$ and $H\colon U \to \mathbf{R}^{n+2}$ such that $f = H \circ \tilde{f}$.

Proposition 6. *Any conformally flat submanifold $f\colon M^n \to \mathbf{R}^{n+2}$, $n \geq 5$, without flat points is locally along an open dense subset either generic or a composition.*

Notice that compositions, as in the above result, can easily be described parametrically by putting together Cartan's parametrization of conformally flat hypersurfaces with the Gauss parametrization. We now consider the generic case.

Theorem 7. *Let $h\colon V^2 \to \mathbf{S}^{n+1}(1) \subset \mathbf{R}^{n+2}$ be a surface of first or second type with conjugate coordinates (u, v). For a given associated function τ, let Θ^* be the adjoint to the tensor $\Theta\colon TV \to TV$ defined by*

$$\Theta\partial_u = \frac{1}{\theta}\partial_u, \quad \Theta\partial_v = -\theta\partial_v,$$

where $\theta = \sqrt{-\tau}$. Moreover, let ρ be a solution of the differential equation

$$\rho_{uv} + \theta^2\Gamma^2\rho_v + \frac{1}{\theta^2}\Gamma^1\rho_u + \rho\langle\partial_u, \partial_v\rangle = 0, \tag{8}$$

and let $\beta\colon V^2 \to \mathbf{R}^{n+2}$ be a solution, unique up to translations, of the completely integrable system of first order

$$\begin{cases} \beta_u = \theta\rho h_u - \dfrac{\rho_u}{\theta}h \\ \beta_v = -\dfrac{\rho}{\theta}h_v + \theta\rho_v h. \end{cases}$$

Then, on the open subset of regular points, the map $\varphi\colon \mathcal{N}_1 \to \mathbf{R}^{n+2}$, defined on the unit normal bundle $\mathcal{N}_1$ of h in the sphere and given by

$$\varphi(w) = \beta - \Theta^* grad\,\rho + \sqrt{\rho^2 - \|\Theta^* grad\,\rho\|^2}\; w \tag{9}$$

is a parametrization of a generic n-dimensional conformally flat submanifold of $\mathbf{R}^{n+2}$. Conversely, for $n \geq 5$, any generic conformally flat submanifold $f\colon M^n \to \mathbf{R}^{n+2}$ can be locally parametrized this way.

It is an interesting fact that the above description can also be obtained independently of our previous results. This (not straightforward) computation makes use of the fact that the second fundamental form along the orthogonal complement of the umbilical direction has rank 2.

Our parametrization and an observation due to Cartan ([Ca$_1$]) now enable us to explicitly construct a large family of what seems to be the first known generic examples.

Examples 8. Take a spherical surface $h\colon V^2 \to \mathbf{S}^{n+1}(1) \subset \mathbf{R}^{n+2}$ of first type satisfying the additional condition

$$\Gamma^1\Gamma^2 + F = 0, \quad F = \langle\partial_u, \partial_v\rangle. \tag{10}$$

As already observed by Cartan, in this situation h can be integrated as

$$h(u,v)=\frac{(\alpha_1(u),\alpha_2(v))}{\sqrt{\|\alpha_1(u)\|^2+\|\alpha_2(v)\|^2}},$$

where $\alpha_j : I_j \to E_j$ are regular curves, with $E_1 \oplus^{\perp} E_2 = \mathbf{R}^{n+2}$.

Assume, in addition, that the α_j's are spherical curves, i.e., $\|\alpha_j\| = c_j$, $c_j \in \mathbf{R}_+$ with $c_1^2 + c_2^2 = 1$. Then, $F = \Gamma^1 = \Gamma^2 = 0$, $\tau = -k^2$ is constant and

$$\rho(u,v) = \rho_1(u) + \rho_2(v).$$

We conclude that

$$\begin{aligned}\varphi(w) = \frac{1}{k}\left(k^2\rho\alpha_1 - (1+k^2)\int \rho_1'\alpha_1 du\ , -\rho\alpha_2 - (1+k^2)\int \rho_2'\alpha_2 dv\right)\\ + \phi + \sqrt{\rho^2 - \|\phi\|^2}\, w,\end{aligned}$$

where

$$\phi = \left(\frac{\rho_1'\alpha_1'}{k\|\alpha_1'\|^2}, -\frac{k\rho_2'\alpha_2'}{\|\alpha_2'\|^2}\right),$$

parametrizes a 1–parameter family of generic conformally flat submanifolds.

The above set of examples contains very simple ones obtained by taking $\rho = 1$. These examples are even simpler if the α_j's are taken to be circles.

Finally, in order to complete our classification, we now describe parametrically all flat Euclidean submanifolds in codimension 2 which cannot be obtained as compositions. Arguments here will be quite sketchy in regard of their similarity with the ones in [**CD**].

We call h: $V^2 \to \mathbf{S}^{n+1}(1)$ a *surface of type C* when there exists a conjugate coordinate system (u,v) such that

$$\Gamma^1 = 0 \quad \text{and} \quad \alpha_h(\partial_v,\partial_v) \neq 0.$$

We can state our last result.

Theorem 9. *Let h: $V^2 \to \mathbf{S}^{n+1}(1)$ be a surface of type C, and let γ be any solution of equation*

$$Hess_\gamma(\partial_u,\partial_v) + \langle\partial_u,\partial_v\rangle\gamma = 0.$$

Let $\delta \in T_h^\perp V$ and $\theta \in C^\infty(V)$ be given by

$$\delta = \frac{\alpha_h(\partial_v, \partial_v)}{\|\alpha_h(\partial_v, \partial_v)\|}, \quad \theta = \frac{Hess_\gamma(\partial_v, \partial_v) + \langle \partial_v, \partial_v \rangle \gamma}{\|\alpha_h(\partial_v, \partial_v)\|}.$$

Then, on the open subset of regular points, the map φ: $N_\delta \to \mathbf{R}^{n+2}$, defined on $N_\delta = \{\beta \in T_h^\perp V : \langle \delta, \beta \rangle = 0\}$ and given by

$$\varphi(\beta) = \gamma h + grad\gamma + \theta\delta + \beta,$$

is a parametrization of an n-dimensional flat submanifold of $\mathbf{R}^{n+2}$ which is nowhere a composition. Conversely, any such submanifold can be locally parametrized this way along an open dense subset.

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