

SOME RESULTS ON NONNEGATIVELY CURVED FOUR MANIFOLDS

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1. Introduction

It is a classical problem in Riemannian geometry to study the topology of manifolds which admit a metric with nonnegative curvature. In the case that the curvature operator is nonnegative, the results of several authors ([13], [17], [19], [27]) lead to a topological classification of such manifolds. They are covered by Riemannian products of manifolds of the following types: homeomorphic to spheres, diffeomorphic to Euclidean spaces, biholomorphic to complex projective spaces or symmetric spaces of compact type. This classification can be found in [16].

If the dimension of the manifold is three, the nonnegativity of the sectional curvatures implies the nonnegativity of the curvature operator, because the Weyl tensor is identically zero. If the dimension is four, the work of Walschap [22] gives a thorough understanding of complete noncompact 4-manifolds with nonnegative sectional curvatures. If M is compact, it follows from Theorem 3 in [6] that the universal cover \tilde{M} splits isometrically as $\tilde{M}^{4-k} \times \mathbf{R}^k$, where \tilde{M} is compact. For $k = 1$, the topological classification of compact 3-manifolds with nonnegative Ricci curvature in [13], implies that \tilde{M}^3 is homeomorphic to the sphere S^3 . For $k = 2$, \tilde{M}^2 is homeomorphic to S^2 , by the classical Theorem of Gauss-Bonnet. Hence, if the fundamental group $\pi_1(M)$ is infinite, M is covered by either \mathbf{R}^4 or by $S^{4-k} \times \mathbf{R}^k$ for $k = 1, 2$. However, if \tilde{M} is compact the manifold M has only been understood so far under additional assumptions, and the Hopf conjecture remains unsolved: does $S^2 \times S^2$ admit a positively

curved Riemannian metric?

The aim of this article is to present and generalize some of the known results concerning compact nonnegatively curved 4-manifolds. It is organized as follows. In Section 2, we review some basic facts about 4-manifolds. In Section 3, we collect some known results on nonnegatively curved 4-manifolds satisfying some geometric conditions which imply definiteness. We show that they still hold, assuming the following weaker condition on the sectional curvatures:

$$(*) \quad K(X_i, X_j) + K(X_k, X_m) \geq 0, \text{ whenever } X_i, X_j, X_m, X_k \\ \text{are orthonormal vectors of } T_x M.$$

Therefore, if the universal cover of such manifolds is compact, the description of the topology of \tilde{M} will then follow from [9] and [11]. The results generalized under the condition $(*)$ are the ones in [22], [23] and [21].

In Section 4, we consider first compact 4-manifolds with positive sectional curvature that are Einstein. In this case, the result of several authors ([2], [8], [12], [14], [24]) combined imply that *either M is homeomorphic to one of S^4 , \mathbf{RP}^4 , \mathbf{CP}^2 or the group of all isometries of M is finite*. Einstein manifolds have harmonic curvature and, in [5], Bourguignon proved a beautiful theorem for 4-manifolds, namely, *a compact 4-manifold with harmonic curvature and non-zero signature is Einstein*. We present an outline of Bourguignon's proof and weakening the assumption of harmonic curvature, we study 4-manifolds with harmonic traceless Ricci tensor. Changing slightly the arguments, we prove that *4-dimensional Riemannian manifolds whose sectional curvatures satisfy condition $(*)$ and such that the traceless Ricci tensor is harmonic and has constant norm are either Einstein or the first Pontrjagin form is zero*. This implies the following results.

Theorem. *Let M be a compact nonnegatively curved Riemannian 4-manifold such that traceless Ricci tensor is harmonic and has constant norm. Then one of the following holds:*

- (a) M is Einstein

- (b) M is homeomorphic to S^4 or \mathbf{RP}^4
- (c) M is covered by $S^{4-k} \times \mathbf{R}^k$ for $k = 1, 2$ or $S^2 \times S^2$.

For the case of positive sectional curvature we obtain:

Corollary. *Let M be a compact positively curved Riemannian 4-manifold such that traceless Ricci tensor is harmonic and has constant norm. Then one of the following holds:*

- (a) M is homeomorphic to S^4 , or \mathbf{RP}^4 , or \mathbf{CP}^2
- (b) The group G of all isometries of M is finite.

2. The Weintzenböck formula.

Let M be an oriented Riemannian manifold of dimension 4, and let Λ^2 denote the bundle of exterior 2-forms and $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ the eigenspace splitting for the Hodge \star -operator. The 2-forms in Λ_+^2 are called *self-dual* and in Λ_-^2 , *anti-self-dual*.

The Riemann curvature tensor defines a symmetric operator $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$ given by

$$\mathcal{R}(e_{ij}) = \frac{1}{2} \sum_{k,l} R_{ijkl} e_{kl}$$

where $\{e_i\}$ is a local orthonormal basis of 1-forms, e_{ij} denotes the 2-form $e_i \wedge e_j$ and $R_{ijkl} = \langle R(e_i, e_j), e_l, e_k \rangle$. The operator \mathcal{R} can be decomposed as

$$\mathcal{R} = \mathcal{R}_+^+ + \mathcal{R}_-^+ + \mathcal{R}_+^- + \mathcal{R}_-^-$$

with respect to the decomposition $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$. This decomposition gives the irreducible components of \mathcal{R} (see [22]). They are $\text{trace} \mathcal{R}_+^+ = \text{trace} \mathcal{R}_-^- = \frac{S}{4}$, where S is the scalar curvature, the two components \mathcal{R}_+^+ and \mathcal{R}_-^+ of the traceless Ricci tensor, and the two components of the Weyl tensor $W^+ = \mathcal{R}_+^+ - \frac{S}{12}$ and $W^- = \mathcal{R}_-^- - \frac{S}{12}$. Note that the metric is *Einstein* if and only if $\mathcal{R}_+^+ = \mathcal{R}_-^+ = 0$.

