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ESCHENBURG SPACES AND BUNDLES OVER $\mathbb{C}P^2$

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Abstract

In 1982, J.-H. Eschenburg produced an infinite family of compact 7-dimensional positively curved manifolds that admit no homogeneous structure. We show here that an infinite subfamily of these manifolds fiber over $\mathbb{C}P^2$ with lens spaces as fibers.

1. Introduction.

Ever since O'Neill's paper [O] showing that Riemannian submersions increase sectional curvature and certainly after Cheeger and Gromoll's [C-G] work on open manifolds with non negative sectional curvature, it became very interesting to know which manifolds with positive or non negative curvature fiber over simpler ones with the same property. At the same time one would like to know which bundles over manifolds with positive curvature admit metrics with a similar property [S-W], [D-R], [R_1], [R_2], [Y; problem 6]. Though recent revival of interest produced some results [S-W], [C-D-R], [Wal], the answers to these questions are not known. Theorems to be proved to this effect in the future, may have to be conceived through examples that are, somehow, in short supply. In this note we show that some of the most interesting examples of compact manifolds with positive sectional curvature [A-W], [K-S], [E_1], [E_2] fiber over $\mathbb{C}P^2$.

These examples should be contrasted with the scarsity of principal bundles with positive curvature [We], [D-R], [C_1].

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We want to thank Prof. J.-H. Eschenburg for bringing to our attention in late September 1994 his paper [E3] containing similar results.

Definition 1. *The Wallach spaces $M_{k,\ell}$ are the quotient of $SU(3)$ by 1-parameter subgroup*

$$T_{k,\ell} = \left\{ \begin{pmatrix} z^k & 0 & 0 \\ 0 & z^\ell & 0 \\ 0 & 0 & z^{-(k+\ell)} \end{pmatrix}, \ z \text{ in } S^1, \ \ k, \ell \text{ in } \mathbb{Z} \right\}.$$

They constitute a 7-dimensional family of positively curved homogeneous spaces with infinitely many distinct topological types [A-W] and very interesting properties of their diffeomorphism classes [K-S].

Proposition [E2]. *The spaces $M_{k,\ell}$ fiber over $\mathbb{C}P^2$.*

Proof. Consider the canonical bundle

$$U(2) - - - - SU(3) \to \mathbb{C}P^2 ,$$

observe that $T_{k,\ell} \subseteq U(2)$ and that the associate bundle

$$U(2)\big/_{T_{k,\ell}} - - - - SU(3) \underset{U(2)}{\times} \left(U(2)\big/_{T_{k,\ell}}\right) \to \mathbb{C}P^2$$

has total space $M_{k,\ell}$ and fibers lens spaces for $k + \ell \neq 0$. □

Eschenburg [E1] constructed another family of 7-dimensional spaces that contains $M_{k,\ell}$ together with some topologically more complex spaces. His idea is to act on $SU(3)$ freely by S^1 as follows:

Let $S^1 = T_{k,\ell,p,q} \subseteq SU(3) \times SU(3), \quad k, \ell, p, q$ in $\mathbb{Z}$, as

$$T_{k,\ell,p,q} = \left\{ \left(\begin{pmatrix} z^k & 0 & 0 \\ 0 & z^\ell & 0 \\ 0 & 0 & \overline{z}^{(k+\ell)} \end{pmatrix}, \begin{pmatrix} z^p & 0 & 0 \\ 0 & z^q & 0 \\ 0 & 0 & \overline{z}^{(p+q)} \end{pmatrix} \right), \ z \text{ in } S^1 \right\}.$$

$SU(3)\times SU(3)$ acts on itself by $(A,B)C = ACB^{-1}$ and $T_{k,\ell,p,q}$ acts as a subgroup of $SU(3)\times SU(3)$. This action is not free in general.

