Matemática Contemporânea, Vol. 9, 1–14
http://doi.org/10.21711/231766361995/rmc91

TOY TOPS, GYROSTATS AND GAUSS-BONNET

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Abstract

A *Gyrostat* consists of a main rigid body with one or more attached rotors. We show that Poinsot's description of rigid body motion can be extended and we find the holonomy angle around the angular momentum vector, in this generalized description, after traversing one periodic solution of the corresponding reduced equation. In order to obtain this result, we follow the steps of M.Levi's study on rigid body motion (*Arch. Rat. Mech. Anal.* **122**, *213-229, 1993*).

1. Introduction.

The aim of this note is to present an example where Geometry and Mechanics come together. *Gyrostats* are mechanical devices, composed of more than one body, yet having the rigid body property that its inertia components are time independent constants ([20], [11]). Such systems consist of a main rigid body, called the *carrier*, together with one, or more, rigid symmetric rotors (which we may call *flywheels*), supported by rigid bearings on the carrier. See Figure 1. Gyrostats have important technological applications, e.g., as stabilizers for artificial satelites [15], [8].

To warm up we consider a toy top (M.Levi [12] prefers a bicycle wheel, which is just fine). Mark a meridian on the top (or a spoke on the wheel); with the top initially at rest, make its tip describe a closed curve C on a surface S, in such a way that the top axis of symmetry is always normal to S. Guess what happens:

* Partially supported by CNPq, Brazil.
key words: holonomy, gyrostats, Hannay-Berry phases. AMS classification numbers: 58F05, 70E05, 70E15. PACS Scheme (1990): 03.20+i, 46.10.+z .

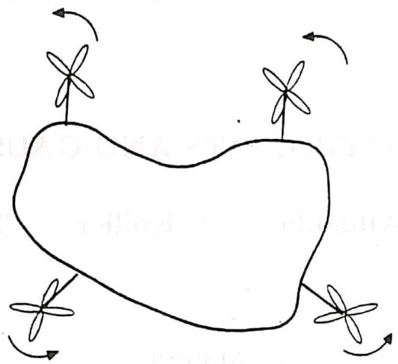

Figure 1: A gyrostat

the meridian does not come back to the original position! Surprisingly (or not), it turns through an angle, precisely that given by Levi Civita's parallel transport, $\int_C \kappa_g ds$, where κ_g is the geodetic curvature. [1] Consider now an arbitrary space curve C and let the top axis of symmetry follow the normal vector in the Frenet frame: the holonomy is $\int_C \kappa ds$, where κ is the curvature of C; moreover, if the top axis follow the tangent vector to the curve, the meridian turns through an angle which is equal to the total torsion $\int_C \tau ds$ ([9]). Light traversing an optical fiber, changes phase by this amount, as was experimentally verified with an interferometer [5]. What happens if we start with two equal wheels, rotating with the same angular velocity, one of which is left at the starting point $q_0 \in C$, and the other traverses the curve? If curve C is traversed *slowly* with respect to the angular motion then, to first order, the holonomy (here the phase difference between marked spokes in the two wheels) is given by the above formulas.

Many such phase changes (generically called "Berry phases") were recently found both in classical and in quantum mechanical problems. B.Simon and J.Hannay discovered that they are related to geometrical/topological invariants in the quantum and classical cases, respectively. For a survey on the field, we refer to the colletanea [19].

In the M.Sc. thesis of MF, [6] holonomies for gyrostats were derived by two

[1] This fact was rediscovered several times; it seems first found out by J. Radon [14].

different methods. One follows a paper by R. Montgomery, using symplectic geometry; our generalization was submited elsewhere. Montgomery revisited the old and honorable Euler's problem of the motion of a free rigid body, and found a Berry phase from the 18th century.

M.Levi presented another derivation for the holonomy of the rigid body, using elementary differential geometry [12]. Mark Levi pushed the "bycicle wheel idea" quite far. For instance, he obtained a "mechanical proof" of the Gauss-Bonnet theorem on 2-dimensional surfaces [13]. This is a beautiful paper; which we think should be mandatory reading on any elementary Differential Geometry class. Levi found other applications, such us: computing areas using a wheel; twisted beams, ropes, springs, ribbons, waveguides, optical fibers, etc.; the writhing number of linked curves.

