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# STRICT CALIBRATIONS, CONSTANT MEAN CURVATURE AND TRIPLE JUNCTIONS

**Frank Morgan**

**Abstract**

Strict calibrations have comass strictly less than one off the calibrated surface $S$ and hence prove $S$ *uniquely* area-minimizing. Ordinary and strict calibrations, with the usual closure condition relaxed, can prove constant-mean-curvature surfaces area-minimizing for fixed volume constraints. Strict calibrations are sufficiently adaptable to prove minimizing properties of certain triple junctions of constant-mean-curvature surfaces. Related questions about minimal surfaces crossing in $\mathbf{R}^4$ remain open.

## 1. Introduction

**1.1. Strict calibrations.** Suppose a smooth submanifold $S$ with boundary in $\mathbf{R}^n$ has a *calibration*, i.e., a closed differential form $\psi$ on $\mathbf{R}^n$ with comass $||\psi(x)||^* \leq 1$, with equality on tangent planes to $S$. Then $S$ is area-minimizing: given any other candidate $S'$ with the same boundary,

$$\text{area } S' \geq \int_{S'} \psi = \int_S \psi = \text{ area } S.$$

If $\psi$ is a *strict* calibration, i.e., $||\psi(x)||^* \leq 1$ for $x \notin S$, then $S$ is uniquely area-minimizing.

For example, the unit interval $I$ on the $x$-axis in $\mathbf{R}^2$ is calibrated (not strictly) by $dx$ and by $dr$. Note that the level curves $\{x = c\}$ remain equidistant after they leave $I$, as do the level curves $\{r = c\}$, as depicted in Figure 1.1.

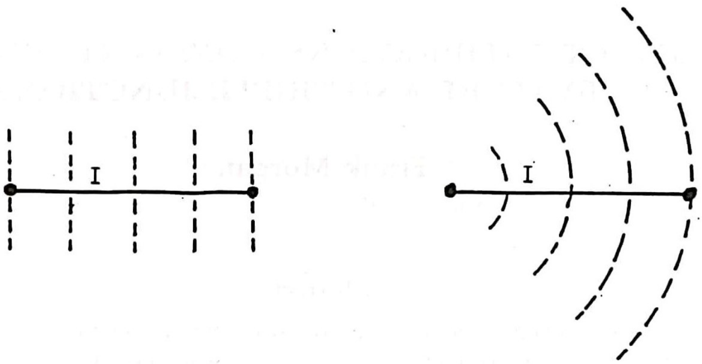


**Figure 1.1.**
The level curves $\{x = c\}$ and $\{r = c\}$ corresponding to the calibrations $dx$ and $dr$ remain equidistant.

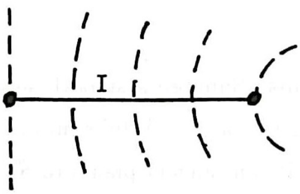


**Figure 1.2.**
The level curves of a strict calibration fan out as they leave $I$.

For a strict calibration, the level curves need to fan out as they leave $I$, as in Figure 1.2. Such a calibration is provided by

$$\psi = d\left(\left(1-\frac{\epsilon}{2}y^2\right)x\right) = \left(1-\frac{\epsilon}{2}y^2\right)dx - \epsilon xy\,dy$$

(for $x, y$ not too big). Note how the $\left(1-\frac{\epsilon}{2}y^2\right)$ factor makes $||\psi||^* < 1$ off the $x$-axis while the second term keeps $\psi$ closed.

Strict calibrations are less rigid than calibrations with comass identically 1 and are often easier to find and more adaptable. Gary Lawlor [L., Thm. 6.3.1.] used strict calibrations to prove any regular and many singular minimal surfaces locally minimizing.

For a general introduction with further references, see [M2].

