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LAPLACE TRANSFORMATION FOR CARTAN SUBMANIFOLDS

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1. Introduction

In this paper, we briefly describe some of the main geometrical and analytical aspects of the higher-dimensional Laplace transformation for Cartan manifolds. The results will be given without proof. For further details and complete proofs, we refer the reader to [KT1] and [KT2].

The classical two-dimensional Laplace transformation, not to be confused with the Laplace transform in harmonic analysis, is a transformation of second-order scalar hyperbolic partial differential equations in the plane which was originally introduced as a method of closed form integration. To any such equation, one associates two Laplace invariants h and k. If one of these invariants is zero, then the equation can be integrated by quadratures as a succession of two parametrized first-order linear ordinary differential equations. If the Laplace invariants are non-zero, then there are well-defined Laplace transformations and the principle of the method of Laplace is to iterate the transformations until one possibly arrives at an equation with a vanishing invariant, which one integrates by quadratures. One then transforms back to obtain a solution of the original equation depending on two arbitrary functions on one variable and finitely many of their derivatives. The classical treatises of Darboux [D], Goursat [G] and Forsyth [F] contain extensive treatments of this method of integration. Recall [D] that the classical Laplace transformation is the analytic expression of

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a geometrically defined transformation of surfaces isometrically immersed in $\mathbf{R}^3$, with a coordinate net which is conjugate for the second fundamental form. One constructs the edge of regression of the tangent developable of the one-parameter family of tangent planes along of one of the coordinate curves of the surface. By varying this curve within its family, one obtains generically another surface with a conjugate coordinate net. This surface will reduce to a curve if and only if the corresponding Laplace invariant is identically zero.

There has recently been a resurgence of interest in the Laplace transformation, motivated for the greatest part by some interesting connections with the theory of completely integrable systems. For example, it has been observed that if the Laplace invariants of a transformed hyperbolic equation are zero, then the Laplace invariants of the original equation give rise to solutions of the A_2 Toda field equations depending on four arbitrary functions of one variable [Va]. Likewise, the Darboux-Crum transformation [Cr], [DT], of Sturm-Liouville operators appears naturally in this context [DOV].

The search for truly higher-dimensional examples of completely integrable equations has only led to a few classes of examples, which have now been extensively studied (see for example [BT], [CT], [TT]). A conclusion that has been drawn from these works is that complete integrability appears to be a rare phenomenon in more than two independent variables. In view of this fact, the problem of generalizing the Laplace transformation to more than two independent variables is of considerable interest.

An effective guiding principle in the search for this generalization is to put the problem in its proper geometric context. This principle proved to be very successful the generalization of Bäcklund's Theorem to higher dimensions [TT].

The higher-dimensional generalization of this transformation was first obtained by Chern [C], who considered n-dimensional submanifolds of projective space, all of whose second fundamental forms are simultaneously diagonalized by the coordinate net. Such manifolds are called *Cartan manifolds* by Chern, since they were first considered by E. Cartan [Ca] in the context of projective differential geometry. By working out analytically the Euclidean version of

Chern's transformation, we [KT1] were able to obtain an n-dimensional generalization of the classical Laplace transformation which applies to overdetermined systems of $n(n-1)/2$ linear second-order partial differential equations , for a scalar function of n independent variables. We showed that any such system admits $n(n-1)^2$ higher-dimensional Laplace invariants and, generically, $n(n-1)$ Laplace transformations. We then proved a reduction theorem to the effect that the systems whose higher-dimensional Laplace invariants are all zero in one direction can be integrated by quadratures in terms of the solutions of an $(n-1)$-dimensional system of the same type.

Even in the 2-dimensional case, it is not a simple matter to determine whether an equation is integrable by the method of Laplace. This is because in general, one cannot predict the number of iterations of the Laplace transformation required in order to obtain an equation with a vanishing Laplace invariant. A considerable amount of attention has therefore been directed in the classical 2-dimensional theory to the question of periodicity. An equation is said to be r-periodic if it is equivalent to its r-th Laplace iterate under a rescaling of the dependent variable and a reparametrization of the independent variables separately. Periodic equations thus cannot be integrated by applying the Laplace transformation in their direction of periodicity, no matter how many times one iterates the transformation. Darboux [D] proved that every 1-periodic equation in the plane is equivalent under one of the above transformations to the Klein-Gordon equation

$$z_{,xy} = \epsilon z, \quad \epsilon^2 = 1.$$

In particular, if an equation is 1-periodic with respect to one of the transforms, then it is also 1-periodic with respect to the other. In [KT2], we obtained the generalization of Darboux's normal form result for 1-periodic equations in the plane to systems in n dimensions. In Section 2, we review the classical Laplace transformation for second-order hyperbolic equations in the plane, together with its geometric interpretation. In Section 3, we recall from [KT1] the Euclidean version of Chern's higher-dimensional Laplace transformation of