We present here a simple derivation for the gyrostat holonomy from the differential geometric perspective. We dedicate this note to Prof. Manfredo do Carmo. We hope he finds it amusing.

2. Euler's free rigid body.

In Mechanics one learns that a free rigid body, as far as its dynamics is concerned, can be replaced by an "inertia ellipsoid". Fix a frame $\vec{e}_i(t), i = 1,2,3$ on the moving body. Any configuration can be represented by an orthogonal matrix R, whose columns are $\vec{e}_i, i = 1,2,3$. At any given time the rigid body is instantaneously rotating with *angular velocity* $\vec{\omega}$. The total energy is given by $T = (\vec{\omega}, \vec{m})$, where $\vec{m}$ is the total angular momentum vector. General considerations (Noether's theorem) imply that the latter is a conserved vector. From the viewpoint of an observer living in the body, however, $\vec{m}$ seems to describe a closed curve $\vec{M}(t)$ in a sphere, since, as a matter of fact, $\vec{m} = R(t)M(t)$. Similarly, one defines the angular velocity *seen from the body* as $\vec{\Omega}$, where $\vec{\omega}(t) = R(t)\vec{\Omega}(t)$ [2]

Consider the dual ellipsoid

$$\mathcal{E}_{dual} = \{\vec{M}\ ;\ \ (A^{-1}\vec{M}, \vec{M}) = E\ \}$$

living in the dual of the Lie Algebra. The intersection of $\mathcal{E}_{dual}$ with the spheres $||\vec{M}|| = J$ are closed curves (except four separatrices) representing the solutions $\vec{M}(t)$. Going back to the inertia ellipsoid, we get closed curves (except the separatrices) $\vec{\Omega}(t)$. Euler derived a set of ODEs for $\vec{M}$,

$$\frac{d\vec{M}}{dt} = \vec{M} \times A^{-1}\vec{M} \tag{1}$$

whose solutions, together with those for the pendulum, are standard examples of elliptic functions. See Fig. 2.

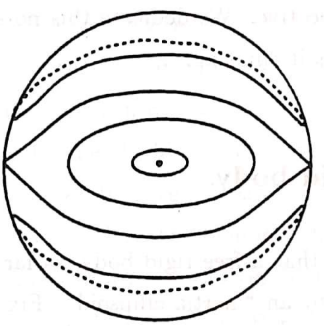

Figure 2: Phase portrait of Euler's reduced system

As the name says, the inertia ellipsoid is inert, it does not move. Actually, the rigid body motion is given by

$$R(t)\mathcal{E}.$$

[2]Writing entities in body coordinates by capital letters is an idea due to Arnold, who immeduately souped it up to arbitrary Lie groups.

How to recover the complete solutions $R(t)$ from the reduced solutions $\vec{\Omega}(t)$? A look on the expressions for $\vec{\Omega}$ in terms of the Euler angles parametrizing $SO(3)$ shows that it amounts to solve a linear set of ODEs with periodic coefficients. This can be done using complicated special functions ([10]). As we will not discuss here, it is equivalent to explicitly solve a problem of lifting a closed curve in the base S^2, to the total space $SO(3)$, using a connection with group S^1 ([17]). At any rate, there is a very nice geometric way to visualize this.

