

MEASURE-VALUED SOLUTIONS FOR MULTIPHASE FLOW IN POROUS MEDIA: EXISTENCE AND WEAK ASYMPTOTIC STEADINESS OF WATERFLOODING

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Abstract

We establish the existence of physically admissible measure-valued solutions for the initial boundary value problem modeling the flow of water, gas and oil in a one-dimensional reservoir submitted to the constant injection of pure water. We get the existence by two distinct ways: vanishing viscosity method and finite difference schemes. We also prove the weak convergence of the time averages of these m-v solutions to Dirac measures concentrated at the state representing pure water, for a.e. space variable. In particular, we get that the time averages of the expected values of the saturations of water and gas converge to the state which represents pure water, at a rate which depends only on the shape of the flux functions. The equations modeling this flow form a system of conservation laws which can admit umbilic points, curves where genuine nonlinearity fails, and also elliptic regions. We also present some useful general results about the weak convergence of probability measures to Dirac measures.

1. Introduction

Here we consider an initial-boundary value problem for a system which models the flow of water, gas and oil in a one-dimensional reservoir viewed as a porous medium. Let us denote by u_1 the saturation of water, that is, the fraction of the pore volume filled with water, u_2 , the saturation of gas, i.e., that fraction

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filled with gas, and u_3 the saturation of oil, defined analogously. Assuming that each pore is completely filled by the three immiscible fluids, we have

$$u_1 + u_2 + u_3 = 1, \quad 0 \leq u_i \leq 1, \quad i = 1, 2, 3.$$

When the effects of gravity, capillarity, and those of thermodynamical nature are neglected, and we take the porosity and the total velocity as constants and equal to one, a model for three-phase flow, extending the Buckley-Leverett model for two-phase flow [1], is provided by a 2×2 system of conservation laws of the form (see e.g. [9])

$$\frac{\partial}{\partial t} \mathbf{u} + \frac{\partial}{\partial x} \mathbf{f}(\mathbf{u}) = 0, \quad (1.1)$$

where $\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t))$, and $\mathbf{f} = (f_1, f_2)$ is given by

$$f_i = \frac{\Lambda_i(\mathbf{u})}{\Lambda_1(\mathbf{u}) + \Lambda_2(\mathbf{u}) + \Lambda_3(\mathbf{u})}, \quad i = 1, 2 \quad (1.1 - \mathbf{f})$$

where $\Lambda_i(\mathbf{u})$ is the so called mobility of the i -th phase, i.e., the phase whose saturation is u_i , $i = 1, 2, 3$. The mobilities Λ_i are defined as the quotient of the corresponding permeability functions $k_i(\mathbf{u})$, by the viscosity of the phase, which is a constant that we denote by μ_i , that is, $\Lambda_i(\mathbf{u}) = \frac{k_i(\mathbf{u})}{\mu_i}$.

Two-phase experiments provide the form of the permeability functions when one of the phases is absent. By these experiments it is known that the functions $\bar{k}_i(s) = k_i(s \mathbf{e}_i)$, $s \in [0, 1]$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the canonical basis of \mathbf{R}^2 , are increasing, convex, satisfying $\bar{k}_i(0) = 0$, for $i = 1, 2$, the same occurring with the functions $k_3((1-s)\mathbf{e}_i)$, $s \in [0, 1]$, $i = 1, 2$, and $k_3(s, 1-s) \equiv 0$, $s \in [0, 1]$. In reservoir simulation, the form of the permeability functions for the remaining values of the saturations is normally guessed in such a way that must coincide with the two-phase experimental issues on the boundary lines $u_i = 0$, $i = 1, 2, 3$. In simple models it is assumed $k_1(\mathbf{u}) = k_1(u_1)$, and $k_2(\mathbf{u}) = k_2(u_2)$, that is, the permeability functions depend only on the saturation of the respective phase. As an example we have the Stone's model [12] where $k_1(\mathbf{u}) = u_1^2$, $k_2(\mathbf{u}) = u_2^2$, and $k_3(\mathbf{u}) = (1 - u_1 - u_2)(1 - u_1)(1 - u_2)$. In [6], Isaacson, Marchesin, Plohr

and Temple present the study of the Riemann problem for the system (1.1) taking the same k_1 and k_2 but $k_3(\mathbf{u}) = (1 - u_1 - u_2)^2$. Even in this last simple model, where the resulting system is hyperbolic, we have the appearance of four umbilic points in the region of physical interest, Δ , given by

$$\Delta = \{\mathbf{u} \in \mathbf{R}^2 \mid 0 \leq u_1 + u_2 \leq 1, 0 \leq u_i \leq 1, i = 1, 2\},$$

three of them are the vertices of this triangle, and one is in its interior. There is also a number of curves where genuine nonlinearity fails.