Cartan manifolds. In Sections 4 and 5, we review from [KT1] and [KT2] the fundamentals of the higher-dimensional Laplace transformation and of the (i,j)-higher-dimensional Laplace invariants $m_{ij}^{ij}, m_{kk}^{ij}, 1 \leq i,j,k \leq n, \quad i,j,k$ distinct, for the systems considered. These are compatible overdetermined systems of linear partial differential equations of the form

$$y_{,k\ell} + a_{k\ell}^{k} y_{,k} + a_{k\ell}^{\ell} y_{,\ell} + c_{k\ell} y = 0, \quad 1 \leq k \neq \ell \leq n \tag{1.1}$$

where the coefficients are smooth functions of the independent variables $x_1, \cdots, x_n$, satisfying certain conditions of compatibility. In contrast with the 2-dimensional case, the higher-dimensional Laplace invariants cannot be prescribed freely in $n \geq 3$ dimensions. They are highly constrained by the compatibility conditions for the overdetermined system (1.1). This fact is one of the main sources of the complexity of the periodicity problem in higher dimensions. We recall the reduction theorem for the systems whose higher-dimensional Laplace invariants vanish in one direction (Theorem 4). In Section 6 we first define the notion of periodicity in higher dimensions. We then give a minimal set of necessary and sufficient conditions, in terms of the higher-dimensional Laplace invariants and their derivatives, for a system (1.1) to be 1-periodic in a given direction. These conditions are integrated in Theorem 5 to give a normal form for the most general system (1.1) which is 1-periodic. Theorem 5 is the n-dimensional generalization of Darboux's theorem on the Klein-Gordon equation in two dimensions. In dimension three, we have the remarkable fact that a system which is 1-periodic with respect to one of the transforms and transformable in all other directions will also be 1-periodic in all other directions (Theorem 6).

2. The classical Laplace transformation and its geometrical interpretation

Consider a second-order partial differential equation for a real valued function $z(u,v)$ given by

$$z_{uv} + az_u + bz_v + cz + l = 0 \tag{2.1}$$

where a, b, c and l are differentiable functions of u and v. The classical Laplace method for solving this equation is given as follows. Consider the Laplace invariant

$$h = a_u + ab - c.$$

If $h = 0$, then (2.1) reduces to

$$\frac{\partial}{\partial u}(z_v + az) + b(z_v + az) + l = 0.$$

Letting

$$s = \int b \, du, \quad \tilde{s} = \int a \, dv,$$

we obtain

$$z = e^{-\tilde{s}} \left[-\int e^{\tilde{s}-s} \left(\int e^{s} l \, du - F(v) \right) dv + G(u) \right],$$

where F and G are arbitrary differentiable functions. The functions F and G are determined by the initial conditions $z(u_0, v)$ and $z(u, v_0)$.

Similarly, if the Laplace invariant

$$k = b_v + ab - c$$

vanishes, we can also explicitly solve (2.1).

The classical transformation of Laplace is a transformation theory for differential equations of the form (2.1). Assume $h \neq 0$ and define

$$z_1 = z_v + az.$$

Then one can easily see that (2.1) is transformed into a differential equation of the same form for z_1. Similarly, if $k \neq 0$ we consider

$$z_{-1} = z_u + bz$$

which also transforms (2.1) into an equation of the same type for z_{-1}. Those two functions z_1 and z_{-1} are said to be the $\mathcal{L}_1$ and $\mathcal{L}_{-1}$ Laplace transforms of z. Moreover, one can invert those transformations. In fact, it is not difficult to see that when $h \neq 0$ and $k \neq 0$,

$$z = [\mathcal{L}_{-1}(z_1) + l]/h \quad and \quad z = [\mathcal{L}_1(z_{-1}) + l]/k.$$

The basic idea for this method is to apply a sequence of Laplace transformations to a given equation (2.1) until eventually it is transformed into one which has a vanishing Laplace invariant. This equation is integrated and then using the inverse transformation one obtains a solution for the given initial differential equation.

There is a geometrical construction for surfaces [D], which corresponds to the Laplace method described above.

Consider a parametrized surface $X(u,v)$ in $\mathbf{R}^3$ or $\mathbf{R}^4$ such that the coordinate curves form a conjugate net. The mixed second order derivative of X is then given by

$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v$$

where $\Gamma^i_{12}, i = 1,2$, are the Christoffel symbols of the surface X. Hence, the vector valued function X satisfies an equation of the form (2.1). The surfaces we consider will be *generic*, in the sense that we shall assume X_u, X_v, X_{uu}, and X_{vv} to be linearly independent in the ambient space.

Suppose that $\Gamma^1_{12} \neq 0$ and consider the ruled surface defined by

$$Y(t,u,v_0) = X(u,v_0) + tX_v(u,v_0).$$

The tangent plane to this surface at $t = 0$ is generated by the vectors $X_u(u,v_0)$ and $X_v(u,v_0)$. Letting u vary, we obtain a one-parameter family of tangent planes given by

$$T_i(p,u) =< p - X(u,v_0), N_i(u,v_0) >= 0 \quad p \in \mathbf{R}^3 or \mathbf{R}^4,$$

where the N_i, $i = 1$ or $i = 1,2$ (according to whether the surface is immersed in $\mathbf{R}^3$ or $\mathbf{R}^4$) are linearly independent normal vector fields to the surface X spanning the normal space of X. The characteristic line of the surface Y is the intersection of the planes

$$\begin{aligned} T_i(p,u) &= 0, \\ T_{i,u}(p,u) &= 0. \end{aligned}$$

It follows that the direction of the characteristic line is given by $X_v(u,v_0)$. In fact, for each u, the line

$$p(t) = X(u,v_0) + tX_v(u,v_0)$$

is contained in the tangent plane and also in the neighboring planes. This follows from the identity $< X_v, N_{i,u} >= 0$, which is a consequence of the fact that the surface X is parametrized by conjugate curves.