**1.2. Calibrating constant-mean-curvature surfaces.** We say that a hypersurface $S$ with boundary has a *d-constant calibration* $\psi$ if $d\psi$ is *constant* and $||\psi(x)||^* \leq 1$, with equality on tangent plane to $S$. It follows that $S$ minimizes area for fixed algebraic volume: given any other hypersurface $S'$ such that $S-S'$ encloses net signed volume 0,

$$\text{area} \quad S' \geq \int_{S'} \psi = \int_S \psi = \text{ area } \; S.$$

In particular, $S$ has constant mean curvature.

For example let $S$ be the graph of a real-valued function on a smooth compact domain $D \subset \mathbf{R}^n$ satisfying the constant-mean-curvature equation, so $S$ has constant mean curvature $H$. Then vertical translations of the covector dual to $S$ provide a $d$-constant calibration $\psi$ with $d\psi = HdV$ and $||\psi(x)||^* = 1$. It follows that $S$ minimizes area in competition with hypersurfaces in $D \times \mathbf{R}$ with the same boundary, enclosing the same net signed volume above $D$. (See [M2, remark at end of 6.1]).

A nicer, strict, local, $d$-constant calibration of a small arc of the unit circle $\{r = 1, |\theta| \leq \theta_0\}$ is given by

$$\psi = (1 - (\Delta r)^2) r d\theta - 3r\Delta r\theta dr,$$

where $\Delta r = r - 1$.

Similar arguments doubtless have been used many times. Since announcing over results, we have come across such applications by Rossman [R, Lemma 2.5] and Duzaar and Steffen [DS, Thm. 1].

**1.3. Calibrating triple junctions.** The paired calibrations of [LM] provide a new proof that three hyperplanes in $\mathbf{R}^{n+1}$ meeting at $120^o$ angles minimize area. Theorem 2.1 uses strict $d$-constant calibrations to generalize the proof to three small portions of constant-mean-curvature hypersurfaces meeting along a surface $C$ at $120^o$ angles, with mean curvatures summing to 0. Competitors are assumed to have the same boundary, locally to partition space into regions of the same volume, and to meet along a surface $C'$ which is $C^1$ close to $C$. (This last hypothesis on $C'$ would follow from appropriate boundary regularity results).

Theorem 2.1 is new for $\mathbf{R}^3$ and above. For arcs of circles in $\mathbf{R}^2$, the result follows from the theorem of Foisy, Alfaro, Brock, Hodges, and Zimba [F, Thm. 2.9] that the standard double bubble in $\mathbf{R}^2$ globally provides the least-perimeter way to enclose and separate two regions of prescribed areas, without any additional assumption about the set of points $C'$ where arcs meet. Their result does, however, depend on existence and regularity theory. In $\mathbf{R}^3$ and above, it remains an open question whether the standard double bubble provides the least-area way to enclose and separate two regions of prescribed volumes, despite notable recent progress by Michael Hutchings [H]. Our results establish local minimizing properties. Our methods readily generalize to more complicated singularities, as when six surfaces and four curves meet at a point, as in a triple bubble in $\mathbf{R}^3$.

Our calibrations require an extra term to guarantee that perturbing $C$ will not save area. Strict calibrations provide the flexibility to add such a term.

An alternative method to similar (probably weaker) results might be to compute that the second variation of area, for fixed volume constraints, is positive. We understand that such variational formulas will appear in the Ph.D. thesis of Claire Chan under Brian White at Stanford.

We conjecture in 2.4 that soap bubble clusters are locally minimizing without any smoothness or topological restrictions on the comparison surfaces.

**1.4. Free boundary problems.** There cannot be a calibration proof that the polar cap is the least-area surface with free boundary to enclose given volume in the extended cone over the Arctic circle. A calibrated surface is minimizing in competition with surfaces of fractional multiplicity, and a cap with twice the radius, multiplicity 1/8, and the same enclosed "volume" (8 times as big with multiplicity 1/8), has half the "area" (4 times as big with multiplicity 1/8). On the other hand, K. Brakke has observed that the "divergence theorem proof" [Be, 12.11.4] of the isoperimetric inequality immediately generalizes to the polar cap. Indeed, the argument proves that a cap of a hypersphere in $\mathbf{R}^n$ inside any convex cone with vertex at its center uniquely provides the least-area surface enclosing given volume. It further generalizes to show that for a general norm $\Phi$, a cap of the "Wulff shape" is $\Phi$-minimizing (see [M4, 10.6]). (The perspective of Zia, Avron, and Taylor [ZAT] is that a cap of a Wulff shape is itself a Wulff shape; cf. [BrM, Intro.]).