Proposition [E₁]. *$T_{k,\ell,p,q}$ acts freely on $SU(3)$ if and only if the following six pairs of integers are relatively prime.*

$$\text{(E)} \qquad \begin{cases} \text{(i) } (k-p,\ell-q), & \text{(ii) } (k-p,\ell+p+q), \quad \text{(iii) } (k+p+q,\ell-p) \\ \text{(iv) } (k-q,\ell-p), & \text{(v) } (k-q,\ell+p+q), \quad \text{(v) } (k+p+q,\ell-q). \end{cases}$$

Under these conditions the Eschenburg spaces $M_{k,\ell,p,q} := SU(3)\big/_{T_{k,\ell,p,q}}$ are defined and are not, in general, homogeneous spaces, which makes the study of their topology considerably more complicated than that of the Wallach spaces. Their "metric proximity" to the Wallach spaces, in a sense defined in [E₁], implies that an infinite number of Eschenburg spaces admit Riemannian metrics of positive sectional curvature. Following up the analogy a little further, it is natural to ask whether they fiber over $\mathbb{C}P^2$, just as the Wallach spaces do and in the afirmative case, what are the fibers?

The subgroup $U(2)$ of $SU(3)$ consisting of all $C = \begin{pmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & \overline{a}\overline{d}-\overline{b}\overline{c} \end{pmatrix}$ is invariant under Eschenburg's action, which is free if and only if $(k-q,\ell-p)$ and $(k-p,\ell-q)$ are relatively prime pairs of integers. Assuming they are we have the principal bundle

$$T_{k,\ell,p,q} ----U(2) \to N_{k,\ell,p,q} = U(2)\big/_{T_{k,\ell,p,q}}.$$

In order to investigate the quotients we proceed to examine the intersection of the orbit $\mathcal{O}(A)$ of an arbitrary element A in $U(2)$ with the subgroup $SU(2)$ of $U(2)$. To this effect observe that any A in $U(2)$ can be written in the form

$$A = \begin{pmatrix} wx & -\overline{y} \\ wy & \overline{x} \end{pmatrix}, \quad w = \det(A) \text{ in } S^1 \text{ and } x\overline{x}+y\overline{y} = 1.$$

The orbit $\mathcal{O}(A)$ becomes then

$$\begin{pmatrix} wxw^{k-p} & -\overline{y}z^{k-q} \\ wyw^{\ell-p} & \overline{x}z^{\ell-q} \end{pmatrix} \quad \text{with } z \text{ in } S^1$$

or

$$\begin{pmatrix} wz^{k+\ell-p-q}xz^{-\ell+q} & -\overline{y}z^{k-q} \\ wz^{k+\ell-p-q}yz^{-k+q} & \overline{x}z^{\ell-q} \end{pmatrix}$$

and its intersection with $SU(2)$, that consists of matrices of the form $\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$, occurs when

$wz^{k+\ell-p-q} = 1$, i.e., for z a $(k+\ell-p-q)$-root of $\det(A)^{-1}$. We must note that $k+\ell-p-q=0$ cannot occur by i) and iv) of Eschenburg's condition (E). As $k+\ell-p-q \neq 0$ there are $k+\ell-p-q$ roots (we have assumed, without loss of generality that $p+q > k+\ell$). This says that each S^1-orbit of the subgroup $U(2)$ intersects $SU(2)$ at precisely the same number of points implying that $N_{k,\ell,p,q}$ is diffeomorphic to $SU(2)/\mathbb{Z}_{p+q-k-\ell}$, i.e., a generalized lens space ([D-N-F], p. 60).

It is not however true, in general, that any S^1-orbit of A in $SU(3)$ is contained in a $U(2)$-orbit of the same A. Immitating the case of the Wallach examples, we have the diagram:

$$\begin{array}{ccccc} T_{k,\ell,p,q} & \cdots & U(2) & \longrightarrow & N_{k,\ell,p,q} \\ & & \vdots & & \\ T_{k,\ell,p,q} & \cdots & SU(3) & \xrightarrow{\pi_1} & M_{k,\ell,p,q} \\ & & \pi_2 \downarrow & & \\ & & \mathbb{C}P^2 & & \end{array}$$