Poinsot noticed that, via $R(t)$, the inertia ellipsoid rolls, without slipping, on a fixed plane perpendicular to the angular momentum vector ([3], [7]). The *polhode* is the curve on the inertia ellipsoid that develops, via $R(t)$, over the *herpolhode*, the corresponding plane curve. The holonomy is depicted in Fig. 3 as the angle $\Delta\varphi$. [3]

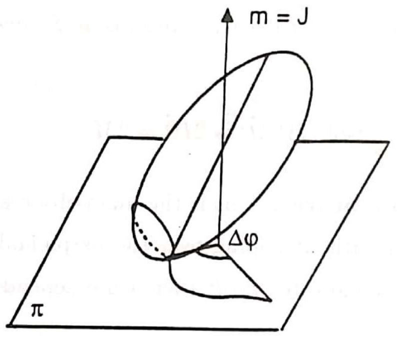

Figure 3: polhode and herpolhode

Theorem 1 *(Montgomery's holonomy formula).*

$$\Delta\phi = \frac{2ET}{J} - \Upsilon \tag{2}$$

[3]Poinsot's theorem is one of the mathematical gems of classical mechanics treasure chest. This amazing fact is a consequence of just the equation $(\vec{\omega}, \vec{m}) = 2E = const.$, as one realizes after some thought.

E is the energy of the trajectory, J the modulus of the angular momentum vector, T is the period of the polhode, and Υ is the (signed) solid angle, swept by the polhode on the inertia ellipsoid.

Thus, an important information about the full motion (the herpolhode angular shift) can be given in terms of the reduced Euler system (the polhode), in a very geometric way, circumnventing the use of complicated special functions ([10]).

M.Levi's derivation starts with the trivial observation that $\Delta\varphi$ is the integral of the herpolhode curvature k_{Γ_π} (as a planar curve). Next, he relates it to the polhode geodetic curvature k_{Γ_c}. That is the crux of his paper. There are two more ingredients, first the fact that geodetic curvature is invariant under Gauss' normal mapping, and then that there is a mechanical meaning for this mapping, namely:

Proposition 1 *(Mechanical interpretation of Gauss' mapping).*
Gauss mapping takes the polhode, living inside $\mathcal{E}$, over the curve $M(t)$ in the sphere of $\|\vec{M}\| = J$.

Proof. The gradient of $(I\vec{\Omega}, \vec{\Omega})$ is $2I\vec{\Omega} = 2\vec{M}$.

2.1. Relating curvatures. This is the main observation of M.Levi: since the polhode (Γ_c) rolls without sliping over the herpolhode (Γ_π), their arc lengths are identical. We denote by ds/dt their time dependence as Poinsot's motion takes place.

Lemma 1 . *The geodetic curvature k_{Γ_c} of the polhode and the planar curvature k_{Γ_π} of the herpolhode at the contact point are related by*

$$k_{\Gamma_\pi} ds/dt = k_{\Gamma_c} ds/dt + \omega_3$$

where ω_3 is the vertical component of the angular velocity vector.

Remark: This formula holds for any convex body rolling without slipping. Try to prove it! In our case, $\omega_3 = 2\frac{E}{J}$ (since $2E = (\vec{m}, \vec{w})$ and $\vec{m} = \vec{J}$).

2.2. Nailing the proof. The holonomy angle $\Delta\varphi$ is given by

$$\Delta\varphi = \int_0^T k_{\Gamma_\pi}(s)ds$$

where T is the time interval which a point in the inertia ellipsoid takes to touch again the horizontal plane. Thus

$$\begin{aligned}\Delta\varphi &= \int_0^T k_{\Gamma_c}(s)ds + \int_0^T \omega_3 ds \\ &= \int_0^T k_{\Gamma_c}(s)ds + 2\frac{ET}{J}\end{aligned}$$

Clearly the geodetic curvature of a surface curve Γ is invariant under Gauss mapping:

$$\int_\Gamma k_\Gamma ds = \int_c k_c ds$$

where k_c is the geodetic curvature of the image curve c. In our case, $\Gamma = \Gamma_c$ is the polhode, and $c = M(t)$ is Euler's solution curve in the momentum sphere. Using Gauss-Bonnet, we get

$$\begin{aligned}\int_c k_c ds &= 2\pi - \int_R K dS \\ &= 2\pi - \Upsilon\end{aligned}$$

where

$K = 1/J^2$ is the gaussian curvature of the sphere.

R is the spherical region inside $c = M(t)$.

Υ is the corresponding solid angle.