**1.5. Calibrating a pair of intersecting surfaces in $\mathbf{R}^4$.** We seek in vain to settle an old question [Br, Problem 1.8] by certifying by calibration a first example of a singularity in an area-minimizing surface in $\mathbf{R}^4$ which is not complex analytic. It is known that an area-minimizing surface in $\mathbf{R}^4$ is a classical branched immersed minimal surface [C], complex analytic to first order at any singularity [M3, Cor. 4].

Perhaps any two minimal surfaces crossing orthogonally at $\mathbf{0}$ are locally area-minimizing. Section 3 describes an unsuccessful attempt to prove such an example with strict calibrations.

**1.6. IX EGD, Vitória.** We announced some of these results at the IX Escola de Geometria Diferencial in Vitoria, July, 1994. We would like to thank the gracious Brasileiros, especially Florencio, Lúzia, Luiz Pedro, and Valmecir, for their kind hospitality.

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## 2. Calibrating Triple Junctions

The following theorem uses paired, strict, $d$-constant calibrations to prove minimizing properties of the kind of triple junctions of surfaces which occur in soap bubble clusters.

**2.1. Theorem.** *Consider three constant-mean-curvature hypersurfaces $M_i$ in $\mathbf{R}^n$ of mean curvature $H_i \in \mathbf{R}$ meeting real analytically along an $(n-2)$-dimensional surface $C$ as in Figure 2.1. Suppose the $M_i$ meet at $120^o$ angles and $\Sigma H_i = 0$. Then in a small ball $B$ centered at a point $p \in C$, the surfaces $M_i$ uniquely minimize total area among triples of rectifiable currents*

(1) *meeting along a variable surface $C'$ which is $C^1$ close to $C$,*

(2) *with the same boundary in the sphere $\partial B$,*

(3) *dividing $B$ into three regions of fixed volumes.*

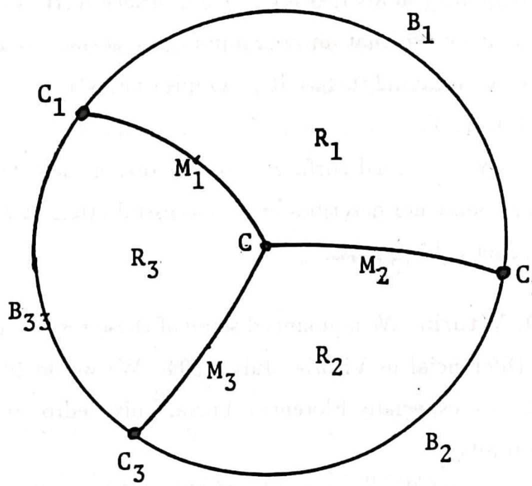


**Figure 2.1.**
Three hypersurfaces $M_i$, meeting along a surface $C$, divide $B$ into regions $R_i$ bounded by $M_{i+1} - M_i + B_i$.

**Proof.** Let $\rho$ denote distance from $p$; let $r$ denote distance from $C$. We now write $M_i$ for the restriction of $M_i$ to $B$, with $\partial M_i = C_i - C$. Let $ds$ denote the area element along $C$ and more generally its pullback to $B$ via nearest point projection onto $C$. Let $\overline{M}_i$ denote the analytic continuation of $M_i$ in $B$. Let $h_i$ denote distance from $\overline{M}_i$. Let $z_i$ denote displacement above the tangent plane to $\overline{M}_i$ at $p$. Let $\psi_i$ denote the unit $(n-1)$-form dual to $\overline{M}_i$, extended to $B$ by translation normal to $T_p\overline{M}_i$; then $d\psi_i = H_i dV$, where $H_i$ is the constant mean curvature of $\overline{M}_i$ (cf. 1.2). Since $d(\psi_i + (-1)^n H_i z_i(\nabla z_i \rfloor dV)) = 0$, there is a smooth $(n-2)$-form $\Psi_i$ with $d\Psi_i = \psi_i + (-1)^n H_i z_i(\nabla z_i \rfloor dV)$ and $\Psi_i(p) = 0$. For small $\epsilon > 0$, let