Let $F : \Lambda^2(T_x M) \rightarrow \Lambda^2(T_x M)$ be the *Weitzenböck operator* given by

$$\langle F(e_{ij}), e_{kl} \rangle = Ric(e_i, e_k)\delta_{jl} + Ric(e_j, e_l)\delta_{ik} - Ric(e_i, e_l)\delta_{jk} - Ric(e_j, e_k)\delta_{il} - 2R_{ijkl}$$

where Ric denotes the Ricci curvature. This operator satisfies the well known *Weitzenböck formula*, e.g., $\Delta\omega = -div \nabla\omega + F(\omega)$. Moreover, F is a symmetric operator.

Proposition 2.1. Λ_+^2 and Λ_-^2 are F -invariant, i.e., $\star F = F\star$.

Proof. Since F is symmetric, let ω denote an eigenvector of F with corresponding eigenvalue r . Then we have

$$F(\omega^+) + F(\omega^-) = r\omega^+ + r\omega^-$$

where ω^\pm denotes the projection onto Λ_\pm^2 , respectively. Therefore it is enough to show that $\langle F(\omega^+), \omega^- \rangle = 0$. In fact, since there is an orthonormal basis $\{X_1, X_2, X_3, X_4\}$ of $T_x M$ such that

$$\omega^\pm = \frac{\sqrt{2}}{2} \|\omega^\pm\| (X_{12} \pm X_{34}),$$

substituting the above in the definition of F , we obtain the desired conclusion.

Now, since both Δ and F commute with the Hodge \star -operator, the Weitzenböck formula can be written as

$$\Delta\omega^\pm = -div \nabla\omega^\pm + F(\omega^\pm).$$

If M is compact, integrating by parts we obtain

$$(\Delta\omega^\pm, \omega^\pm) = (\nabla\omega^\pm, \nabla\omega^\pm) + \int_M \langle F(\omega^\pm), \omega^\pm \rangle dV \quad (2.2)$$

where (\cdot, \cdot) is the inner product on $\Lambda^2(M)$ given by

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle dV$$

dV is the volume form of M and \langle, \rangle is the naturally induced inner product on the space of 2-forms $\Lambda^2(T_x M)$.

Let $H^2(M; \mathbf{R})$ denote the 2^{nd} de Rham cohomology group of M . If M is compact, it follows from the Hodge's Theorem that $H^2(M; \mathbf{R})$ is isomorphic to the space of harmonic 2-forms denoted by \mathcal{H} , and because $\star \Delta = \Delta \star$, we obtain the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. We will denote by $b_2^\pm = \dim \mathcal{H}^\pm$. Therefore the second Betti number $b_2 = b_2^+ + b_2^-$. From (2.2) we get immediately:

Proposition 2.3. *Let M be a compact oriented 4-manifold. Then we have*

- (i) *if $F^\pm \geq 0$, then a 2-form ω^\pm is harmonic if and only if is parallel*
- (ii) *if $F^\pm \geq 0$ and there is a point $p \in M$ such that $F^\pm(p) > 0$ then $b_2^\pm = 0$.*

Recall that a compact, oriented, smooth 4-manifold is called *positive* (respectively, *negative*) *definite* if $b_2^- = 0$ (respectively $b_2^+ = 0$). We call such a manifold *definite manifold*.

The seminal work of Donaldson, [9], and Freedman, [11], give the following topological classification for definite, smooth, 4-manifolds.

Theorem 2.4. (Donaldson-Freedman) *Let M be a definite, smooth, simply connected, compact 4-manifold. Then M is homeomorphic to one of*

- i) S^4 , *if the the second Betti number $b_2 = 0$*
- ii) *the connected sum $\mathbf{CP}^2 \# \dots \# \mathbf{CP}^2$, (b_2 times), if $b_2 > 0$.*

In order to study nonnegatively curved compact 4-manifolds for which the universal covering \tilde{M} is compact, we search then for conditions that imply that M is definite. In view of Proposition (2.3), we look for hypotheses that will imply that the operators $F^\pm \geq 0$. For that, since $\star F = F \star$, we find for each point of M a normal form for F (as in [25] for R). Since this normal form will be used in the rest of the article, we repeat the arguments used in [25].

Proposition 2.5. *Let M be an oriented four-manifold. Then for each $x \in M$ there exists a positively oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_x M$ such*

that relative to the corresponding basis $\{e_{12}, e_{34}, e_{13}, e_{42}, e_{14}, e_{23}\}$, F takes the form

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} \eta_i & \mu_i \\ \mu_i & \eta_i \end{pmatrix}$$

Proof. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ be the orthonormal bases of eigenvectors of F^+ and F^- respectively, and r_i and $s_i, i = 1, 2, 3$, be the corresponding eigenvalues. Let us define the planes $P_i = \frac{\alpha_i + \beta_i}{\sqrt{2}}$ and $P_i^\perp = \frac{\alpha_i - \beta_i}{\sqrt{2}}$. Therefore $\{P_1, P_2, P_3, P_1^\perp, P_2^\perp, P_3^\perp\}$ is an orthonormal basis of $\Lambda^2(T_x M)$ and $F(P_i) = \eta_i P_i + \mu_i P_i^\perp$ and $F(P_i^\perp) = \eta_i P_i^\perp + \mu_i P_i$ where $\eta_i = \frac{r_i + s_i}{2}$ and $\mu_i = \frac{r_i - s_i}{2}$. In addition, since $\star P_i = P_i^\perp$ we have that $P_i \wedge P_i = 0 = P_i^\perp \wedge P_i^\perp$ which implies that P_i and P_i^\perp are decomposable. We also have $P_1 \wedge P_2 = 0$ and hence $P_1 \cap P_2 \neq \{0\}$. Let $e_1 \in P_1 \cap P_2$ be a unit vector and e_2 and e_3 such that $\{e_1, e_2\}$ and $\{e_1, e_3\}$ are oriented bases for P_1 and P_2 respectively. Choose e_4 to complete a positively oriented orthonormal basis of $T_x M$. Then $P_1 = e_1 \wedge e_2$, $P_2 = e_1 \wedge e_3$ and $e_1 \wedge e_4$ is either $\pm P_3$ or $\pm P_3^\perp$. The matrix of F relative to $\{e_{12}, e_{34}, e_{13}, e_{42}, e_{14}, e_{23}\}$ is of the above type.