The edge of regression of the surface Y is determined by the intersection of the planes

$$\begin{aligned} T_i(p,u) &= 0, \\ T_{i,u}(p,u) &= 0, \\ T_{i,uu}(p,u) &= 0. \end{aligned}$$

It follows that for each u, there exists a unique point $p(t_0)$ on the line $p(t)$ such that $Y(t_0, u, v_0)$ is on the edge of regression. In fact,

$$T_{i,uu}(p(t),u) = -(1 + t\Gamma^1_{12}) < X_u, N_{i,u} > .$$

Since we are assuming the surface is generic, we conclude that $t = -1/\Gamma^1_{12}$. As u and v vary, we obtain in general, a map $X_1(u,v)$ given by

$$X_1 = X - X_v/\Gamma^1_{12},$$

whose differential satisfies

$$\begin{pmatrix} X_{1,u} \\ X_{1,v} \end{pmatrix} = \begin{pmatrix} h/\Gamma^1_{12} & 0 \\ \star & -1 \end{pmatrix} \begin{pmatrix} X_v \\ X_{vv} \end{pmatrix} \tag{2.2}$$

where

$$h = -\Gamma^1_{12,u} + \Gamma^1_{12}\Gamma^2_{12}$$

Since X_v and X_{vv} are assumed to be linearly independent, it follows from (10) that $X_1(u,v)$ will be a parametrized surface if and only if h is not zero. Moreover, the coordinate curves will also form a conjugate net on X_1. We will say that $X_1 = \mathcal{L}_1(X)$ is the $\mathcal{L}_1$-*Laplace transform* of the surface X. From (2.2) we see that X_1 reduces to a curve if and only if $h = 0$. Similarly, one can do the same construction by interchanging u and v obtaining what is called the $\mathcal{L}_{-1}$-*Laplace transform* of the surface, $X_{-1} = \mathcal{L}_{-1}(X)$. The two Laplace transformations $\mathcal{L}_1$ and $\mathcal{L}_{-1}$ are inverses of each other. If $h \neq 0$, then $\mathcal{L}_{-1}(\mathcal{L}_1(X)) = X$. Similarly, when $k \neq 0$ we have $\mathcal{L}_1(\mathcal{L}_{-1}(X)) = X$.

Examples:

a) Consider the surface (see Fig.1 a) given by

$$X(u,v)=\left(\frac{e^{uv}(uv-1)}{v^2},\frac{e^{uv}(u^2v^2-2uv+2)}{v^3},\cos v\right)$$

It is easy to see that u, v are conjugate coordinates for the surface, $\Gamma^1_{12}=u$ and $\Gamma^2_{12}=0$. Moreover, we get the invariant $h=1$ By applying the Laplace transformation $\mathcal{L}_1$ to X, we obtain the surface (see Fig. 1 b) given by

$$X_1=\left(\frac{e^{uv}(uv-2)}{uv^3},\frac{e^{uv}(u^2v^2-4uv+6)}{uv^4},\frac{u\cos v+\sin v}{u}\right)$$

In Fig. 1, we can also see some of the straight lines of the line congruence between the surfaces X and X_1.

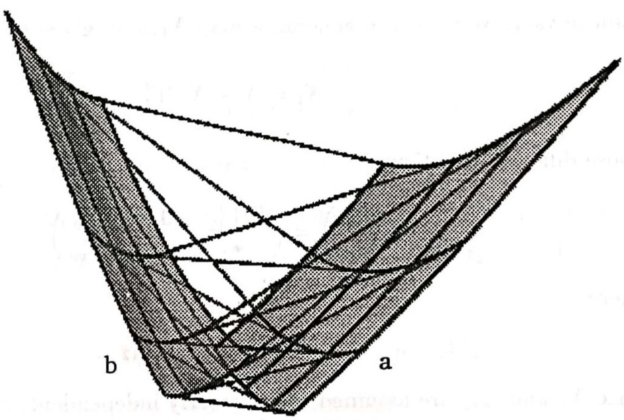

Fig. 1

b) Let $X(u,v)$ be a surface of rotation

$$X(u,v)=(f(u)\cos v,f(u)\sin v,g(u)),$$

generated by a regular plane curve $(f(u),g(u))$, where $f(u)>0$. Assume $f'\neq 0$, then its Laplace transform $\mathcal{L}_{-1}(X)$ reduces to a subset of the rotation axis. In fact, $\Gamma_{12}^1 = 0$ and $\Gamma_{12}^2 = f'/f$. Hence,

$$X_{-1} = (0, 0, g - \frac{fg'}{f'}).$$

We observe that both invariants h and k vanish for a surface of rotation. In Fig. 2, we can see the Laplace transform (vertical segment) of the upper-half torus and the line congruence which takes points of the torus into its Laplace transform.