The basic properties about the flux map \mathbf{f} we will need for the study developed here are the following, which are trivially presented by those in models of three-phase flow in porous media, by what was said above:

- (i) \mathbf{f} is a C^2 map $\Delta \rightarrow \Delta$;
- (ii) $f_i(\mathbf{u}) = 0$, if $u_i = 0$, $i = 1, 2$, and $f_1(\mathbf{u}) + f_2(\mathbf{u}) = 1$, if $u_1 + u_2 = 1$;
- (iii) Setting $\mathbf{e} = (1, 0)$ and, for $\varepsilon_0 > 0$,

$$\Delta(\mathbf{e}, \varepsilon_0) = \Delta \cap (\mathbf{e} + [-\varepsilon_0, \varepsilon_0] \times [-\varepsilon_0, \varepsilon_0])$$

we assume that, for some ε_0 , \mathbf{f} is a homeomorphism of $\Delta(\mathbf{e}, \varepsilon_0)$ onto its image.

We will call a C^2 map \mathbf{f} satisfying (i), (ii), (iii) above, a Δ -flux, and denote by $\mathcal{F}(\Delta)$ the set of Δ -fluxes.

Here we will be interested in the initial-boundary value problem given by system (1.1), where we assume \mathbf{f} to be a Δ -flux, and the conditions

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad 0 < x < L, \quad (1.2)$$

$$\mathbf{u}(0, t) = \mathbf{e} \stackrel{\text{def}}{=} (1, 0), \quad t > 0. \quad (1.3)$$

The boundary condition (1.3) represents, in models of three-phase flow in porous media, the constant injection of pure water into the reservoir.

As we mentioned above, even simple examples of Δ -fluxes, as that appearing in the system whose Riemann problem is studied in [6], can give rise to such a

number of degeneracies that would become attempting to get a weak solution to the initial-boundary value problem (1.1), (1.2), (1.3), a challenge which seems far from being achievable with the technics which have been developed so far. Here, we deal with the weaker concept of measure-valued solution which were introduced by DiPerna in [3], in the context of Cauchy problems.

Definition 1.1. Let $\mathbf{P}(\mathbf{R}^2)$ denote the set of the probability Borel measures on \mathbf{R}^2 . A measure-valued solution to (1.1), (1.2), (1.3) is a measurable map $\nu: [0, L] \times [0, \infty) \rightarrow \mathbf{P}(\mathbf{R}^2)$, denoted by $\nu_{x,t}$, satisfying

$$\text{Supp} \nu_{x,t} \subset \Delta, \quad \text{for a.e. } (x, t) \in (0, L) \times (0, \infty), \quad (1.4)$$

and such that for all $\phi \in C_0^1([0, L] \times [0, \infty))$ (i.e., $\phi \in C^1((0, L) \times (0, \infty))$ and $\text{Supp} \phi$ is a compact subset of $[0, L] \times [0, \infty)$) we have

$$\begin{aligned} \int_0^\infty \int_0^L \{ \langle \nu_{x,t}, \mathbf{u} \rangle \phi_t + \langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle \phi_x \} dx dt \\ + \int_0^L \mathbf{u}_0(x) \phi(x, 0) dx + \int_0^\infty \mathbf{e} \phi(0, t) dt = (0, 0). \end{aligned} \quad (1.5)$$

An important consequence of the above definition is given by the following proposition, which is a key result for the study of the asymptotic behavior of the m-v solutions of (1.1), (1.2), (1.3) as it will be seen further on.

Proposition 1.2. Let $\nu_{x,t}$ be a measure-valued solution to (1.1), (1.2), (1.3). Given any $T > 0$, for each $\zeta \in C_0^1((0, T))$ and a.e. $x \in (0, L)$ we have

$$\left| \int_0^T \langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle - \mathbf{e} \rangle \zeta(t) dt \right| \leq C x \text{Var}(\zeta), \quad (1.6)$$

where $C > 0$ is a constant independent of T and x . In particular, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle \nu_{x,t}, \mathbf{f} \rangle dt = \mathbf{e}, \quad (1.7)$$

for a.e. $x \in (0, L)$, with a rate of convergence $O(T^{-1})$.