It follows from the above Proposition that the self-dual 2-forms

$$\alpha_1 = \frac{\sqrt{2}}{2}(e_{12} + e_{34}), \quad \alpha_2 = \frac{\sqrt{2}}{2}(e_{13} - e_{24}), \quad \alpha_3 = \frac{\sqrt{2}}{2}(e_{14} + e_{23})$$

are the eigenvectors of F^+ , with corresponding eigenvalues $r_i = \eta_i + \mu_i$, and the anti-self-dual 2-forms

$$\beta_1 = \frac{\sqrt{2}}{2}(e_{12} - e_{34}), \quad \beta_2 = \frac{\sqrt{2}}{2}(e_{13} + e_{24}), \quad \beta_3 = \frac{\sqrt{2}}{2}(e_{14} - e_{23})$$

are the eigenvectors of F^- , with corresponding eigenvalues $s_i = \eta_i - \mu_i$

Proposition 2.6. Let $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis of Proposition 2.5. Then the 2-forms $\{\alpha_i\}$ are the eigenvectors of \mathcal{R}_+^\perp and the 2-forms $\{\beta_i\}$

are the eigenvectors of \mathcal{R}_- . Moreover, if the corresponding eigenvalues are denoted by λ_i and φ_i respectively, we have

$$r_1 = 2(\lambda_2 + \lambda_3), \quad r_2 = 2(\lambda_1 + \lambda_3), \quad r_3 = 2(\lambda_1 + \lambda_2)$$

$$s_1 = 2(\varphi_2 + \varphi_3), \quad s_2 = 2(\varphi_1 + \varphi_3), \quad s_3 = 2(\varphi_1 + \varphi_2).$$

Proof. We will show that $\langle \mathcal{R}(\alpha_i), \alpha_j \rangle = 0$ and $\langle \mathcal{R}(\beta_i), \beta_j \rangle = 0$ for $i \neq j$. For simplicity, we will show that $\langle \mathcal{R}(\alpha_1), \alpha_2 \rangle = 0$ and the other ones are proved in a similar manner. Since $\langle F(\alpha_1), \alpha_2 \rangle = 0$, we have

$$\langle F(e_{12}), e_{13} \rangle - \langle F(e_{12}), e_{24} \rangle + \langle F(e_{34}), e_{13} \rangle + \langle F(e_{34}), e_{24} \rangle = 0.$$

From the definition of F we get that

$$0 = Ric(e_2, e_3) - 2R_{1231} + Ric(e_1, e_4) + 2R_{1242} - Ric(e_1, e_4) - 2R_{3431}$$

$$- Ric(e_2, e_3) + 2R_{3442} = -2R_{1231} + 2R_{1242} - 2R_{3431} + 2R_{3442}$$

$$= -4\langle \mathcal{R}(\alpha_1), \alpha_2 \rangle.$$

Now, the eigenvalues λ and φ are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(K_{12} + K_{34}) - R_{1234} & \varphi_1 &= \frac{1}{2}(K_{12} + K_{34}) + R_{1234} \\ \lambda_2 &= \frac{1}{2}(K_{13} + K_{24}) + R_{1324} & \varphi_2 &= \frac{1}{2}(K_{13} + K_{24}) - R_{1324} \\ \lambda_3 &= \frac{1}{2}(K_{14} + K_{23}) - R_{1423} & \varphi_3 &= \frac{1}{2}(K_{14} + K_{23}) + R_{1423} \end{aligned} \quad (2.7)$$

where K_{ij} denotes the curvature of the plane $\{e_i, e_j\}$. On the other hand, from the definition of F we have

$$r_1 = \langle F(\alpha_1), \alpha_1 \rangle = \frac{1}{2}(Ric(e_1) + Ric(e_2) + Ric(e_3) + Ric(e_4) - 2K_{12}$$

$$- 2K_{34} + 4R_{1234}) = K_{13} + K_{24} + K_{14} + K_{23} + 2R_{1234}.$$

Using the first Bianchi identity, we conclude

$$r_1 = K_{13} + K_{24} + 2R_{1324} + K_{14} + K_{23} - 2R_{1423} = 2(\lambda_2 + \lambda_3).$$

Similarly we obtain

$$\begin{aligned}
 s_1 &= K_{13} + K_{24} + K_{14} + K_{23} - 2R_{1234} = 2(\varphi_2 + \varphi_3) \\
 r_2 &= K_{12} + K_{34} + K_{14} + K_{23} - 2R_{1324} = 2(\lambda_1 + \lambda_3) \\
 s_2 &= K_{12} + K_{34} + K_{14} + K_{23} + 2R_{1324} = 2(\varphi_1 + \varphi_3) \\
 r_3 &= K_{12} + K_{34} + K_{13} + K_{24} + 2R_{1423} = 2(\lambda_1 + \lambda_2) \\
 s_3 &= K_{12} + K_{34} + K_{13} + K_{24} - 2R_{1423} = 2(\varphi_1 + \varphi_2).
 \end{aligned} \tag{2.8}$$

From the equations above, we conclude that $r_i + 2\lambda_i = s_i + 2\varphi_i = \frac{S}{2}$, where S is the scalar curvature. Therefore, we can state:

Proposition 2.9 *The Weitzenböck operator is given in terms of the scalar curvature by*

$$F^+ = \frac{S}{2} - 2\mathcal{R}_+^+ = \frac{S}{3} - 2W^+ \quad F^- = \frac{S}{2} - 2\mathcal{R}_-^- = \frac{S}{3} - 2W^-$$

We finish this section showing a condition that implies $F^\pm \geq 0$.