Putting everything together, Montgomery's formula is proven:

$$\Delta\varphi = 2\frac{ET}{J} - \Upsilon \ (\mathrm{mod} 2\pi)$$

3. Holonomy for gyrostats.

3.1. Extending Montgomery's formula. For a gyrostat with n flywheels, the configuration space is $SO(3)\times S^1 \times \ldots \times S^1$. We denote the additional degrees of freedom $\theta_1, \ldots, \theta_n$. Here, in addition to *left (or spatial)* $SO(3)$ invariance, there is also *right (or material)* $S^1 \times \ldots \times S^1$ symmetry. In this case Noether's theorem implies the existence of n additional conserved scalar momenta $I_1, \ldots, I_n$, besides the conserved vector $\vec{m}$. The reduced equations of motion also give rise to trajectories on the sphere $\|\vec{M}\| = J$. They are given by

$$\frac{d\vec{M}}{dt} = \vec{M} \times \nabla_M H(\vec{M}, I) \quad , \quad \frac{d\theta}{dt} = \nabla_I H(\vec{M}, I). \tag{3}$$

The reduced Hamiltonian $H(\vec{M}, I)$ is given by a quadratic function of $3 + n$ variables $M_1, M_2, M_3, I_1, \ldots, I_n$. In the first set of equations, the conserved momenta $I_1, \ldots, I_n$ are thought as parameters. We have shortened the notations, $\theta = (\theta_1, \ldots, \theta_n)$ and $I = (I_1, \ldots, I_n)$ for the flywheels coordinates and momenta. Once a closed trajectory $C : \vec{M} = \vec{M}(t)$ is found, say with period T, *the phases associated to the flywheels can be found by quadratures*:

$$\Delta\theta_j = \int_0^T \nabla_{I_j} H(\vec{M}(t), I) dt. \tag{4}$$

Here is the subject of our note:

Theorem 2 . *The geometric phase $\Delta\phi$ (around $\vec{m}$) of the main body is given by:*

$$\Delta\phi = \frac{2ET}{J} - \Upsilon - \frac{1}{J} \sum_{j=1}^{n} I_j \Delta\theta_j. \tag{5}$$

Remark: The holonomies $\Delta\theta_j$ around the flywheels must be calculated case by case, using (4), as the example below shows.

3.2. An example. In this section we exhibit the phase portrait of the reduced equations, for a special configuration of the gyrostat with one rotor (Fig.4). For simplicity, we place the flywheel on the Z axis of the inertia ellipsoid of the main body, assuming that the X, Y, Z axis are the principal axis of inertia of the carrier. The Hamiltonian consists of kinetic energy only and thus is a quadratic form in the Lie algebra $s0(3) \times R$, given by

$$\begin{aligned} E &= \frac{1}{2}(I\vec{\Omega},\vec{\Omega}) + \frac{1}{2}\lambda_3\dot{\theta}^2 \\ &\quad + \frac{1}{2}(\lambda + \mathcal{M}d^2)(\Omega_1^2 + \Omega_2^2) \\ &\quad + \frac{1}{2}\lambda_3\Omega_3^2 + \lambda_3\dot{\theta}\Omega_3 \end{aligned}$$

where λ is the double eigenvalue of the flywheel inertia matrix , λ_3 its third eigenvalue; $\mathcal{M}$ is the flywheel mass, $\dot{\theta}$ its angular velocity and d is the distance beetween the centers of mass of the carrier and flywheel.

Denoting by I_1, I_2 e I_3 the eigenvalues of I_{mb}, the inertia matrix of the main body,

$$E = \frac{1}{2}(\dot{\theta},\Omega_1,\Omega_2,\Omega_3)I_g(\dot{\theta},\Omega_1,\Omega_2,\Omega_3)^t$$

where I_g, the inertia operator of the *gyrostat*, is given by:

$$I_g = \begin{pmatrix} \lambda_3 & 0 & 0 & \lambda_3 \\ 0 & \lambda + \mathcal{M}d^2 + I_1 & 0 & 0 \\ 0 & 0 & \lambda + \mathcal{M}d^2 + I_2 & 0 \\ \lambda_3 & 0 & 0 & \lambda_3 + I_3 \end{pmatrix}$$