Proof. Fix $x_0 \in (0, L)$ and let $\delta \in C_0^1((-1, 1))$, $\delta \geq 0$, satisfying $\int_{-1}^1 \delta(s) ds = 1$. Set $\delta_h(s) = h^{-1} \delta(h^{-1}s)$. In (1.5), choose $\phi(x, t) = \zeta(t) \chi_h^0(x)$, where

$$\chi_h^0(x) = 1 - \int_0^x \delta_h(s - x_0) ds,$$

with $h < \min\{x_0, L - x_0\}$. Then, if x_0 is a Lebesgue point of the measurable function (see [10])

$$\int_0^T \langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle \zeta(t) dt,$$

we get that (cf. [7])

$$\int_0^L \delta_h(x) \left\{ \int_0^T \langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle \zeta(t) dt \right\} dx \xrightarrow{h \rightarrow 0} \int_0^T \langle \nu_{x_0,t}, \mathbf{f}(\mathbf{u}) \rangle \zeta(t) dt.$$

So, making $h \rightarrow 0$ in (1.5) for $\phi(x, t) = \chi_h^0(x) \zeta(t)$, we obtain

$$\int_0^T \{ \langle \nu_{x_0,t}, \mathbf{f}(\mathbf{u}) \rangle - \mathbf{e} \} \zeta(t) dt = \int_0^{x_0} \int_0^T \langle \nu_{x,t}, \mathbf{u} \rangle \zeta'(t) dt dx. \quad (1.8)$$

Now, taking modulus on both sides of (1.8), using (1.4) and making obvious estimates, we get (1.6). More yet, we can choose a sequence $T_n \rightarrow \infty$ and then find a null set so that, for x not belonging to this set, (1.6) holds for all $\zeta \in C_0^1((0, \infty))$. To obtain (1.7), we observe that (1.6) holds, by passing to limit, for $\zeta(t) = \chi_{(0,T)}(t)$, for any $T > 0$, for a.e. $x \in [0, L]$. With this choice for ζ , where we have $\text{Var}(\zeta) = 2$, we divide (1.6) by T and let $T \rightarrow \infty$ to get (1.7) with the asserted rate of convergence. □

The main results of this work can be summarized in the following statement.

Theorem 1.3. (Existence and Weak Asymptotic Steadiness) *There exists a globally defined measure-valued solution to (1.1), (1.2), (1.3). Let $\nu_{x,t}$ be such a m-v solution to (1.1), (1.2), (1.3) and set*

$$\langle \mu_T^x, \cdot \rangle = \frac{1}{T} \int_0^T \langle \nu_{x,t}, \cdot \rangle dt.$$

Then, for a.e. $x \in [0, L]$, $\mu_T^x \rightharpoonup \delta_{\mathbf{e}}$, as $T \rightarrow \infty$, where $\delta_{\mathbf{e}}$ is the Dirac measure concentrated at \mathbf{e} . Further, the limits for a.e. $x \in [0, L]$

$$\lim_{T \rightarrow \infty} \langle \mu_T^x, u_1 \rangle = 1, \quad \lim_{T \rightarrow \infty} \langle \mu_T^x, u_2 \rangle = 0,$$

are attained at a rate of convergence for which one can obtain estimates using only properties of the flux functions f_1, f_2 and the fact that the limit for a.e.

$x \in [0, L]$

$$\lim_{T \rightarrow \infty} \langle \mu_T^x, \mathbf{f} \rangle = \mathbf{e},$$

is realized at a rate $O(T^{-1})$.

The above theorem will be proved along the following three sections. In section 2, we prove the existence of m-v solutions to (1.1), (1.2), (1.3) using the vanishing viscosity method. In section 3, we prove the existence of these solutions using finite difference schemes. In section 4, we prove the results about the asymptotic behavior of these solutions.

2. Existence: Vanishing Viscosity Method.

We consider the approximation of system (1.1) given by the vanishing viscosity method, that is, we introduce an artificial viscosity term and consider the resulting parabolic system

$$\frac{\partial}{\partial t} \mathbf{u} + \frac{\partial}{\partial x} \mathbf{f}(\mathbf{u}) = \varepsilon \frac{\partial^2}{\partial x^2} \mathbf{u}. \quad (2.1)$$

The purpose of this section is the proof of the following theorem.