$$\varphi_i = (1 - h_i^2)\psi_i + d(h_i^2) \wedge \Psi_i - (-1)^n H_i h_i^2 z_i(\nabla z_i \rfloor dV) + \epsilon\, ds \wedge d(h_i^2).$$

Then

$$\begin{aligned} d\varphi_i &= H_i dV - h_i^2 H_i dV - 2h_i dh_i \wedge \psi_i + 2h_i dh_i \wedge (\psi_i + (-1)^n H_i z_i(\nabla z_i \rfloor dV)) \\ &\qquad -(-1)^n H_i(2h_i) z_i dh_i \wedge (\nabla z_i \rfloor dV) + H_i h_i^2 dV \\ &= H_i dV. \end{aligned}$$

Since $d(h_i^2)$ is small and orthogonal to $\psi_i$ along $\overline{M}_i$ and $|\Psi_i| = O(\rho)$,

$$\begin{aligned} |\varphi_i|^2 &= |\text{component parallel to } \psi_i|^2 + |\text{component orthogonal to } \psi_i|^2 \\ &\le (1 - h_i^2 + O(h_i^2)O(\rho) + O(h_i^2)\epsilon)^2 + (O(h_i)O(\rho) + \epsilon\, O(h_i))^2 \\ &\le 1 - 2h_i^2 + (O(\rho) + \epsilon)O(h_i^2) \le 1 \end{aligned}$$

for $B$ and $\epsilon$ small, with equality only on $\overline{M}_i$.

Since by hypothesis $\Sigma H_i = 0$, therefore $\Sigma d\varphi_i = 0$. Let $\Phi$ be a smooth $(n-2)$-form with $\Phi(p) = 0$ and $d\Phi = \Sigma\varphi_i$.

We claim that for a surface $C'$ which is $C^1$ close to $C$,

$$\int_C \Phi \ge \int_{C'} \Phi.$$

Let $R$ be a nice $C^1$ surface bounded by $C - C'$. Then

$$\begin{aligned} \int_C \Phi &- \textstyle\int_{C'} \Phi = \int_R \sum \varphi_i \\ &= \textstyle\int_R \sum [\psi_i - h_i^2\psi_i + d(h_i^2) \wedge \Psi_i - (-1)^n H_i h_i^2 z_i(\nabla z_i \rfloor dV) + \epsilon\, ds \wedge d(h_i^2)] \\ &= \textstyle\int_R O(r^2) + \epsilon \sum ds \wedge 2y_i dy_i, \end{aligned}$$

where in the 2-plane normal to $C$ at $q$, the coordinates $x_i, y_i$ give orthogonal coordinates with the $y_i$-axis orthogonal to $T_q\overline{M}_i$. Indeed, on $C$, $\Sigma\psi_i = 0$, changes in $\psi_i(\vec{R})$ are $O(r^2)$, $h_i = O(r)$, $|\vec{R}\lfloor dh_i| = O(r)$, $|\pm h_i - y_i| = O(r)$, and $|\pm dh_i - dy_i| = O(r)$. As a homogeneous quadratic expression invariant under $120°$ rotations, $\Sigma y_i^2$ must be some multiple of $r^2$; by symmetry, $\Sigma y_i^2 = \frac{1}{2}\Sigma(x_i^2 + y_i^2) = \frac{3}{2}r^2$.

Therefore,

$$\int_C \Phi - \int_{C'} \Phi = \int_R O(r^2) + 3\epsilon\, r ds dr \geq 0$$

for $r$ small.