Lemma 2.10. *Let $\|W^\pm\|^2$ denote the norm of the components of the Weyl tensor. Let us suppose that the scalar curvature $S \geq 0$. Therefore*

i) *if $\|W^\pm\|^2 \leq \frac{S^2}{24}$ then $F^\pm \geq 0$.*

ii) *if at some point $p \in M$, we have $\|W^\pm\|^2(p) < \frac{S^2}{24}$, then $F^\pm(p) > 0$.*

Proof. Let W_i^\pm be the eigenvalues of W^\pm . Using that $\text{trace}W^\pm = 0$ we obtain that $(W_i^\pm)^2 \leq \frac{2}{3}\|W^\pm\|^2$. Then, $(W_i^\pm)^2 \leq \frac{S^2}{36}$, for $i = 1, 2, 3$. This together with Proposition 2.9 concludes the Lemma.

3. On the topology of nonnegatively curved 4-manifolds.

In [17], Micallef and Moore introduced the concept of curvature on totally isotropic two-planes for manifolds of dimension ≥ 4 . We will call it, for brevity, *isotropic curvature*. This curvature plays a similar role in the study of the second variation of area of minimal surfaces that the sectional curvature does in

the study of geodesics. We say that M has *nonnegative isotropic curvature* if for all sets of orthonormal vectors e_i, e_j, e_m, e_k in $T_x M$ we have

$$K_{ik} + K_{im} + K_{jk} + K_{jm} \pm 2R_{ijkm} \geq 0.$$

Notice that for 4-manifolds, the nonnegativity of the isotropic curvature is equivalent to the nonnegativity of the Weitzenböck operator F . The special feature of nonnegative sectional curvature in dimension four is that for each point in M either F^+ or F^- is nonnegative. Actually, this is true even under the weaker condition

$$(*) \quad K(X_i, X_j) + K(X_k, X_m) \geq 0, \text{ whenever } X_i, X_j, X_m, X_k \\ \text{are orthonormal vectors of } T_x M$$

as we will show in the next Lemma. Note that $(*)$ implies $S \geq 0$.

Lemma 3.1. *Let M be an oriented 4-manifold satisfying the condition $(*)$. If $\|W^+\|^2 \geq \frac{S^2}{24}$ then $\|W^-\|^2 \leq \frac{S^2}{24}$. Moreover, if the first inequality is strict, so is the second.*

Proof. It follows from $\text{trace } \mathcal{R}_+^+ = \text{trace } \mathcal{R}_-^- = \frac{S}{4}$ that if $\mathcal{R}_+^+ \geq 0$ then $\|W^+\|^2 \leq \frac{S^2}{24}$ and if $\mathcal{R}_-^- \geq 0$ then $\|W^-\|^2 \leq \frac{S^2}{24}$. Therefore, if $\|W^+\|^2 \geq \frac{S^2}{24}$, then \mathcal{R}_+^+ has one nonpositive eigenvalue. We show now that the condition $(*)$ implies that $\mathcal{R}_-^- \geq 0$ which finishes the lemma. For that, suppose that after reordering the basis $\{\alpha_i\}$ we have $\lambda_1 \leq \lambda_2 \leq \lambda_3$. If $\lambda_1 \leq 0$, then (2.7) implies that $R_{1234} \geq 0$ and hence $\varphi_1 \geq 0$. In order to show that $\varphi_2 \geq 0$ and $\varphi_3 \geq 0$, we consider the planes $P = \frac{\alpha_1 + \beta_2}{\sqrt{2}}$ and $P^\perp = \frac{\alpha_1 - \beta_2}{\sqrt{2}}$. The proof of Proposition 2.5 shows that there is an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of the tangent space such that $f_{12} = \frac{\alpha_1 + \beta_2}{\sqrt{2}}$ and $f_{34} = \frac{\alpha_1 - \beta_2}{\sqrt{2}}$. Hence,

$$K(f_1, f_2) + K(f_3, f_4) = \lambda_1 + \varphi_2 \geq 0$$

and if $\lambda_1 \leq 0$ then $\varphi_2 \geq 0$. In a similar manner we show that $\varphi_3 \geq 0$. This proof also shows that if $\lambda_1 < 0$ then if $\varphi_i > 0$ for $i = 1, 2, 3$.

Theorem 3.2. *Let M be a compact oriented 4-manifold with sectional curvatures satisfying $(*)$. Let us suppose that $\|W^+\|^2 \leq \frac{S^2}{24}$ or $\|W^+\|^2 \geq \frac{S^2}{24}$ for all points of M . Therefore*

- i) *if for some point in M , $\|W^+\|^2 \neq \frac{S^2}{24}$, then M is definite.*
- ii) *if $\|W^+\|^2 = \frac{S^2}{24}$ for all points and the fundamental group $\pi_1(M)$ is finite, then one of following holds:*
 - a) *M is homomorphic to the sphere S^4*
 - b) *\tilde{M} is a Riemannian product of two compact surfaces and one of them is homomorphic to the sphere S^2*
 - c) *M is a Kähler manifold biholomorphic to \mathbf{CP}^2 .*

Proof. If $\|W^+\|^2 \leq \frac{S^2}{24}$ and for some point $p \in M$, we have $\|W^+\|^2(p) < \frac{S^2}{24}$, then by (2.3) and (2.10) we conclude that M is negative definite. If $\|W^+\|^2 \geq \frac{S^2}{24}$ and for some point the inequality is strict, we obtain that M is positive definite, using Lemma 3.1.

Now consider the case that $\|W^+\|^2 = \frac{S^2}{24}$, for all points of M . In this case, (3.1) and (2.10) together imply that the operator F is nonnegative and hence the isotropic curvature is nonnegative. If \tilde{M} is a Riemannian product, then Theorem 3.1 in [18] gives case (b). If M is locally irreducible and $b_2(M) > 0$, we conclude from Theorem 2.1 in [18] that M is a simply connected Kähler manifold and hence biholomorphic to \mathbf{CP}^2 by Theorem 1 in [20]. This also shows that if $b_2(M) = 0$ then we cannot have $b_2(\tilde{M}) > 0$, since this would imply that \tilde{M} is biholomorphic to \mathbf{CP}^2 and therefore cannot cover M . Then $b_2(\tilde{M}) = 0$ and by Freedman's solution of the Poincare conjecture, \tilde{M} is homomorphic to S^4 . Since we are supposing that M is oriented, M itself is homeomorphic to S^4 .