and we check that the fibers of π_1 are not, in general, included in the fibers of π_2. To that efect, since the fibers are the orbits, we have the $U(2)$-orbit of A in $SU(3)$, $\mathcal{O}_{U(2)}A = \{AB, \ B \text{ in } U(2)\}$, and the S^1, or $T_{k,\ell,p,q}$, orbit of A,

$$\mathcal{O}_{S^1}(A) = \{(z^{k,\ell})A(\overline{z}^{p,q}) \ , \quad z \text{ in } S^1\},$$

where $(z^{k,\ell})A(\overline{z}^{p,q})$ denotes Eschenburg's action, and $\mathcal{O}_{S^1}(A) \subseteq \mathcal{O}_{U(2)}(A)$ if and only if for every z in S^1 the element

$$A^{-1}(z^{k,\ell})A(\overline{z}^{p,q}) \text{ is in } U(2) \subseteq SU(3).$$

Obviously this occurs for all A in $U(2)$ and for some others outside $U(2)$. This condition would be valid for all A if and only if $p=3s, k=\ell=2s$ and $q=-6s$ for some s in $\mathbb{Z}$, which violates conditions (E).

There is, therefore, no direct analogy with the fibering of the Wallach spaces over $\mathbb{C}P^2$. In some cases there is, however, an indirect one and we proceed to exhibit the calculations for a characteristic case.

To do this we replace the 4-dimensional subgroup $U(2)$ by one of its finite coverings H, a subgroup of $U(3) \times U(3)$ that acts freely on $SU(3)$ (see also §42 of [E₂]).

Let $C_z = \begin{pmatrix} z^2 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D_z = \begin{pmatrix} z & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z \end{pmatrix}$ and define

$$H = \{(C_z, D_z E) \ , \quad z \ \text{ in } \ S^1 \quad , \quad E \ \text{ in } \ SU(2)\}.$$

Observe that H is closed under multiplication:

$$(C_z, D_z E)(C_w, D_w F) = (C_{zw}, D_{zw} G)$$

where $G = \phi(\overline{w})(E)F$ and ϕ is the following action of S^1 on $SU(2)$:

$$\phi(s)\begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix} = \begin{pmatrix} x & -\overline{s}\,\overline{y} \\ sy & \overline{x} \end{pmatrix}.$$

This turns H into a subgroup of $U(3) \times U(3)$, that is a semi-direct product of S^1 and $SU(2)$.

Consider now the obvious action of H on $SU(3)$

$$\Phi : H \times SU(3) \to SU(3)$$

by

$$\Phi((C_z, D_z E), A) = C_z A E^* D_{\overline{z}} \ ,$$

where E^* is the adjoint matrix of E.

Proposition. *Φ is free.*

Proof. Suppose $C_z A E^* D_{\overline{z}} = A$ and consider the equality of the last columns of these two matrices. □

Lemma. *The quotient of the free action Φ is diffeomorphic to $\mathbb{C}P^2$.*

Proof. Observe first that the quotient of S^5 by the action $z\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} zq \\ zb \\ \overline{z}c \end{pmatrix}$ of S^1 is diffeomorphic to $\mathbb{C}P^2$ and denote the quotient class by $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define the map $\psi : SU(3)\big/_{H} \to \mathbb{C}P^2$ by $\psi([A]) = [\ \overline{A_3}\ ]$, i.e., the equivalence class of the conjugate of the third column and observe it is well defined since

$$(\Phi(h)A)_3 = \begin{pmatrix} za_{13} \\ za_{23} \\ \overline{z}a_{33} \end{pmatrix}, \quad \text{for any} \quad h = (C_z, D_z E) \quad \text{in} \quad H.$$

It is obvious that ψ is smooth and surjective. The differential $d\psi$ can be a readily checked to be non singular at all points of $SU(3)\big/_{H}$, which implies that ψ is a covering map and therefore a diffeomorphism since $\mathbb{C}P^2$ is simply connected. □

We would like now to translate the subgroups $T_{k,\ell,p,q}$ of $SU(3) \times SU(3)$ so that they are included in H, preserving the freeness of the action, without modifying the orbit space. One could use the center of $U(3)$ for the translation or move the group H itself.