We are ready to apply the "paired calibration" argument of [LM, 1.1, 2.1]. The $M_i$ divide $B$ into regions $R_i$ with

$$\partial R_i = M_{i+1} - M_i + B_i.$$

Consider competing hypersurfaces $M_i'$ meeting along some $C'$. Since on $M_i$, $h_i = 0$ and $\varphi_i = \psi_i = dA$,

$$\begin{aligned}
3\textstyle\sum \text{ area } M_i &= 3\textstyle\sum\int_{M_i}\varphi_i = \sum\int_{M_i}[(\varphi_i - \varphi_{i+1}) - (\varphi_{i-1} - \varphi_i)] + \sum\int_{M_i} d\Phi \\
&= -\textstyle\sum\int_{R_i}(d\varphi_i - d\varphi_{i+1}) + \sum\int_{B_i}(\varphi_i - \varphi_{i+1}) + \sum\int_{C_i}\Phi - 3\int_C\Phi \\
&= -\textstyle\sum\int_{R_i}(H_i - H_{i+1}) + \sum\int_{B_i}(\varphi_i - \varphi_{i+1}) + \sum\int_{C_i}\Phi - 3\int_C\Phi \\
&\leq -\textstyle\sum\int_{R_i'}(H_i - H_{i+1}) + \sum\int_{B_i}(\varphi_i - \varphi_{i+1}) + \sum\int_{C_i}\Phi - 3\int_{C'}\Phi,
\end{aligned}$$

because vol $R_i'$ = vol $R_i$ and $\int_C\Phi \geq \int_{C'}\Phi$. Hence

$$\begin{aligned}
3\textstyle\sum \text{ area } M_i &\leq -\textstyle\sum\int_{R_i'}(d\varphi_i - d\varphi_{i+1}) + \sum\int_{B_i}(\varphi_i - \varphi_{i+1}) + \sum\int_{C_i}\Phi - 3\int_C\Phi \\
&= \textstyle\sum\int_{M_i'}[(\varphi_i - \varphi_{i+1}) - (\varphi_{i-1} - \varphi_i)] + \sum\int_{M_i'} d\Phi \\
&= 3\textstyle\sum\int_{M_i'}\varphi_i \leq 3\sum \text{ area } M_i'.
\end{aligned}$$

Therefore the $M_i$ minimize area. Suppose equality holds in the last inequality. Since $|\varphi_i| \leq |$, with equality only on $\overline{M}_i$, the asserted uniqueness follows.

**2.2. Triple junctions of minimal surfaces.** For the case of minimal hypersurfaces ($H_i = 0$), the proof of Theorem 2.1 simplifies and does not require any hypothesis of fixed volumes. Three minimal hypersurfaces in $\mathbf{R}^n$ meeting real analytically along an $(n-2)$-dimensional surface at 120 degree angles are uniquely locally area-minimizing (assuming the singular surface remains $C^1$ close).

**2.3. Other types of singularities.** The proof of Theorem 2.1 can be generalized to other types of singularities, such as the other soap bubble cluster singularity, where four singular curves meet at a point (or in fact to any number of singular curves meeting at a point).

The following conjecture implies that all equilibrium soap bubble clusters in $\mathbf{R}^3$ are locally area-minimizing, without any unnecessary restrictions on the comparison surfaces.

**2.4. Conjecture.** *Consider an almost everywhere smooth compact hypersurface $S$ in $\mathbf{R}^n$. Suppose that at every point of $S$ the measure-theoretic tangent cone is strictly area-minimizing among separators of regions. Suppose the mean curvatures sum to 0 around any closed path in $\mathbf{R}^n$ which intersects only regular pieces of $S$ (in particular, regular pieces have constant mean curvature). Then $S$ is locally area-minimizing among separators of regions of prescribed volumes.*

Note that local competing surfaces must maintain the identity and volume of the regions of space, as well as their own boundary.