Corollary 3.3. *Let M be a compact oriented 4-manifold with sectional curvatures satisfying $(*)$. Let us suppose that $\|W^-\|^2 \leq \|W^+\|^2$.*

- a) *If at some point we have that $\|W^-\|^2 < \|W^+\|^2$, then M is definite.*

- b) If $\|W^-\|^2 = \|W^+\|^2$ for all points of M and the fundamental group $\pi_1(M)$ is finite, then M is either homeomorphic to S^4 or is locally Riemannian product of two surfaces.

Proof. Suppose first that $\|W^+\|^2 > \frac{S^2}{24}$ for some point $p \in M$. Then Lemmas 3.1 and 2.10 imply that $F^- > 0$ at p . For the points that $\|W^+\|^2 \leq \frac{S^2}{24}$, the hypothesis implies that $\|W^-\|^2 \leq \frac{S^2}{24}$ and therefore we have that F^- is nonnegative and positive on neighborhood of p . Now from Proposition 2.3, we conclude that $b_2^- = 0$. If $\|W^+\|^2 \leq \frac{S^2}{24}$, the hypothesis in (a) implies again $b_2^- = 0$. Now, we suppose that $\|W^-\|^2 = \|W^+\|^2$ for all points of M . Therefore Lemma 3.1 gives that $\|W^-\|^2 = \|W^+\|^2 \leq \frac{S^2}{24}$ and M has nonnegative isotropic curvature. Moreover, recall that the first Pontrjagin form on an oriented 4-manifold is given by $p_1 = (\|W^-\|^2 - \|W^+\|^2)dV$, where dV is the volume form of M . Then from Theorem 3.2, the only possibilities for M are S^4 or locally product of two surfaces, since $p_1 = 0$ implies that the signature of M is zero and hence excludes the other possibilities.

We show now that some of the known results for nonnegatively curved 4-manifolds can be generalized. It is well known that *normal homogeneous spaces have nonnegative sectional curvatures*. Simply connected homogeneous spaces of dimension four are classified. They are diffeomorphic to one of S^4 , \mathbf{CP}^2 , $S^2 \times S^2$, $S^3 \times \mathbf{R}$, $S^2 \times \mathbf{R}^2$ or \mathbf{R}^4 (see [3] pg.46). On a Riemannian homogeneous space, the scalar curvature, $\|W^+\|$ and $\|W^-\|$ are constant. Supposing the constancy of the norm of only one component of the Weyl tensor and nonnegative curvature we obtain:

Theorem 3.4. *Let M be a simply connected 4-manifold with nonnegative sectional curvature.*

- i) *If M is complete and noncompact, then M is diffeomorphic to one of $S^3 \times \mathbf{R}$, $S^2 \times \mathbf{R}^2$ or \mathbf{R}^4 .*
- ii) *If M is compact and the scalar curvature S and $\|W^-\|$ are constant on*

M, then *M* is homeomorphic to one of S^4 , $\mathbf{CP}^2 \# \dots \# \mathbf{CP}^2$ or $S^2 \times S^2$.

Proof. The first assertion follows from [26]. For compact manifolds, if $\|W^+\|^2 > \frac{S^2}{24}$ for some point p in M , it follows from Lemma 3.1 that $\|W^-\|^2 < \frac{S^2}{24}$ at p . Now the hypotheses imply $\|W^-\|^2 < \frac{S^2}{24}$ for all points of M and therefore $F^- > 0$. By Proposition 2.3 we conclude that $b_2^- = 0$ and M is positive definite. In this case we have that M is either homeomorphic to S^4 or to $\mathbf{CP}^2 \# \dots \# \mathbf{CP}^2$. If $\|W^+\|^2 \leq \frac{S^2}{24}$ for all points of M , the result follows from Theorem 3.2.

In [22], Seaman gave another proof for the result of Frankel, [10], namely, *a compact positively curved Kähler manifold is biholomorphic to \mathbf{CP}^2* . Seaman used the Weitzenböck operator and the existence of a parallel 2-form. Notice that on a Kähler manifold (with the natural orientation) $\|W^+\|^2 = \frac{S^2}{24}$. For the case that the sectional curvatures satisfy condition (*), we obtain:

Theorem 3.5. *Let M be a compact 4-manifold with sectional curvatures satisfying (*). If M admits a parallel 2-form, then either M is flat or is a Kähler manifold biholomorphic to \mathbf{CP}^2 or \tilde{M} splits isometrically as $\Sigma_1 \times \Sigma_2$, where Σ_1 is homomorphic to the sphere S^2 .*

Proof. Let us suppose without losing generality that M is oriented. Since it has a parallel 2-form, M is Kähler. Therefore $\|W^+\|^2 = \frac{S^2}{24}$ and the the proof of Theorem 3.2 shows that in this case M has nonnegative isotropic curvature. Since $b_2 \neq 0$, if M is locally irreducible, then M is biholomorphic to \mathbf{CP}^2 , and if \tilde{M} splits isometrically, the only possibilities are \mathbf{R}^4 or $\Sigma_1 \times \Sigma_2$, where Σ_1 is homomorphic to the sphere S^2 , by Theorem 3.1 of [18].

In [23], Seaman replaced the assumption of the existence of a parallel 2-form by the existence of a harmonic 2-form of constant length and studied this case for positive sectional curvature. We generalize his result for the case that the

sectional curvatures satisfy a strict inequality in (*), i.e.,

$$(**) \quad K(X_i, X_j) + K(X_k, X_m) > 0, \quad \text{whenever } X_i, X_j, X_m, X_k \\ \text{are orthonormal vectors of } T_x M.$$

Theorem 3.6. *Let M be a compact 4-manifold with sectional curvatures satisfying (**). If M admits a harmonic 2-form with constant length, then M is definite.*