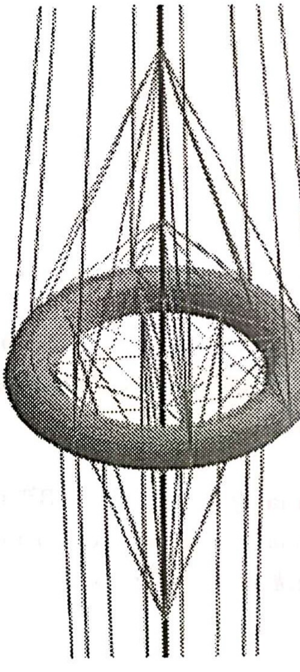

Fig. 2

3. The geometry of Cartan manifolds

In [C], Chern considered a class of manifolds in projective space, which he called Cartan manifolds, since they had been previously considered by Cartan in a different context [Ca]. Chern showed that there exists a transformation that generalizes for these manifolds the classical Laplace transformation for surfaces admitting a conjugate net. In this section, we give the Euclidean version of this transformation.

Definition: A Riemannian n-dimensional manifold M^n isometrically immersed in $\mathbf{R}^{2n}$ is said to be a *Cartan manifold* if there exist local coordinates $(x_1 \ldots x_n)$ such that the net of coordinate curves is conjugate and the osculating space is $2n$-dimensional.

It follows from this definition that when a Cartan manifold is parametrized by such coordinates the second fundamental forms are simultaneously diagonalized. In this case, we say that the manifold is parametrized by conjugate coordinates.

We will use the following range of indices

$$1 \leq i, j, k, l \leq n,$$

and we will denote as usual by Γ_{ij}^k the Christoffel symbols for a given parametrization $X : U \subset \mathbf{R}^n \to \mathbf{R}^{2n}$. Moreover, X_i and X_{jk} will denote the derivative of X with respect to x_i and the second derivative of X with respect to x_j and x_k respectively.

Lemma: *If $X : U \subset \mathbf{R}^n \to \mathbf{R}^{2n}$ is a Cartan manifold parametrized by conjugate coordinates, then for each i, j with $i \neq j$ the vector field X_{ij} lies in the space spanned by X_i and X_j, i. e.*

$$X_{ij} = \Gamma_{ij}^i X_i + \Gamma_{ij}^j X_j \qquad i \neq j.$$

Moreover, the Christoffel symbols satisfy

$$\frac{\partial \Gamma_{ik}^k}{\partial x_l} + \Gamma_{ik}^k \Gamma_{kl}^k - \Gamma_{il}^i \Gamma_{ik}^k - \Gamma_{il}^l \Gamma_{lk}^k = 0 \qquad l, i, k \quad \textit{distinct}.$$

It should be pointed out that the equations above do not provide the full set of integrability conditions for the immersion X to define a Cartan manifold. However, the remaining conditions (see [KT1] and [Ca]) will not be used in our analysis, as they are mainly needed to determine the degree of generality of the immersion.

In what follows, we will associate to each n-dimensional Cartan manifold X, in general a family of $n(n-1)$ manifolds which will also be Cartan manifolds. This will be achieved by considering the edge of regression of ruled manifolds constructed from X.

Consider a Cartan manifold X parametrized by conjugate coordinates. For each $(n-1)$-dimensional submanifold of X, given by fixing $x_j = x_j^0$, consider the ruled manifold defined by

$$Y(t, x_1, .., x_j^0, .., x_n) = X(x_1, .., x_j^0, .., x_n) + tX_j(x_1, .., x_j^0, .., x_n).$$

The tangent space to this manifold at $t = 0$ is generated by the vectors $X_k(x_1, .., x_j^0, ..x_n)$, $1 \le k \le n$. This gives rise to a $n-1$-parameter family of tangent spaces described by the system of equations

$$T_l(p, x_1, .., x_j^0, .., x_n) = < p - X, N_l >= 0 \quad p \in \mathbf{R}^{2n}, \quad 1 \le l \le n,$$

where N_l are normal vector fields to the manifold X and the right hand side is evaluated at $(x_1, .., x_j^0, .., x_n)$. The characteristic line of the manifold Y is the intersection of the spaces

$$\begin{aligned} T_l(p, x_1, .., x_j^0, .., x_n) &= 0, \quad 1 \le l \le n, \\ T_{l,k}(p, x_1, .., x_j^0, .., x_n) &= 0, \quad \forall\, k, k \ne j. \end{aligned}$$

It follows that the direction of the characteristic line is given by X_j. In fact, at each point $(x_1, .., x_j^0, .., x_n)$, the line

$$p(t) = X + tX_j$$

is contained in the tangent space and also in the neighboring spaces. This results from the identity $< X_j, N_{l,k} >= 0 \quad \forall k \ne j, \forall l$ implied by the fact that X is a Cartan manifold.

Now for each $i \ne j$, we define the edge of regression of the manifold Y in the direction i to be the intersection of the spaces

$$\begin{aligned} T_l(p, x_1, .., x_j^0, .., x_n) &= 0, && \forall\, l,\ 1 \le l \le n, \\ T_{l,k}(p, x_1, .., x_j^0, .., x_n) &= 0, && \forall\, ; k,\ k \ne j, \\ T_{l,ki}(p, x_1, .., x_j^0, .., x_n) &= 0, && it for fixed\, i \ne j. \end{aligned}$$

This intersection is a unique point on the characteristic line $p(t)$. In fact, since

$$T_{l,ki}(p(t), x_1, .., x_j^0, .., x_n) = -(1 + t\Gamma_{ij}^i) < X_i, N_{l,k} > \quad \forall\, l \ne j, k \ne j,$$

we conclude that $t = -1/\Gamma_{ij}^i$. As x_j^0 varies, we obtain

$$Y(x_1, \ldots, x_n) = X - X_j/\Gamma_{ij}^i, \qquad i \ne j.$$

The map Y will be called the (i,j)-*Laplace Transform* of X, where (i,j) is an ordered pair.