Some similar results for minimal surfaces with isolated singularities were proved by Hardt and Simon [HS, Thm. 4.4] for one notion of "strictly minimizing" [HS, §3] and by G. Lawlor [L, Thm. 6.3.1] for a related one [L, 6.1.1]. A forthcoming paper of Lawlor and Morgan treats the case of three minimal surfaces meeting at 120° angles along a curve, without any restrictions on the comparison surfaces.

## 3. Calibrating a pair of intersecting surfaces in $\mathbf{R}^4$

*Can a 2-dimensional area-minimizing surface in $\mathbf{R}^n$ have a non-complex-analytic singularity?*

The simplest example of a (complex-analytic) singularity is the sum (i.e. union) $P_1 + P_2$ of the $xy$ and $zw$ planes. One simple candidate for a non-complex-analytic singularity replaces $P_1$ by a non-complex-analytic minimal surface $S_1$ tangent to $P_1$ at $\mathbf{0}$, perhaps lying in $\mathbf{R}^3 \subset \mathbf{R}^4$, given by $z = u(x, y), w = 0$.

$P_1 + P_2$ is calibrated by the Kähler form $dxdy + dzdw$, which has comass exactly 1 at all points. Attempts to perturb it to calibrate some $S_1 + P_2$, as by replacing $dxdy$ by the standard $z, w$-invariant calibration $\varphi_1$ of $S_1$, typically yield forms of comass greater that 1 (see §3.1).

There is, however, a nice strict calibration $\psi_1 + \psi_2$ of $P_1 + P_2$; with

$$\psi_1 = d\left(\frac{1}{2}r_1^2(1 - r_2^2 - r_1^2 r_2^2)d\theta_1\right),$$
$$\psi_2 = d\left(\frac{1}{2}r_2^2(1 - r_1^2 - r_1^2 r_2^2)d\theta_2\right),$$
$$||\psi_1 + \psi_2||^* \leq 1 - r_1^2 r_2^2$$

for $r_1 r_2$ small. Each $\psi_i$ alone calibrates $P_i$. Without the $r_1^2 r_2^2$ term in the definitions of the $\psi_i$, the comass of $\psi_1 + \psi_2$ would be identically 1. Even with the $r_1^2 r_2^2$ term, the leeway is too small to admit perturbations of $P_1$ and $\psi_1$ (as in [L, §6], for example). Indeed, $dxdy + \psi_2$ has comass exceeding 1, and any $\psi_1'$ along $S_1$, where it must have length 1, will sometimes be less favorably tilted with respect to $\psi_2$ than $dxdy$ and hence yield $||\psi_1' + \psi_2||^* > 1$. Therefore first modify $\psi_2$ to a $\psi_2'$ which is orthogonal to $S_1 = \{z = u(x, y), w = 0\}$, such as

$$\psi_2' = d\left((1 - r_1^2)(z - u)\right) dw,$$

which near $\mathbf{0}$ has length $|\psi_2'| \leq 1 - r_1^2$. The best corresponding perturbation of $\psi_1$ I have been able to find is

$$\psi_1' = \left((1 - (z - u)^2)\right)\varphi_1 + 2\lambda x(z - u)\varphi_2 + 2\lambda y(z - u)\varphi_3,$$

where the $\varphi_i(x,y)$ are independent of $z, w$; $\varphi_1$ is the unit dual to $S_1$ as before, and

$$\varphi_2 = \frac{u_x dxdy + dydz}{\sqrt{1+u_x^2+u_y^2}}, \qquad \varphi_3 = \frac{u_y dxdy - dxdz}{\sqrt{1+u_x^2+u_y^2}},$$

are orthogonal to $\varphi_1$, with common normalizing denominators for convenience. $\psi_1'$ is not closed, but $d\psi_1'$ is favorably signed if $\lambda > 1/2$. $||\psi_1' + \psi_2'||^* \leq 1$ if $\lambda < 1/4$. Unfortunately those two conditions on $\lambda$ are contradictory.