Using the coordinates $\omega_g = (\dot{\theta},\Omega_1,\Omega_2,\Omega_3)$ and $\mathbf{m}_g = (I, M_1, M_2, M_3)$, respectively for g (Lie algebra of $S^1 \times SO(3)$) and its dual g^*, it follows that $\mathbf{m}_g = I_g\omega_g$ (I_g is the Legendre transformation). Consider the pairing $g \times g^*$, given by 2 $h = (\mathbf{m}_g, \omega_g)$. The energy of the gyrostat is again expressed in terms of the momenta as $h = (\mathbf{m}_g, I_g^{-1}\mathbf{m}_g)$.

Inverting I_g, we get:

$$I_g^{-1} = \begin{pmatrix} \frac{1}{\lambda_3} + \frac{1}{I_3} & 0 & 0 & -\frac{1}{I_3} \\ 0 & \frac{1}{I_1+\lambda+\mathcal{M}d^2} & 0 & 0 \\ 0 & 0 & \frac{1}{I_2+\lambda+\mathcal{M}d^2} & 0 \\ -\frac{1}{I_3} & 0 & 0 & \frac{1}{I_3} \end{pmatrix}$$

Hence

$$h = \frac{1}{2}\left(\frac{M_1^2}{I_1 + \lambda + \mathcal{M}d^2} + \frac{M_2^2}{I_2 + \lambda + \mathcal{M}d^2} + \frac{(M_3 - I)^2}{I_3} + \frac{I^2}{\lambda_3}\right) \tag{6}$$

Moreover, since $\omega_g = I_g^{-1}\mathbf{m}_g$, we get

$$\Omega_1 = \frac{M_1}{I_1 + \lambda + \mathcal{M}d^2}$$

$$\begin{aligned} \Omega_2 &= \frac{M_2}{I_2 + \lambda + \mathcal{M}d^2} \\ \Omega_3 &= -\frac{1}{I_3} + \frac{M_3}{I_3} \end{aligned} \tag{7}$$

$$\dot{\theta} = (\frac{1}{I_3} + \frac{1}{\lambda_3})I - \frac{1}{I_3}M_3 \tag{8}$$

Trajectories are intersections of a fixed momentum sphere $\|\vec{M}\| = J$ with the energy ellipsoids with varying energies E. Note that the centers of the ellipsoids belong to the Z axis. Fig.4 below depicts the possible phase portraits as a function of the parameter I. The reader should be able to dedect the critical values. For details, see [6]