Theorem 2.1. *There exists a subsequence of solutions of (2.1) in $(0, L) \times (0, \infty)$, satisfying suitable initial-boundary conditions, which generates, as $\varepsilon \rightarrow 0$, a measure-valued solution to (1.1), (1.2), (1.3).*

Before proving theorem 2.1 we want to state a preliminary result about the solutions of initial-boundary value problems for parabolic systems like (2.1). So, let us set the following initial and boundary conditions for system (2.1)

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in (0, L), \quad (2.2)$$

$$\mathbf{u}(0, t) = \mathbf{u}(L, t) = \mathbf{e}, \quad (2.3)$$

where $\mathbf{u}_0(x)$, \mathbf{e} , are the same as in (1.1), (1.2), (1.3).

We will now state a simple but useful lemma on invariant regions for non-linear parabolic systems, which is in fact an easy corollary of the general result in [2] (see also [11]).

We consider, for the moment, an initial (or initial-boundary) value problem for a general system like (2.1) with $\mathbf{f} \in C^2(\mathbf{R}^n)$, $(x, t) \in (a, b) \times (0, \infty)$, where we can eventually have $a = -\infty$, $b = \infty$, and we assume that there are given appropriate boundary conditions.

Now, let $B \subset \mathbf{R}^n$ be given by

$$B = \bigcap_{i=1}^N B_i,$$

with

$$B_i = \{\mathbf{u} \in \mathbf{R}^n : G_i(\mathbf{u}) \leq 0\}, \quad (2.4)$$

for certain smooth functions $G_i : \mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \dots, N$.

We assume that $\mathbf{u}(x, t)$ is a smooth solution of (2.1) in $(a, b) \times (0, T)$, satisfying certain initial (or initial-boundary) conditions, and we also suppose as in [2] that $\mathbf{u}(x, t)$ satisfies:

$$\begin{aligned} &\text{for each fixed } t, \text{ there exists a compact interval } I \subset (a, b) \\ &\text{such that if } x \notin I \text{ then } \mathbf{u}(x, t) \in \text{int}(B). \end{aligned} \quad (2.5)$$

Condition (2.5) apparently does not allow boundary data taking values on the boundary of B . In this case, the usual procedure is to perturb slightly the boundary conditions in order to satisfy condition (2.5), get the invariance for the new problem and then obtain the invariance for the original problem by a limit process.

Lemma 2.2. *Assume that the functions G_i satisfy the quasiconvexity condition*

$$\begin{aligned} \xi^T \nabla^2 G_i(\xi) &\geq 0, \quad \text{for every } \xi \text{ such that } \langle \nabla G_i, \xi \rangle = 0, \\ &\text{in } \partial B_i \cap \partial B, \quad i = 1, \dots, N, \end{aligned} \quad (2.6)$$

and let the map \mathbf{f} in (2.1) satisfy

$$T_{\mathbf{f}(\mathbf{u})}(\mathbf{f}(\partial B_i)) \subset T_{\mathbf{u}}(\partial B_i), \quad (2.7)$$

for each $\mathbf{u} \in \partial B_i \cap \partial B$, $i = 1, \dots, N$ (where we look at both affine spaces as translated to the origin). Then B is invariant for the solution of the nonlinear parabolic system (2.1); that is, if the initial and boundary conditions are in B , then $\mathbf{u}(x, t) \in B$ for all $(x, t) \in (a, b) \times (0, T)$.

Proof. In order to apply Chueh-Conley-Smoller theorem on invariant regions [2] we need only to prove the following:

$$\nabla G_i \text{ is a left-eigenvector of } \nabla \mathbf{f} \text{ on } \partial B_i \cap \partial B, \text{ for each } i = 1, \dots, N. \quad (2.8)$$

Now, if $\xi \in T_{\mathbf{u}}(\partial B_i)$, with $\mathbf{u} \in \partial B_i \cap \partial B$, then $\nabla \mathbf{f} \cdot \xi \in T_{\mathbf{f}(\mathbf{u})}(\mathbf{f}(\partial B_i))$. By hypothesis (2.7) we have that, since $\nabla G_i(\mathbf{u})$ is normal to $T_{\mathbf{u}}(\partial B_i)$, then it is also normal to $T_{\mathbf{f}(\mathbf{u})}(\mathbf{f}(\partial B_i))$, if $\mathbf{u} \in \partial B_i \cap \partial B$, $i = 1, \dots, N$. So, we have,

$$\langle \nabla \mathbf{f}(\mathbf{u})^\top \cdot \nabla G_i(\mathbf{u}), \xi \rangle = \langle \nabla G_i(\mathbf{u}), \nabla \mathbf{f}(\mathbf{u})\xi \rangle = 0,$$

for all $\xi \in T_{\mathbf{u}}(\partial B_i)$, $\mathbf{u} \in \partial B_i \cap \partial B$, $i = 1, \dots, N$. Then,