**3.1. Attempts to perturb the Kähler form to calibrate $\mathbf{S_1 + P_2}$.** The Kähler form $dxdy + dzdw$ calibrates the orthogonal axis planes $P_1 + P_2$. One calibration $\varphi$ of a minimal surface $S_1 \subset \mathbf{R}^3 \subset \mathbf{R}^4$ tangent to $P_1$ at $\mathbf{0}$ comes from vertically translating the unit dual covector. For $S_1 + P_2$ one candidate calibration is $\varphi + dzdw$, but is has comass greater that 1. A second candidate is $\varphi + \varphi^{\perp}$. Unfortunately,

$$d(\varphi + \varphi^{\perp}) = 0 + d\varphi^{\perp} \neq 0.$$

Indeed, the following proposition holds.

**3.2. Proposition.** *Let $\varphi$ be a smooth unit 2-covector field on $\mathbf{R}^4$ such that $d(\varphi + \varphi^{\perp}) = 0$. Then $\varphi + \varphi^{\perp}$ is constant and hence orthogonally equivalent to the Kähler form.*

**Proof.** Using coordinates $x_i$ on $\mathbf{R}^4$,

$$\begin{aligned}\varphi + \varphi^{\perp} = f(dx_1dx_2 + dx_3dx_4) \ &+ g(dx_1dx_3 - dx_2dx_4)\\ &+ h(dx_1dx_4 + dx_2dx_3).\end{aligned}$$

The condition $d(\varphi + \varphi^{\perp}) = 0$ means that

(1) $f_3 - g_2 + h_1 = 0,$

(2) $f_4 - g_1 - h_2 = 0,$

(3) $f_1 + g_4 - h_3 = 0,$

(4) $f_2 + g_3 + h_4 = 0.$

Appropriately differentiating and adding the four equations yields that $\Delta f = \Delta g = \Delta h = 0$. Since $\varphi$ is a *unit* covector, $|\varphi + \varphi^{\perp}| = \sqrt{2}$ and

$$\begin{aligned} 0 &= \frac{1}{2}\Delta(f^2+g^2+h^2) = f\Delta f + g\Delta g + h\Delta h + |\nabla f|^2 + |\nabla g|^2 + |\nabla h|^2 \\ &= |\nabla f|^2 + |\nabla g|^2 + |\nabla h|^2. \end{aligned}$$

Therefore $\varphi + \varphi^{\perp}$ is constant.

**3.3. Comass of a sum** (cf. [M1]). In the preceding study, the following formulas for the comass of a sum are useful. Let $\psi = \psi_1 + \psi_2$ be the sum of two simple 2-covectors in $\mathbf{R}^4$. We may assume they are in normal form:

$$\begin{aligned} \psi_1 &= |\psi_1| dx_1 dx_2 \\ \psi_2 &= |\psi_2|(\cos\theta_1 dx_1 + \sin\theta_1 dx_3) \wedge (\cos\theta_2 dx_2 + \sin\theta_2 dx_4) \end{aligned}$$

with $0 \leq \theta_1 \leq \pi/2$, $\theta_1 \leq \theta_2 \leq \pi - \theta_1$.

[M1, Cor. 4.4] gives a formula for the comass of any 2-covector in $\mathbf{R}^4$. If $\theta_2 = \pi/2$, the formula implies

(1) $2||\psi||^* = \sqrt{|\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2|\sin\theta_1} + \sqrt{|\psi_1|^2 + |\psi_2|^2 - 2|\psi_1||\psi_2|\sin\theta_1}$

Alternatively if $|\psi_1| = |\psi_2| = 1$,

(2) $||\psi||^* = \cos\dfrac{\theta_1 - \theta_2}{2} + \cos\dfrac{\theta_1 + \theta_2}{2}.$

If $\theta_2 = \pi/2$ and $|\psi_1| = |\psi_2| = 1$, both (1) and (2) become

(3) $||\psi||^* = \sqrt{2}\cos\frac{\theta_1}{2}.$

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Department of Mathematics
Williams College
Williamstown, Massachusetts 01267
USA
*Email:* Frank.Morgan@williams.edu