Our next result shows that a Laplace Transform of a Cartan manifold is generically also a Cartan manifold.

Theorem 1: *Let $X(x_1,\ldots,x_n)$ be a Cartan manifold in $\mathbf{R}^{2n}$ parametrized by conjugate coordinates. Consider an ordered pair $(i,j), i \neq j$ such that $\Gamma^i_{ij} \neq 0$. Then the map*

$$Y = X - \frac{1}{\Gamma^i_{ij}} X_j$$

defines generically a Cartan manifold.

Explicit examples of Cartan manifolds and their Laplace transforms can be found in [KT1]. It follows from the definition there are at most $n(n-1)$ Laplace transforms for a given Cartan manifold. Moreover, it is particularly simple to characterize the Cartan manifolds X for which the (i,j)-Laplace transform degenerates to a curve.

In order to state our next result, we need to introduce some notation. We consider an n-dimensional Cartan manifold in $\mathbf{R}^{2n}$ parametrized by conjugate coordinates. For each ordered pair $(i,j), i \neq j$ for which Γ^i_{ij} is nonzero, we define the functions:

$$\begin{aligned} M_{ij} &= \frac{\Gamma^i_{ij,i}}{\Gamma^i_{ij}} - \Gamma^j_{ij}, \\ M_{ll} &= \Gamma^i_{ij} - \Gamma^l_{jl}, \quad \forall\, l,\ l \neq j, \end{aligned}$$

where $\Gamma^i_{ij,i}$ denotes the derivative of Γ^i_{ij} with respect to x_i.

Corollary: *The (i,j)-Laplace Transform of a Cartan manifold X reduces to a curve if and only if*

$$M_{ij} = M_{kk} = 0 \quad \forall\, k, k \neq i, k \neq j.$$

As it occurs with the 2-dimensional case, generically the Laplace transformation is invertible.

Proposition: *If $M_{ij} \neq 0$, the inverse of the (i,j)-Laplace transform exists and it is given by the (j,i)-Laplace transform.*

4. The Higher-Dimensional Laplace Invariants

The higher-dimensional method of Laplace applies to linear systems of second-order partial differential equations, of the form

$$y_{,k\ell} + a^k_{k\ell} y_{,k} + a^\ell_{k\ell} y_{,\ell} + c_{k\ell} y = 0, \quad 1 \leq k \neq \ell \leq n \tag{4.1}$$

where y is a scalar function of the independent variables $x_1, \cdots, x_n$ and the coefficients a and c are smooth functions of $x_1, \cdots, x_n$ which are symmetric in the pair of lower indices. One could also study systems with a non-homogeneous term $h_{k\ell}$ added to the left-hand side of (4.1) (see [KT1]), but this will not be done here.

We first recall from [KT1] the definition of the higher-dimensional Laplace invariants of (4.1). We will see that the system (4.1) is determined in an essentially unique way by its Laplace invariants, just as in the classical 2-dimensional case. However, we shall see that in contrast with the planar case, the higher-dimensional invariants cannot be prescribed arbitrarily. They must satisfy differential constraints which are necessarily satisfied for overdetermined systems (4.1) for $n \geq 3$.

We shall consider Cauchy data given by

$$y(x^0_1, \cdots, x_\ell, \cdots, x^0_n) = f_\ell(x_\ell), \quad 1 \leq \ell \leq n. \tag{4.2}$$

Any smooth solution of (4.1) must of course satisfy

$$y_{,k\ell j} = y_{,kj\ell}, \tag{4.3}$$

for k, j and l distinct. Combined with (4.2), these conditions imply that the coefficients a and c in (4.1) must satisfy the following set of compatibility conditions, for k, j and ℓ distinct,

$$\begin{aligned} &a^{\ell}_{\ell k,j} - a^{\ell}_{\ell j,k} = 0, \\ &c_{\ell j} = a^{k}_{\ell k,j} - a^{k}_{\ell k} a^{k}_{kj} + a^{\ell}_{\ell j} a^{k}_{\ell k} + a^{j}_{\ell j} a^{k}_{jk}, \\ &c_{\ell k,j} - c_{\ell j,k} + a^{l}_{\ell j} c_{\ell k} + (a^{j}_{\ell j} - a^{k}_{\ell k}) c_{kj} - a^{\ell}_{\ell k} c_{\ell j} = 0. \end{aligned} \tag{4.4}$$

We observe that the first equation of (4.4) is also a consequence of the second one.

The general form of the system (4.1) is preserved under the *admissible transformations*

$$y = \lambda(x_1, \cdots, x_n)\overline{y}, \tag{4.5}$$

$$x_i = f_i(\overline{x}_i), \quad 1 \leq i \leq n, \tag{4.6}$$

where λ is smooth and non-vanishing and the f_i's are smooth and have non-vanishing derivatives.