$$\nabla \mathbf{f}(\mathbf{u})^\top \nabla G_i(\mathbf{u}) \parallel \nabla G_i(\mathbf{u}),$$

that is, there exists a certain $\mu_i \in \mathbf{R}$, $\mu_i = \mu_i(\mathbf{u})$, such that

$$\nabla \mathbf{f}(\mathbf{u})^\top \nabla G_i(\mathbf{u}) = \mu_i \nabla G_i(\mathbf{u}),$$

and, so, we have (2.8) and the lemma is proved. □

Proof of the theorem 2.1. We will prove that, for each $\varepsilon > 0$, there exists a solution $\mathbf{u}^\varepsilon(x, t)$ of problem (2.1), (2.2), (2.3) defined for all $(x, t) \in [0, L] \times [0, \infty)$ and taking values in Δ . Then, Tartar's theorem on the existence of Young measures (see [13]) will give us the existence of a measurable map $\nu : [0, L] \times [0, \infty) \rightarrow \mathbf{P}(\mathbf{R}^2)$, which we denote by $\nu_{x,t}$, such that $\text{Supp} \nu_{x,t} \subset \Delta$ and for all $h \in C(\mathbf{R}^2)$ we have

$$w^*-\lim h(\mathbf{u}^\varepsilon(x, t)) = \langle \nu_{x,t}, h(\mathbf{u}) \rangle, \quad (2.9)$$

a.e. $(x, t) \in (0, L) \times (0, \infty)$.

Then, after proving the existence of $u^\varepsilon(x, t)$ as above, we prove that the associated Young measure $\nu_{x,t}$ satisfies (1.5) of definition 1.1, and this will conclude the proof of the theorem 2.1.

We begin by recalling the solution of the elementary initial-boundary value problem for the heat equation:

$$\frac{\partial}{\partial t} v = \varepsilon \frac{\partial^2}{\partial x^2} v, \quad (2.10)$$

$$v(x, 0) = v_0(x), \quad (2.11)$$

$$v(0, t) = v(L, t) = 0. \quad (2.12)$$

Its solution can be represented in the form

$$v(x, t) = \int_0^L G(x, y, t) v_0(y) dy, \quad (2.13)$$

with

$$G(x, y, t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \sum_{s \in \mathbb{Z}} (e^{-(x-y-2sL)^2/4\varepsilon t} - e^{-(x+y-2sL)^2/4\varepsilon t}).$$

Similarly, the solution of the problem given by (2.10), (2.11) and

$$v(0, t) = v(L, t) = 1, \quad (2.14)$$

can be expressed by

$$\tilde{v}(x, t) = 1 + \int_0^L G(x, y, t) (v_0(y) - 1) dy.$$

Let us identify any function $h(x, t)$ defined in $[0, L] \times [0, \infty)$ with a map $t \mapsto h(t)$ from $[0, \infty)$ to the space of functions defined in $[0, L]$. Also, let us denote by $T_{G(\tau)}h(t)$ the function defined in $[0, L]$ by

$$[T_{G(\tau)}h(t)](x) = \int_0^L G(x, y, \tau) h(y, t) dy,$$

and similarly for $T_{G_y(\tau)}h(t)$, where $G_y = \frac{\partial}{\partial y}G$. So, a smooth solution $\mathbf{u}^\varepsilon(x, t)$ to (2.1), (2.2), (2.3), defined for $(x, t) \in [0, L] \times [0, T]$, satisfies the integral equation

$$\mathbf{u}^\varepsilon(t) = \mathbf{e} + T_{G(t)}(\mathbf{u}_0 - \mathbf{e}) - \int_0^t T_{G_y(t-s)}\mathbf{f}(\mathbf{u}^\varepsilon(s)) ds. \quad (2.15)$$

Now, we define the operator $\mathbf{v} \mapsto \mathcal{L}(\mathbf{v})$, for $\mathbf{v} \in L^\infty([0, L] \times [0, \infty); \mathbf{R}^2)$, by

$$\mathcal{L}(\mathbf{v})(t) = \mathbf{e} + T_{G(t)}(\mathbf{u}_0 - \mathbf{e}) - \int_0^t T_{G_y(t-s)}\mathbf{f}(\mathbf{v}(s)) ds.$$