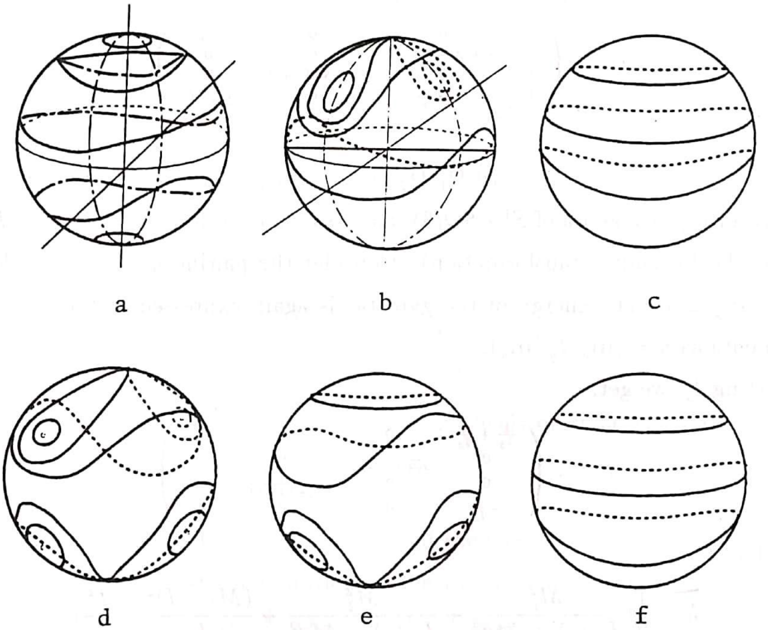

Figure 4: Bifurcations of phase portrait of a special gyrostat

3.3. Geometric phase of a gyroststat. We fix the values of the scalar momenta I_j and angular momentum vector $\vec{m}$ for the rigid body with n flywheels. In contradistinction with the usual Poinsot description, in this case an invariable plane does not exist. Nonetheless, after a period T, the plane perpendicular to $\vec{m}$ and touching the inertia elipsoid attains the *same* height. Thus, although we can still speak about herpolhodes, these are not planar curves.
To fix ideas we add just one flywheel. The generalized inertia elipsoid in the example above is given in terms of the angular velocities by

$$\alpha\Omega_1^2 + \beta\Omega_2^2 + \gamma\Omega_3^2 = 2h - \frac{I^2}{\lambda_3} \tag{9}$$

where

$$\alpha = I_1 + \lambda + \mathcal{M}d^2, \beta = I_2 + \lambda + \mathcal{M}d^2, \gamma = I_3. \tag{10}$$

A short calculation gives

$$(\vec{\omega}, \vec{m}) = constant + \Omega_3(t)I \tag{11}$$

showing that the polhodes are planar curves only when $I = 0$.

Going over the proof of Lemma 1, one realizes that the oscillations of the planeπ, perpendicular to $\vec{J}$ (passing through the lowest point of the inertia ellipsoid) do not hamper the proof. The new expression for ω_3 is given by $\omega_3 = \frac{2h - I\dot{\theta}}{J}$. This follows from $2h = (I, 0, 0, J).(\dot{\theta}, \omega_1, \omega_2, \omega_3) = I\dot{\theta} + (\mathbf{m}, \omega) = I\dot{\theta} + (\vec{M}, \vec{\Omega})$ where $h = (\mathbf{m}_g, \omega_g)$ and $\mathbf{m}_s = (I, 0, 0, J)$. Thus

$$k_{\Gamma_\pi} = k_{\Gamma_c} + \frac{2h - I\dot{\theta}}{J} \tag{12}$$

The rest of the derivation is straightforward. It is easy to extent it for n flywheels; these do not need to be attached to principal axis of the carrier.

4. Directions for further work.

i) J.E. Marsden, R. Montgomery & T. Ratiu [17] present a general "reconstruction formula" for the complete solutions of Hamiltonian systems with group symmetry, once the reduced system is known. Let $J : M \to g^*$ the momentum mappingand $\mu \in g^*$ satisfying the conditions of Marsden-Weinstein reduction procedure ([3]). Any connection on the principal bundle $G_\mu \to J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ can be used; lifting a periodic reduced solution in the base space yields a "geometric phase" (associated to the connection). Conversely, knowing the complete solutions of specific Hamiltonian systems may give us ways to explicitly lift special curves in the base. Another approach for reconstruction is being developed by the Calgary geometric mechanics group [4], and the rigid body is again the basic example.

ii) Consider Kirchhoff's problem of the motion of a solid body through incompressible, inviscid, irrotational fluid. Here one has geodetic motions of a left-invariant metric on the group of rigid motions of three-dimensional space. The problem has sixdegrees of freedom; it is nonintegrable except at exceptional cases. See [2] and references therein. It would be instructive to compute the relevant holonomies for the integrable cases, and see what happens in the nearly integrable situations. A novel feature here is the possibility of *translational holonomy*, i.e, how much the rigid body translates after one period of a periodic orbit of the reduced problem.

iii) In this note, the motion occurs simply by inertia in the configuration space of carrier *plus* flywheels. However, there are interesting associated *control problems*, where one considers "protocolled " flywheel motions, and one desires to prescribe the carrier holonomy. Finding the optimal flywheel trajectory to achieve the reorientation is an interesting variational problem. A similar study could be made also for Kirchhoff's "submarine".

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