Proof. Let ω denote the harmonic 2-form of constant length. Then we can find an orthonormal basis $\{X_1, X_2, X_3, X_4\}$ of $T_x M$ such that

$$\omega^\pm = \frac{\sqrt{2}}{2} \|\omega^\pm\| (X_{12} \pm X_{34}).$$

Substituting in the definition of F we obtain

$$\|\omega^-\|^2 \langle F(\omega^+), \omega^+ \rangle + \|\omega^+\|^2 \langle F(\omega^-), \omega^- \rangle = 2\|\omega^+\|^2 \|\omega^-\|^2 \mathcal{K} \quad (3.7)$$

where $\mathcal{K} = K(X_1, X_3) + K(X_2, X_4) + K(X_1, X_4) + K(X_2, X_3)$ and therefore $\mathcal{K} > 0$, since the sectional curvatures satisfy (**). On the other hand, the Weitzenböck formula implies that

$$\langle \Delta(\omega), \omega \rangle = \frac{1}{2} \Delta(\|\omega\|^2) + \|\nabla \omega\|^2 + \langle F(\omega), \omega \rangle$$

and since ω is a harmonic 2-form of constant length, we have

$$\langle F(\omega), \omega \rangle = \langle F(\omega^+), \omega^+ \rangle + \langle F(\omega^-), \omega^- \rangle \leq 0. \quad (3.8)$$

Suppose that for some point in M we have $\|\omega^+\| = \|\omega^-\|$. In this case, if $\|\omega^+\| \neq 0$, substituting in (3.7) and using the fact that $\mathcal{K} > 0$, we would get $\|\omega^+\|^2 \langle F(\omega), \omega \rangle > 0$, which contradicts (3.8). Also, if $\|\omega^+\| = 0$, since ω has constant length, we would have $\omega = 0$. Therefore we can assume that $\|\omega^+\| > \|\omega^-\|$ everywhere. For the points that $\|\omega^-\| = 0$, (3.8) implies that $\langle F(\omega^+), \omega^+ \rangle \leq 0$. If $\|\omega^-\| \neq 0$ at some point, we conclude that $\langle F(\omega^+), \omega^+ \rangle < 0$, otherwise substituting the inequality $\|\omega^+\| > \|\omega^-\|$ in (3.7),

we would obtain $||\omega^+||^2 \langle F(\omega), \omega \rangle > 0$, which contradicts again (3.8). Therefore, $\langle F(\omega^+), \omega^+ \rangle \leq 0$ for all points of M which implies by Lemma 2.9 that $||W^+||^2 \geq \frac{S^2}{24}$ for all points of M . Moreover, if ω^+ is not parallel, there exists an open set for which $\langle F(\omega^+), \omega^+ \rangle < 0$ which in turn gives $||W^+||^2 > \frac{S^2}{24}$. Now from Proposition 2.3, Lemmas 2.10 and 3.1, we conclude that in this case M is definite. If ω^+ is parallel, then we are in the previous theorem. Now, the only case that satisfies the condition (**) is a Kähler manifold biholomorphic to \mathbf{CP}^2 and therefore M is definite.

Before we finish this section, we would like to point out that in $S^2 \times S^2$ the first Pontrjagin class is zero. Therefore, Corollary 3.3 answers the Hopf conjecture under the stronger assumption of first Pontrjagin form zero, instead of first Pontrjagin class. In addition, if in Theorems 3.5 and 3.6, we assume positive sectional curvature, M is either biholomorphic to \mathbf{CP}^2 or homeomorphic to $\mathbf{CP}^2 \# \dots \# \mathbf{CP}^2$. Therefore, if there exists a metric of positive curvature on $S^2 \times S^2$, such a metric does not admit a harmonic 2-form of constant length.

4. Harmonicity of the traceless Ricci tensor.

Let $Hom(TM, TM) \rightarrow M$ be the bundle of the homomorphisms of the tangent bundle TM . We denote the space of 2-forms with values in $Hom(TM, TM)$ by $\Omega^2(Hom(TM, TM))$. Notice that the curvature tensor R is in $\Omega^2(Hom(TM, TM))$. A Riemannian manifold is said to have *harmonic curvature* if the Laplacian of the curvature tensor R satisfies $\Delta R = 0$. Recall that for 4-manifolds, we have the decomposition $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$, and with respect to this decomposition R can be written as

$$R = R_+^+ + R_+^- + R_-^+ + R_-^-.$$

It is well known that R is harmonic if and only if each of the components is harmonic. Let $Z = R_+^- + R_-^+$. Z is called the *traceless Ricci tensor* of M . In this section we want to consider compact nonnegatively curved 4-manifolds for which $\Delta Z = 0$.

We start with Einstein 4-manifolds with positive sectional curvature. Notice that if M is an Einstein 4-manifold then Z is identically zero. Compact Einstein 4-manifolds are not yet completely understood. The results of Derdzinski in [2] combined with those of Berard-Bergery in [2] prove:

Theorem 4.1. (Berard-Bergery, Derdzinski) *Let M be a compact 4-dimensional Einstein manifold and G the group of all isometries of M . If the dimension of G is at least 4, or $G = T^3$ or $G = SO(3)$ and the principal orbits are S^2 or the real projective space \mathbf{RP}^2 , then M is either locally symmetric or M itself or a double cover is isometric, up to a scaling factor, to $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ endowed with the Page metric.*

The reader is referred to [2] or to [4] for a clear exposition of the Page metric. Therefore the remaining cases are: $G = SU(2)$, $G = SO(3)$ with 3-dimensional principal orbits, $G = T^2$, $G = S^1$ or the group G is finite. The study of these cases turned out to be a difficult problem. However, for positive sectional curvature, the results of several authors lead to a topological classification of most of them and the exceptional cases are only those whose group of isometries is finite.

In fact, if $G = S^1$, then M has a nontrivial Killing vector field, and a theorem of Hsiang and Kleiner in [14], imply that M is homeomorphic to S^4 or \mathbf{RP}^4 or \mathbf{CP}^2 . If $G = T^2$ or $G = SU(2)$, we conclude that M is diffeomorphic to S^4 or \mathbf{RP}^4 or \mathbf{CP}^2 , by the results in [12]. If $G = SO(3)$ and the principal orbits are 3-dimensional, a theorem in [24] implies again that M is diffeomorphic to S^4 or \mathbf{RP}^4 or \mathbf{CP}^2 . Moreover, the Page metric on $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ does not have positive sectional curvature, since it has a non-trivial Killing vector field, and therefore, by the theorem of Hsiang and Kleiner, it would be homeomorphic to S^4 or \mathbf{RP}^4 or \mathbf{CP}^2 . Now, from these results and the classification of locally symmetric spaces, we obtain:

Theorem 4.2. *Let M be a compact, positively curved Einstein 4-manifold.*

Then M is either homeomorphic to S^4 or \mathbf{RP}^4 or \mathbf{CP}^2 or the group of all isometries of M is finite.