By standard arguments (see [5]) we prove that this operator has a fixed point in $L^\infty([0, L] \times [0, T]; \mathbf{R}^2)$ for T sufficiently small, and that this fixed point has suitable regularity properties. We then prove, again by standard arguments (see [5]), that this fixed point $\mathbf{u}^\varepsilon(x, t)$ is a local solution to (2.1)–(2.4). Now, we apply lemma 2.2. It is easy to verify that Δ and \mathbf{f} , being a Δ -flux, satisfy the hypotheses (2.6), (2.7) of lemma 2.2, where G_i are the obvious affine functions defining Δ . So, Δ is an invariant region for the solution of the problem (2.1), (2.2), (2.3). Hence, we can easily extend the local solution $\mathbf{u}^\varepsilon(x, t)$ to a global solution, that is, a smooth solution of (2.1), (2.2), (2.3) defined in $[0, L] \times [0, \infty)$. This global solution $\mathbf{u}^\varepsilon(x, t)$ takes its values in Δ .

Let $\nu_{\varepsilon, t}$ be a Young measure associated to a subsequence of $\mathbf{u}^\varepsilon(x, t)$, when $\varepsilon \rightarrow 0$. We will now prove that $\nu_{\varepsilon, t}$ satisfy (1.5), in definition 1.1.

By (2.9), we have that, for every $\phi \in C_0^1([0, L] \times [0, \infty))$

$$\begin{aligned} & \int \int_{(0, L) \times (0, \infty)} \{ \langle \nu_{\varepsilon, t}, \mathbf{u} \rangle \phi_t + \langle \nu_{\varepsilon, t}, \mathbf{f}(\mathbf{u}) \rangle \phi_x \} dx dt \\ & + \int_0^L \mathbf{u}_0(x) \phi(x, 0) dx = (0, 0). \end{aligned} \quad (2.16)$$

Now, for $\phi \in C_0^\infty([0, L] \times [0, \infty))$, with $\phi \geq 0$ and $\phi_x(0, t) \equiv 0$, we have

$$\begin{aligned} \varepsilon \int \int (u_1^\varepsilon)_{xx} \phi dx dt &= -\varepsilon \int_0^\infty (u_1^\varepsilon)_x(0, t) \phi(0, t) dt \\ &+ \varepsilon \int \int u_1^\varepsilon \phi_{xx} dx dt \geq \varepsilon \int \int u_1^\varepsilon \phi_{xx} dx dt, \end{aligned}$$

since $(u_1^\varepsilon)_x(0, t) \leq 0$, for all $t > 0$. So, for every $\phi \in C_0^\infty([0, L] \times [0, \infty))$, with $\phi \geq 0$, $\phi_x(0, t) \equiv 0$, we have

$$\begin{aligned} \int \int_{[0, L] \times [0, \infty)} \{ \langle \nu_{x,t}, u_1 \rangle \phi_t + \langle \nu_{x,t}, f_1(u) \rangle \phi_x \} dx dt \\ + \int_0^L u_{01}(x) \phi(x, 0) dx + \int_0^\infty \phi(0, t) dt \leq 0. \end{aligned} \quad (2.17)$$

Now, taking $\phi(x, t) = \zeta(t)\chi^n(x)$, with $\zeta \in C_0^\infty((0, \infty))$, $\zeta \geq 0$, and $\chi^n \in C_0^\infty([0, L])$ satisfying $\chi^n(x) = 1$, for $0 \leq x \leq \frac{x_0}{2}$, $0 \leq \chi^n \leq 1$, $\chi^n \rightarrow \chi_{[0, x_0]}$ (the characteristic function of $[0, x_0]$), as $n \rightarrow \infty$, with $x_0 \in [0, L]$ (a sequence of this type was constructed in proposition 1.2), we get

$$0 \leq \int_0^\infty \langle \nu_{x_0,t}, (1 - f_1(u)) \rangle \zeta(t) dt \leq C \text{Var}(\zeta)x_0, \quad (2.18)$$

for some fixed constant $C > 0$, provided that x_0 is a Lebesgue point of the function (cf. [7])

$$\int_0^\infty \langle \nu_{x,t}, f_i(u) \rangle \zeta(t) dt.$$

So, we have

$$\lim_{x_0 \rightarrow 0} \int_0^\infty \langle \nu_{x_0,t}, f_1(u) \rangle \zeta(t) dt = \int_0^\infty \zeta(t) dt, \quad (2.19)$$

for all $\zeta \in C_0^\infty((0, \infty))$, with $\zeta \geq 0$, which gives, by linearity and density, (2.19) for all $\zeta \in C_0^1([0, \infty))$.

An analogous procedure would give us

$$\lim_{x_0 \rightarrow 0} \int_0^\infty \langle \nu_{x_0,t}, f_2(u) \rangle \zeta(t) dt = 0, \quad (2.20)$$

for all $\zeta \in C_0^1([0, \infty))$, (in this case we use the fact that $(u_2^\varepsilon)_x(0, t) \geq 0$, for all $t > 0$).