Following [KT1], we define the *higher-dimensional Laplace invariants* of (4.1) to be the $n(n-1)^2$ functions given by

$$\begin{aligned} m^{ij}_{ij} &= a^{i}_{ij,i} + a^{i}_{ij} a^{j}_{ij} - c_{ij}, \\ m^{ij}_{kk} &= a^{k}_{kj} - a^{i}_{ij}, \quad k \neq i, j. \end{aligned} \tag{4.7}$$

for all ordered pairs (i, j), $1 \leq i \neq j \leq n$. It is readily checked, using (4.7), that under an admissible transformation, we have

$$\overline{m}^{ij}_{ij} = f'_i f'_j m^{ij}_{ij}, \quad \overline{m}^{ij}_{kk} = f'_j m^{ij}_{kk}. \tag{4.8}$$

In particular, the functions m^{ij}_{ij} and m^{ij}_{kk} are invariant under pure rescalings (4.5).

In the 2-dimensional case, the compatibility conditions (4.4) are vacuous, the m^{ij}_{kk}'s are not defined and the m^{ij}_{ij}'s correspond to the classical Laplace invariants h and k of the equation. If the equation is given by

$$z_{xy} + a(x,y) z_x + b(x,y) z_y + c(x,y) z = 0, \tag{4.9}$$

then we have indeed

$$m^{12}_{12} = h = a_x + ab - c, \quad m^{21}_{21} = k = b_y + ab - c, \tag{4.10}$$

It is shown in Darboux [D] that given any two functions h and k of x, y, there exists a linear p.d.e. (4.9) such that h and k are its Laplace invariants. Any such p.d.e. is of course defined up to a rescaling

$$z = \lambda(x,y)\bar{z}, \quad \lambda(x,y) \neq 0. \tag{4.11}$$

The p.d.e. is uniquely determined if we choose λ such that upon the rescaling (4.5), we have $ab - c$ identically zero, a identically zero on a characteristic curve $x = x_0$ and b identically zero on a characteristic curve $y = y_0$. In this case the p.d.e. is given by

$$z_{,xy} + \left(\int_{x_0}^{x} h dx\right) z_{,x} + \left(\int_{y_0}^{y} k dy\right) z_{,y} + \left(\int_{x_0}^{x} h dx \int_{y_0}^{y} k dy\right) z = 0.$$

In order to to state the generalization of the result of Darboux we first determine the necessary conditions which must be satisfied by the higher dimensional Laplace invariants.

Lemma: *The higher dimensional Laplace invariants of a compatible system (4.1) satisfy the following relations:*

$$\begin{aligned}
&m^{ij}_{kk} + m^{kj}_{ii} = 0,\\
&m^{ij}_{kk,k} - m^{jk}_{ii} m^{ij}_{kk} - m^{kj}_{kj} = 0,\\
&m^{ij}_{ij,k} + m^{ij}_{kk} m^{ik}_{ik} + m^{ik}_{jj} m^{ij}_{ij} = 0,\\
&m^{ij}_{kk} - m^{ij}_{\ell\ell} - m^{\ell j}_{kk} = 0,\\
&m^{\ell i}_{kk,j} + m^{ij}_{\ell\ell} m^{ki}_{\ell\ell} + m^{\ell j}_{kk} m^{ki}_{jj} = 0,
\end{aligned} \tag{4.12}$$

for $1 \leq i, j, k, \ell \leq n, \quad i, j, k, \ell$ *distinct.* We then have

Theorem 2: *Given any collection of* $n(n-1)^2$, $n \geq 3$ *smooth functions of* $x_1, \cdots, x_n$,

$$m^{ij}_{ij}, \quad m^{ij}_{kk}, \quad 1 \leq i, j, k \leq n, \quad i, j, k, \text{ distinct},$$

satisfying, the constraints (4.12) there exists a linear system (4.1) whose higher-dimensional Laplace invariants are the given functions m^{ij}_{ij} *and* m^{ij}_{kk}. *Any such* *system is defined up to a rescaling (4.5). A representative is given by*

$$\begin{aligned} &y_{,ij} + A\, y_{,j} - m^{ij}_{ij} y = 0, \\ &y_{,ik} + \left(m^{ji}_{kk} + A\right) y_{,k} - m^{ik}_{ik} y = 0, \\ &y_{,jk} + m^{ik}_{jj} y_{,j} + m^{ij}_{kk} y_{,k} = 0, \\ &y_{,\ell k} + m^{ik}_{\ell\ell} y_{,\ell} + m^{i\ell}_{kk} y_{,k} = 0, \end{aligned} \tag{4.13}$$

where (i,j) is a fixed (ordered) pair, $1 \leq i,j,k,\ell \leq n$ are distinct and A is a function which satisfies the following:

$$A_{,j} = m^{ji}_{ji} - m^{ij}_{ij}, \quad A_{,k} = -m^{jk}_{ii,i}.$$

5. The Higher-Dimensional Laplace Transformation

Consider a system of p.d.e.s (1.1) for y and let (i,j), $1 \leq i,j \leq n$, denote an ordered pair. We say that the system is (i,j)-*transformable* if

$$\Omega^{ij} := m^{ij}_{ij} \prod_{k \neq i,j} m^{ij}_{kk} \neq 0.$$

Observe that the property of (i,j)-transformability is invariant under the admissible transformations (4.5) and (4.6). If the system is (i,j)-transformable, then we define, following [KT1],

$$\tilde{y} = y_{,j} + a^{i}_{ij} y,$$

to be the (i,j)-*Laplace transform* of y, which we denote by

$$\tilde{y} = \mathcal{L}_{(i,j)}(y).$$

The transformation $\mathcal{L}_{(i,j)}$ is a higher-dimensional generalization of the classical Laplace transformation for linear scalar second-order p.d.e.s in the plane [D]. It is based on the geometric transformation which we reviewed in Section 3. We now recall from [KT1] that if (4.1) is (i,j)-transformable, then just as in the two-dimensional case, $\tilde{y}$ will satisfy a system of differential equations of the same type as (4.1):