Translating $T_{k,\ell,p,q}$ by the center of $U(3)$ that consits of all wI, w in S^1 and requiring the result to be in H we obtain the conditions:

$$wz^k = \lambda^2\ ,\ wz^\ell = \lambda^2\ ,\ wz^{-(k+\ell)} = \lambda^0 \quad \text{and} \quad wz^{-(p+q)} = \lambda$$

for some λ in S^1. Consequently, $k = \ell = 2(p+q)$. There are infinitely many integral solutions of these conditions that satisfy (E) as well. One can take, for example, $q = p + 1$, for any integer p, parametrizing so an infinite family of Eschenburg spaces.

There are, of course, other solutions outside this family. The choice of $k = \ell = 2,\ p = 4$ and $q = -3$, for example, is acceptable. .

We will work out this particular example for the sake of clarity.

The Eschenburg space $M_{2,2,4,-3}$ defined as $SU(3)\big/_{T_{2,2,4,-3}}$ is diffeomorphic to the quotient of $SU(3)$ by the translated $\widetilde{T}_{2,2,4,-3}$ which is now a subgroup of

H, since the left and right central elements will cancel each other in each orbit. Consider the associated bundle

$$H\Big/_{\widetilde{T}_{2,2,4,-3}} \cdots SU(3) \underset{H}{\times} \left(H\Big/_{\widetilde{T}_{2,2,4,-3}}\right) \to \mathbb{C}P^2$$

and observe that the total space is $M_{2,2,4,-3}$. The map

$$\varphi : SU(3) \underset{H}{\times} \left(H\Big/_{\widetilde{T}_{2,2,4,-3}}\right) \to SU(3)\Big/_{\widetilde{T}_{2,2,4,-3}}$$

defined by $\varphi\{A, [h_1, h_1']\} = [h_1^{-1} A h_1']$ is a well defined diffeomorphism.

We see that the fiber $H\Big/_{\widetilde{T}_{2,2,4,-3}}$ is a generalized lens space as follows: The orbit of $(C_z, D_z E)$ of H by the action of $\widetilde{T}_{2,2,4,-3}$ intersects $(I, SU(2)) \subseteq H$ at the points where w satisfies the equations at

$$\begin{pmatrix} z^2 & & \\ & z^2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} w^6 & & \\ & w^6 & \\ & & 1 \end{pmatrix} = I \quad \text{and}$$

$$\begin{pmatrix} z & & \\ & z^2 & \\ & & z \end{pmatrix} \begin{pmatrix} x & -\overline{\alpha} & \\ \alpha & \overline{x} & \\ & & 1 \end{pmatrix} \begin{pmatrix} w^8 & & \\ & w & \\ & & w^3 \end{pmatrix}$$

$$= \begin{pmatrix} y & -\overline{\beta} & \\ \beta & \overline{y} & \\ & & 1 \end{pmatrix} \quad \text{for some } y, \beta.$$

The second equality can be written as

$$\begin{pmatrix} zw^3 & & \\ & (zw^3)^2 & \\ & & zw^3 \end{pmatrix} \begin{pmatrix} w^5 x & -\overline{w}^2\overline{\alpha} & 0 \\ w^2\alpha & \overline{w}^5\overline{x} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y & -\overline{\beta} & 0 \\ \beta & \overline{y} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and it contains the first one that resumes to $zw^3 = 1$ and $w^5 x = y, w^2\alpha = \beta$. As y and β are defined by the respective equations, for w one of the three third roots of $\overline{z}$, there are exactly three intersection points of the orbit in question and the sphere $SU(2)$.

The quotient is therefore a generalized lens space as expected. Observe that $\pi_1(N_{2,2,4,-3}) \cong \mathbb{Z}_3$ as follows from the number of full turns the circle $T_{2,2,4,-3}$ twists arount a generator of $\pi_1 U(2)$, i.e., $2 + 2 - (4 - 3) = 3$.