Now, let $\phi \in C_0^1([0, L] \times [0, \infty))$ and set

$$\begin{aligned} \phi_\delta(x, t) &= \phi(x - \delta, t), & \text{for } x \geq \delta \\ \phi_\delta(x, t) &= 0, & \text{for } x < \delta. \end{aligned}$$

We approximate ϕ_δ by $\phi_\delta^n \in C_0^1((0, L) \times [0, \infty))$, setting $\phi_\delta^n = \phi_\delta \chi^n$, with $\chi^n \in C_0^\infty((0, L))$, $0 \leq \chi^n \leq 1$, $\text{Supp} \chi^n \subset [\delta, L]$ and $\chi^n \rightarrow \chi_{[\delta, L]}$, as $n \rightarrow \infty$. Substituting ϕ by ϕ_δ^n in (2.18), assuming that δ is a Lebesgue point of the function

$$\int_0^\infty \langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle dt,$$

we get

$$\begin{aligned} \int \int_{(\delta, L) \times (0, \infty)} \{ \langle \nu_{x,t}, \mathbf{u} \rangle (\phi_\delta)_t + \langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle (\phi_\delta)_x \} dx dt \\ + \int_0^L \mathbf{u}_0(x) \phi(x, 0) dx + \int_0^\infty \langle \nu_{\delta,t}, \mathbf{f}(\mathbf{u}) \rangle \phi(0, t) dt = 0. \end{aligned} \quad (2.21)$$

So, making $\delta \rightarrow 0$, and using (2.19), (2.20), we get (1.5) of definition 1.1. The theorem is then proved. \square

3. Existence: Finite Difference Schemes.

In this section we prove the existence of measure-valued solution to (1.1), (1.2), (1.3) using finite difference schemes.

Let $\Delta x, \Delta t$ be the mesh lengths and $M \in \mathbb{N}$ such that $L = M \Delta x$. Assume to be satisfied the following condition

$$\frac{\Delta x}{\Delta t} \geq \Lambda \stackrel{\text{def}}{=} \sup_{\substack{i=1,2 \\ \mathbf{u} \in \Delta}} |\nabla f_i(\mathbf{u})|. \quad (3.1)$$

We propose the following scheme in order to generate a m-v solution to (1.1)–(1.3):

$$\mathbf{u}^{0,n} = \mathbf{e}, \quad n \in \mathbb{N}, \quad (3.2)$$

$$\mathbf{u}^{k,0} = \mathbf{u}_0(k \Delta x), \quad k \in \mathbb{N}, \quad 1 \leq k \leq M-1, \quad (3.3)$$

and

$$\mathbf{u}^{k,n+1} = \mathbf{u}^{k,n} - \frac{\Delta t}{\Delta x} (\mathbf{f}(\mathbf{u}^{k,n}) - \mathbf{f}(\mathbf{u}^{k-1,n})), \quad (3.4)$$

$n, k \in \mathbf{N}, 1 \leq k \leq M$.

We first prove that

$$u_i^{k,n} \geq 0, \quad n, k \in \mathbf{N}, 1 \leq k \leq M, i = 1, 2, \quad (3.5)$$

and

$$u_1^{k,n} + u_2^{k,n} \leq 1, \quad n, k \in \mathbf{N}, 1 \leq k \leq M. \quad (3.6)$$

The proof of (3.5), (3.6) is made by induction in n and is a consequence of the definition of a Δ -flux, condition (3.1) and the Fundamental Theorem of Calculus applied to suitable functions.

Indeed, we have

$$\begin{aligned} u_i^{k,n+1} &= u_i^{k,n} - \frac{\Delta t}{\Delta x} (f_i(\mathbf{u}^{k,n}) - f_i(\mathbf{u}^{k-1,n})) \\ &= u_i^{k,n} - \frac{\Delta t}{\Delta x} (f_i(\mathbf{u}^{k,n} - u_i^{k,n} \mathbf{e}_i) - f_i(\mathbf{u}^{k-1,n})) \\ &\quad - \frac{\Delta t}{\Delta x} (f_i(\mathbf{u}^{k,n}) - f_i(\mathbf{u}^{k,n} - u_i^{k,n} \mathbf{e}_i)) \\ &= u_i^{k,n} + \frac{\Delta t}{\Delta x} f_i(\mathbf{u}^{k-1,n}) \\ &\quad - \frac{\Delta t}{\Delta x} (f_i(\mathbf{u}^{k,n}) - f_i(\mathbf{u}^{k,n} - u_i^{k,n} \mathbf{e}_i)) \\ &\geq u_i^{k,n} - \frac{\Delta t}{\Delta x} \Lambda u_i^{k,n} \geq 0, \end{aligned}$$

if $\mathbf{u}^{k,n}, \mathbf{u}^{k-1,n} \in \Delta$, by (3.1), where we have used the fact that $f_i(\mathbf{u} - u_i \mathbf{e}_i) = 0$, $\mathbf{u} \in \Delta, i = 1, 2$.