Theorem 3: *Consider a system (4.1) for y, whose coefficients satisfy the compatibility conditions (4.4). If the system is (i,j)-transformable, then $\tilde{y} =$*

$\mathcal{L}_{(i,j)}(y)$ satisfies a system of the same type, whose coefficients satisfy the compatibility conditions (4.4)

The transformation laws, under the (i,j)-transform, for the coefficients and the higher-dimensional Laplace invariants of any (i,j)-transformable system (4.1) are explicitly given in [KT1] and [KT2].

We also recall from [KT1] that the (i,j) transform can be generically inverted. Indeed, consider a system (4.1) for y and let $\tilde{y} = \mathcal{L}_{(i,j)}(y)$. If $m^{ij}_{ij} \neq 0$, then the inverse of the (i,j) transform exists and is given by

$$y = \left[\mathcal{L}_{(j,i)}(\tilde{y})\right] / m^{ij}_{ij}.$$

In section 2, we recalled [D] that in the planar case ($n = 2$), the vanishing of one of the Laplace invariants $h = m^{12}_{12}$ or $k = m^{21}_{21}$ implies that the p.d.e. (4.9) factors into two parametrized first-order o.d.e.s in x and y respectively, so that it can be integrated by quadratures in terms of two arbitrary functions of one variable. The principle of the classical method of integration of Laplace for (4.9) is to iterate the transforms $\mathcal{L}_{(1,2)}$ or $\mathcal{L}_{(2,1)}$ until one possibly obtains a transformed p.d.e. with one of its Laplace invariants equal to zero. One then transforms back using the inversion formula to obtain solutions of the original equation.

The generalization to higher dimensions of the reduction to parametrized o.d.e.s which occurs for (4.9), when either h or k is identically zero, was obtained in [KT1]. We now briefly recall the main content of this result. A system of p.d.e.s (4.1), whose coefficients a and c satisfy (4.4), is said to be (i,j)-*reducible*, for an ordered pair (i,j), $1 \leq i,j \leq n$, if

$$\Delta^{ij} := \left(m^{ij}_{ij}\right)^2 + \sum_{k \neq i,j} \left(m^{ij}_{kk}\right)^2 = 0.$$

The conditions of (i,j) reducibility is invariant under the admissible transformations (4.5), (4.6). In the case $n = 2$, any p.d.e. (2.1) is either (i,j)-transformable or (i,j)-reducible for some open subset of $\mathbf{R}^2$. This is no longer true in the overdetermined case $n \geq 3$, where a system (4.1) could be neither

(i,j)-transformable nor (i,j) reducible.

In [KT1], we considered systems with an additional non-homogeneous term $h_{k\ell}$ in (4.1), i.e.

$$y_{,k\ell} + a^k_{k\ell} y_{,k} + a^\ell_{k\ell} y_{,\ell} + c_{k\ell} y = h_{k\ell} = 0, \quad 1 \le k \ne \ell \le n \tag{5.1}$$

where the coefficients satisfy (4.4) and $h_{k\ell}$ satisfy

$$h_{\ell k,j} - h_{\ell j,k} + a^l_{\ell j} h_{\ell k} + (a^j_{\ell j} - a^k_{\ell k}) h_{kj} - a^\ell_{\ell k} h_{\ell j} = 0. \tag{5.2}$$

For such systems, we proved the following reduction theorem which generalizes the classical result for p.d.e.s in the plane:

Theorem 4: *Consider a system (5.1) for y whose coefficients a, c and h satisfy (4.4) and (5.2). If the system is (i,j)-reducible, then the general solution of the system is given by*

$$y = Q + e^{-J} G(\hat{x}_j),$$

where

$$Q = -e^{-J} \int e^{J-I} \left[\int e^I h_{ij} dx_i - F(x_j) \right] dx_j,$$

$$I = \int a^j_{ij} dx_i, \qquad J = \int a^i_{ij} dx_j,$$

where F is an arbitrary function of x_j, $G(x_1, \cdots, \hat{x}_j, \cdots, x_n)$ does not depend on x_j and where the antiderivative I is such that $I_{,k} = a^j_{jk}$ for $k \ne i, k \ne j$. Then G satisfies a linear system in $n-1$ independent variables $x_1, \cdots, \hat{x}_j, \cdots, x_n$ of the form

$$G_{,k\ell} + g^k_{k\ell} G_{,k} + g^\ell_{k\ell} G_{,\ell} + b_{k\ell} G + r_{k\ell} = 0, \quad k \ne \ell \ \textit{distinct from } j.$$

where

$$g^l_{lk} = a^l_{lk} - J_{,k}, \qquad g^k_{lk} = a^k_{lk} - J_{,l}$$

$$b_{lk} = c_{lk} + J_{,k} J_{,l} - J_{,lk} - a^l_{lk} J_{,l} - a^k_{lk} J_{,k}$$

and

$$r_{lk} = e^J (h_{lk} + Q_{,lk} + a^l_{lk} Q_{,l} + a^k_{lk} Q_{,k} + c_{lk} Q).$$

The proof of Theorem 4 is given in Theorem 2 of [KT1], where the reader can also find applications to explicit examples. It is a non-trivial result that if one applies Theorem 4 to a homogeneous reducible system in its normal form as in (4.13), then the reduced system that one obtains is also homogeneous. We refer the reader to [KT2] for details.