Now we suppose that $\Delta Z = 0$. This condition implies that, if $Z = 0$ on an open set of M , then Z is zero everywhere, since it satisfies an elliptic system (according to [1]), and then M is Einstein manifold. Therefore we will study the case that $Z \neq 0$ on a dense set of M . For that we consider naturally induced metrics on $\text{Hom}(TM, TM)$ and $\Omega^2(\text{Hom}(TM, TM))$. For the Riemannian vector bundle $\text{Hom}(TM, TM) \rightarrow M$ with connection ∇ , the Weitzenböck formula states (see [15] pg. 95) :

$$(\Delta R)(V, W) = (\nabla^* \nabla R)(V, W) + \rho(R)(V, W)$$

where

$$\rho(R)(V, W) = R(\text{Ric}(V), W) + R(V, \text{Ric}(W)) + (\tilde{\mathcal{R}}R)(V, W) + (\bar{\mathcal{R}}R)(V, W)$$

and if $\{e_i\}$, $i = 1, \dots, 4$ is an orthonormal basis of $T_x M$,

$$\text{Ric}(V) = \sum_{k=1}^4 R(V, e_k), e_k$$

$$(\tilde{\mathcal{R}}R)(V, W) = \sum_{k=1}^4 R(e_k, R(V, W), e_k)$$

$$(\bar{\mathcal{R}}R)(V, W) = \sum_{k=1}^4 2[R(e_k, V), R(e_k, W)]$$

A straightforward computation yields the following result:

Proposition 4.3. *Let $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$ (defined as in section 3) be an orthonormal basis diagonalizing the symmetric operator F . Then the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ diagonalizes the symmetric operator $\rho(R_+^+)$ and their corresponding eigenvalues are*

$$\rho_i = \frac{S}{2} \lambda_i - 2\lambda_i^2 - 4\lambda_j \lambda_k.$$

Similarly, the basis $\{\beta_1, \beta_2, \beta_3\}$ diagonalizes the symmetric operator $\rho(R_-)$ and their corresponding eigenvalues are

$$\sigma_i = \frac{S}{2}\varphi_i - 2\varphi_i^2 - 4\lambda_j\varphi_k.$$

Moreover, $\rho(R_-^+) : \Lambda_+^2 \rightarrow \Lambda_-^2$ is given by

$$\rho(R_-^+)(\alpha_i) = \left(\frac{S}{2} - 2\lambda_i\right)R_-^+(\alpha_i) - 2\sqrt{2}[R_-^+(\alpha_j), R_-^+(\alpha_k)]$$

and $\rho(R_+^-) : \Lambda_-^2 \rightarrow \Lambda_+^2$ is given by

$$\rho(R_+^-)(\beta_i) = \left(\frac{S}{2} - 2\varphi_i\right)R_+^-(\beta_i) - 2\sqrt{2}[R_+^-(\beta_j), R_+^-(\beta_k)].$$

If $\Delta Z = 0$, then $\Delta R_-^+ = 0$ and $\Delta R_+^- = 0$. Substituting this in the Weitzenböck formula, and using Propositions 2.9 and 4.3, we get that

$$\nabla^* \nabla R_-^+ = R_-^+ F^+ - 4(R_-^+)^{\sharp}$$

$$\nabla^* \nabla R_+^- = R_+^- F^- - 4(R_+^-)^{\sharp}$$

where $(R_-^+)^{\sharp}$ as a matrix, is the matrix of cofactors of R_-^+ with respect to the basis $\{\alpha_i, \beta_i\}$. Since the transpose of R_+^- is R_-^+ , transposing the second equation and comparing to the first, we get that $R_-^+ F^+ = F^- R_-^+$, which is equivalent to

$$R_-^+ W^+ = W^- R_-^+ \quad (4.4)$$

by Proposition 2.9. This equation (4.4) was obtained by [5] (theorem 5.1) and [18] (equation 4.1b), where they considered harmonic curvature. We point out here, that the hamonicity of the traceless Ricci tensor is enough to imply (4.4).

Lemma 4.5. *Suppose that the components of the Weyl tensor satisfy $R_-^+ W^+ = W^- R_-^+$. If $R_-^+(p) \neq 0$ then at this point, either W^+ and W^- have the same spectra or the Ricci operator has exactly two eigenvalues of multiplicity 2.*

Proof. Let $\{\alpha_i, \beta_i\}$ be as in section 2. Let $R_{ij} = \langle \mathcal{R}(\alpha_i), \beta_j \rangle$ and W_i^{\pm} denote the eigenvalues of W^{\pm} . Let us suppose then that $R_{11} \neq 0$. If R_{11} is the only non

null entry in the matrix R_-^+ , we claim that the basis $\{e_1, e_2, e_3, e_4\}$ of Proposition 2.6 diagonalizes the Ricci operator. In fact, from the definition of F and the fact that $\langle F(\alpha_i), \beta_j \rangle = 0$, we get that

$$2\langle \mathcal{R}(\alpha_1), \beta_2 \rangle = Ric(e_2, e_3) - Ric(e_1, e_4)$$

$$2\langle \mathcal{R}(\alpha_2), \beta_1 \rangle = Ric(e_2, e_3) + Ric(e_1, e_4)$$

which gives that $Ric(e_2, e_3) = Ric(e_1, e_4) = 0$, since we are supposing $\langle \mathcal{R}(\alpha_2), \beta_1 \rangle = \langle \mathcal{R}(\alpha_1), \beta_2 \rangle = 0$. Similarly, we show that $Ric(e_i, e_j) = 0$ for $i \neq j$. From the assumption that $R_{ii} = 0$ for $i = 1, 2$ we obtain that $Ric(e_1, e_1) = Ric(e_2, e_2)$ and $Ric(e_3, e_3) = Ric(e_4, e_4)$.