The same reasoning applies to each element of the infinite family defined above. It is interesting to notice that for this same family of Eschenburg spaces Astey, Micha and Pastor [A-M-P] have been able to obtain examples of non-diffeomorphic homeomorphic spaces, using technics developped by Kreck and Stolz [K-S].

It would be also interesting to classify all $M_{k,\ell,p,q}$ that fiber over $\mathbb{C}P^2$ with a generalized lens space as a fiber. As a final observation, we recall

Proposition [E₂]. *Eschenburg's six dimensional positively curved non-homogeneous manifold M_6, is an S^2-bundle over $\mathbb{C}P^2$.*

Proof. Consider the 2-dimensional subgroup U of H, defined by

$$U = \left\{ \begin{pmatrix} \overline{z}^2 & & \\ & \overline{z}^2 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} z\,\overline{w} & & \\ & \overline{z}^2 w & \\ & & \overline{z} \end{pmatrix}, \quad z, w \text{ in } S^1 \right\}.$$

The associated bundle to $H \ldots SU(3) \to \mathbb{C}P^2$ with fiber $H/U \cong S^2$ has total space M_6. □

References

[A-M-P] L. Astey, E. Micha and G. Pastor, *Homeomorphism and diffeomorphism types of Eschenburg spaces*, preprint, Instituto Tecnológico Autónomo de México (1994).

[A-W] S. Aloff and N.R. Wallach, *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. 81, (1975) 93-97.

[C₁] L.M. Chaves, *A theorem of finiteness for fat bundles*, Topology 33 (3) (1994), 493-497.

[C₂] L.M. Chaves, Doctoral thesis, UNICAMP (1992).

[C-D-R] L.M. Chaves, A. Derdzinski and A. Rigas, *A condition for positivity of curvature*, Boletim Soc. Bras. Mat. 23(1,2) (1992), 153-165.

[C-G] J. Cheeger and D. Gromoll, *On the structure of complete open manifolds of non-negative curvature*, Ann. Math. 96, (1972), 413-443.

[D-R] A. Derdzinski and A. Rigas, *Unflat connections in 3-sphere bundles over* S^4, Trans. Amer. Math. Soc. 265(2) (1981), 485-493.

[D-N-F] B. Doubrovine, S. Novikov and A. Fomenko, *Modern Geometry*, Vol. III, Springer-Verlag (1992).

[E$_1$] J.-H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. Math. 66 (1982), 469-480.

[E$_2$] J.-H. Eschenburg, *Freie Isometrische Aktionen auf kompacten Lie-Gruppen mit positiv gekrümten Orbitraumen*, Schriften reihe des Math. Instituts, Univ. Münster (1984).

[E$_3$] J.-H. Eschenburg, *Inhomogeneous spaces of positive curvature*, Differential Geometry and its Applications 2 (1992) 123-132.

[K-S] M. Kreck and S. Stolz, *Some nondiffeomorphic homeomorphic homogeneous 7-manifolds with positive sectional curvature*, J. Diff. Geom. 33 (1991), 465-486.

[O] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. 23 (1966), 459-469.

[R$_1$] A. Rigas, *Geodesic Spheres as generators of $\pi_q O, \pi_{q+1} BO$*, Jour. Diff. Geom. 13 (4) (1978), 527-545.

[R$_2$] A. Rigas, *Some bundles of non negative curvature*, Math. Ann. 232 (1978), 187-193.

[S-W] M. Strake and G. Walschap, *Connection metrics of nonnegative curvature on vector bundles*, preprint, UCLA (1989).

[Y] S.-T. Yau, *Seminar on Differential Geometry*, Ann. of Math. Studies, 102, Princeton Univ. Press. (1982).

[Wal] G. Walschap, *Nonnegatively curved manifolds with souls of codimension two*, J. Diff. Geom. 27 (1988), 525-537.

[We] A. Weinstein, *Fat bundles and symplectic manifolds*, Advances in Math. 37(3) (1980), 239-250.

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