6. Periodic Systems

Suppose that we are given a system S of the form (4.1) whose coefficients a and c satisfy (4.4). If the system is (i,j)-reducible for some ordered pair (i,j), then we may apply Theorem 4 to reduce the number of independent variables from n to $n-1$. If the system S is not (i,j)-reducible for any ordered pair (i,j) and it is (k,ℓ)-tranformable for some direction (k,ℓ), then the application of the (k,ℓ) transformation to S will lead to a new system which could again be considered for reducibility. We will thus say that S is *reducible after r steps* if there exists a path of ordered pairs $(I,J)=((i_1,j_1),\cdots,(i_r,j_r),(i,j))$ such that the composition $\tilde{S}=\mathcal{L}_{(i_r,j_r)}\circ\cdots\circ\mathcal{L}_{(i_1,j_1)}(S)$ exists and is (i,j)-reducible. Given that there are a priori $n(n-1)$ directions in which to transform at each stage, it is important to have certain criteria by which one can eliminate certain pairs (i_r,j_r) in the sequence (I,J). One such criterion is provided by the notion of periodicity. Given a system S which is (i,j)-transformable for some pair (i,j), we say that S is *1-periodic in $\mathcal{L}_{(i,j)}$* if S and $\mathcal{L}_{(i,j)}(S)$ are equivalent under an admissible transformation (4.5), (4.6). Thus, one should never transform a system in a direction in which it is 1-periodic when constructing a path which is to lead to an (i,j)-reducible system.

In the case $n=2$, we only have two transforms $\mathcal{L}_{(1,2)}$ and $\mathcal{L}_{(2,1)}$, any p.d.e. (4.9) which is 1-periodic in the (1,2) direction will also be periodic in the (2,1) direction, and a p.d.e. which is 1-periodic is not reducible after any number of steps. Darboux [D] proved the remarkable fact that every 1-periodic p.d.e. (4.9) is equivalent under an admissible transformation (4.5), (4.6) to the Klein-

Gordon equation

$$z_{xy} = \varepsilon z \quad , \quad \varepsilon^2 = 1.$$

We state a theorem which gives the n-dimensional generalization of Darboux's result.

Theorem 5: *An (i,j)-transformable system (4.1) is 1-periodic in $\mathcal{L}_{(i,j)}$ if and only if it is equivalent under an admissible transformation to a system of the form*

$$\begin{aligned}
&y_{,ij} - \varepsilon_i \varepsilon_j y = 0 \\
&y_{,ki} + \varepsilon_i \varepsilon_k y_{,k} + a^i_{ki} y_{,i} + \varepsilon_i \varepsilon_k a^j_{kj} y = 0 \,, \quad 1 \le k \ne i,j \le n \\
&y_{,kj} + \varepsilon_j \varepsilon_k y_{,k} + a^j_{kj} y_{,j} + \varepsilon_j \varepsilon_k a^i_{ki} y = 0 \,, \quad 1 \le k \ne i,j \le n \\
&y_{,k\ell} + a^k_{k\ell} y_{,k} + a^\ell_{k\ell} y_{,\ell} + c_{k\ell} y = 0, \quad 1 \le k,\ \ell \le n,
\end{aligned}$$

where $i, j, k\ell$ are distinct

$$a^j_{kj,j} = 0, \quad a^j_{kj,i} = 0, \quad a^i_{ki,j} = 0, \quad a^i_{ki,i} = 0,$$
$$\varepsilon_i^2 = \varepsilon_j^2 = \varepsilon_k^2 = 1.$$

It can be verified that in dimension three, every system which is 1-periodic with respect to one of the higher-dimensional Laplace transforms and transformable in all other directions is also 1-periodic in all other directions. More precisely, we have the following result

Theorem 6: *A compatible system of type (4.1) and $n = 3$ is 1-periodic with respect to one of its higher dimensional Laplace transforms if and only if, after a relabelling of independent variables, it is equivalent under an admissible transformation to a system of the form*

$$\begin{aligned}
&y_{,12} - \varepsilon_1 \varepsilon_2 y = 0 \\
&y_{,31} + \varepsilon_1 \varepsilon_3 y_{,3} + f\ y_{,1} + \varepsilon_1 \varepsilon_3 g\ y = 0 \,, \\
&y_{,32} + \varepsilon_2 \varepsilon_3 y_{,3} + g\ y_{,2} + \varepsilon_2 \varepsilon_3 f\ y = 0 \,,
\end{aligned} \tag{6.1}$$

where $\varepsilon_i^2 = 1$, $1 \le i \le 3$, f and g are differentiable functions of x_3 only. If $f \ne g$, then , the system (6.1) is 1-periodic with respect to all higher-dimensional Laplace transforms.

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