Now, we suppose that there are other non null entries in R_-^+ . First, notice that $R_{11} \neq 0$ implies by (4.4) that $W_1^+ = W_1^-$. If $R_{ij} \neq 0$, for $i, j \in \{2, 3\}$, then by (4.4) we have again that $W_i^+ = W_i^-$ for all $i = 1, 2, 3$. If $R_{ij} = 0$, for $i, j \in \{2, 3\}$, we suppose $R_{1i} \neq 0$ and $R_{i1} \neq 0$, for $i = 2, 3$. In this case (4.4) implies $W^+ = 0 = W^-$. Therefore, if $W \neq 0$ we can suppose that $R_{13} = 0$. If $R_{12} \neq 0$ and $R_{i1} \neq 0$, we would have again that $W^+ = 0 = W^-$. Then either $R_{12} \neq 0$ and $R_{21} \neq 0$ or $R_{12} = 0$. The first case gives $W_1^+ = W_1^- = W_2^+ = W_2^-$, implying then $W_3^+ = W_3^-$. The second case implies that $W^+ = 0$ and $rank R_-^+ = 1$. Therefore the basis $\{\alpha_i\}$ can be chosen so that $\alpha_2, \alpha_3 \in Ker \mathcal{R}_-$. In this case, we have again that the only nonnull entry is R_{11} .

The above Lemma was used by Bourguignon in [5] to prove that a compact 4-manifold with harmonic curvature and non-zero signature is Einstein. Recall that for an oriented compact 4-manifold, the signature is given by

$$\tau = \frac{1}{12\pi^2} \int_M \|W^+\|^2 - \|W^-\|^2 dV$$

Moreover, the harmonicity of the curvature tensor implies that the Ricci tensor is Codazzi. Therefore, on a neighborhood U where the Ricci tensor has two eigenvalues of multiplicity 2, U is a Riemannian product of two surfaces by Lemma 2 of [7] and the de Rham's theorem. Then, in the case of harmonic curvature, $Z \neq 0$ on an open and dense set implies $spectrum W^+ = spectrum W^-$,

which in turn implies $\tau = 0$. Supposing the harmonicity of the traceless Ricci tensor only, we no longer have that the Ricci tensor is Codazzi. Then, we will assume the condition $(*)$ of section 3 and prove:

Proposition 4.6. *Let M be a 4-manifold whose sectional curvatures satisfy the condition $(*)$. Suppose that the traceless Ricci tensor is harmonic and has constant norm. Then either M is Einstein or $\|W^+\| = \|W^-\|$.*

Proof. If $Z = 0$ on an open set then Z is identically zero by [1] and M is Einstein. Then, let us suppose that $Z \neq 0$ on a dense set of M . It follows from the proof of Lemma 4.5 that now we have to consider only the case that the matrix R_-^+ has only one non-null entry on a neighborhood U . Suppose that R_{11} is the only non-null entry. This implies that $W_1^+ = W_1^-$ which gives (with the notation of section 2) $\lambda_1 = \varphi_1$. Then we have in (2.7) that $\lambda_1 = \varphi_1 = \frac{1}{2}(K_{12} + K_{34})$. But the condition $(*)$ implies that $0 \leq \frac{1}{2}(K_{12} + K_{34}) \leq \frac{S}{4}$ and substituting in Proposition 4.3 we get

$$\langle \rho(R_-^+), R_-^+ \rangle = \left(\frac{S}{2} - 2\lambda_1\right)R_{11}^2 \geq 0.$$

Now, from the Weitzenböck formula we obtain that

$$\langle \Delta(R_-^+), R_-^+ \rangle = \frac{1}{2}\Delta(\|R_-^+\|^2) + \|\nabla R_-^+\|^2 + \langle \rho(R_-^+), R_-^+ \rangle.$$

If Z has constant norm, so does R_-^+ , since R_-^+ is the transpose of R_+^- . Therefore we conclude that $\nabla R_-^+ = 0$. This implies that all entries R_{ij} are constant on U and by continuity, constant on M . In addition, we have the local equation

$$(\nabla_X R_-^+)(\alpha_1) = X(R_{11})\beta_1 + R_{11}\nabla_X\beta_1 - R_-^+(\nabla_X\alpha_1) = 0$$

that implies that $\nabla_X\beta_1 = 0$, since R_{11} is constant and $R_-^+(\nabla_X\alpha_1) = 0$ because $\nabla_X\alpha_1$ is orthogonal to α_1 . Similarly, we obtain that $\nabla_X\alpha_1 = 0$. The existence of these local parallel sections in Λ_-^2 and Λ_+^2 implies that $\lambda_1 = \varphi_1 = \frac{S}{4}$ and $\lambda_2 = \varphi_2 = \lambda_3 = \varphi_3 = 0$. Thus $\text{spectrum } W^+ = \text{spectrum } W^-$ and $\|W^+\| = \|W^-\|$.

From this proposition, Corollary 3.3 (b) and the description of the universal cover of compact manifolds with nonnegative sectional curvature we get immediately the following results stated in the Introduction.

Theorem 4.7. *Let M be a compact 4-manifold with nonnegative sectional curvatures. Suppose that the traceless Ricci tensor is harmonic and has constant norm. Then either its universal cover \tilde{M} is homeomorphic to one of S^4 , $S^2 \times S^2$, $S^k \times \mathbf{R}^{4-k}$ for $k = 2, 3$ or M is Einstein.*

Combining the above theorem with Theorem 4.2 we obtain the Corollary:

Corollary 4.8. *Let M be a compact 4-manifold with positive sectional curvatures. Suppose that the traceless Ricci tensor is harmonic and has constant norm. Then either M is homeomorphic to one of S^4 , \mathbf{RP}^4 , \mathbf{CP}^2 or the group G of all isometries of M is finite.*

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