For (3.6), we begin with the following relation obtained by addition

$$\begin{aligned} u_1^{k,n+1} + u_2^{k,n+1} &= u_1^{k,n} + u_2^{k,n} - \frac{\Delta t}{\Delta x} ((f_1(\mathbf{u}^{k,n}) + f_2(\mathbf{u}^{k,n})) \\ &\quad - (f_1(\mathbf{u}^{k-1,n}) + f_2(\mathbf{u}^{k-1,n}))). \end{aligned} \quad (3.7)$$

Then, we consider the line $\mathbf{u}^{k,n}(t)$ given by

$$\begin{aligned} u_1^{k,n}(t) &= u_1^{k,n} + \frac{t}{2}(1 - u_1^{k,n} - u_2^{k,n}), \\ u_2^{k,n}(t) &= u_2^{k,n} + \frac{t}{2}(1 - u_1^{k,n} - u_2^{k,n}). \end{aligned}$$

We set

$$h(t) = f_1(\mathbf{u}^{k,n}(t)) + f_2(\mathbf{u}^{k,n}(t)).$$

We note that

$$h(0) = f_1(\mathbf{u}^{k,n}) + f_2(\mathbf{u}^{k,n}), \quad \text{and} \quad h(1) = 1,$$

since $u_1(1) + u_2(1) = 1$, and also

$$h'(t) = \frac{1}{2}(1 - u_1^{k,n} - u_2^{k,n}) \sum_{i,j=1}^2 \frac{\partial}{\partial u_i} f_j(\mathbf{u}^{k,n}(t)).$$

So, using (3.7) and applying the Fundamental Theorem of Calculus to $h(t)$ we get

$$\begin{aligned} u_1^{k,n+1} + u_2^{k,n+1} &= u_1^{k,n} + u_2^{k,n} - \frac{\Delta t}{\Delta x} (h(1) - (f_1(\mathbf{u}^{k-1,n}) + f_2(\mathbf{u}^{k-1,n}))) \\ &\quad - \frac{\Delta t}{\Delta x} (h(1) - h(0)) \\ &\leq u_1^{k,n} + u_2^{k,n} + \frac{\Delta t}{\Delta x} (1 - f_1(\mathbf{u}^{k-1,n}) - f_2(\mathbf{u}^{k-1,n})) \\ &\quad + \Lambda \frac{\Delta t}{\Delta x} (1 - u_1^{k,n} - u_2^{k,n}) \\ &\leq 1 - (1 - \Lambda \frac{\Delta t}{\Delta x}) (1 - u_1^{k,n} - u_2^{k,n}) \leq 1, \end{aligned}$$

by (3.1), if $\mathbf{u}^{k,n}, \mathbf{u}^{k-1,n} \in \Delta$.

So, the scheme given by (3.2)–(3.4), with condition (3.1), defines $\mathbf{u}^{k,n}$ in such a way that $\mathbf{u}^{k,n} \in \Delta$, for all $k, n \in \mathbb{N}$, $0 \leq k \leq M$. We then define, for $\varepsilon = \Delta x = \Lambda \Delta t$,

$$\begin{aligned} \mathbf{u}^\varepsilon(x, t) &= \mathbf{u}^{k,n}, \quad \text{if } (k - \tfrac{1}{2}) \cdot \Delta x \leq x < (k + \tfrac{1}{2}) \Delta x, \\ &\quad n \Delta t \leq t < (n + 1) \Delta t, \end{aligned} \quad (3.8)$$

for all $k, n \in \mathbb{N}$, $0 \leq k \leq M$.

So, by Tartar's theorem on the existence of Young measures (see [13]), we can obtain $\nu_{x,t}$, a Young measure associated to a subsequence of $\mathbf{u}^\varepsilon(x, t)$, still denoted by $\mathbf{u}^\varepsilon(x, t)$, satisfying $\text{Supp} \nu_{x,t} \subset \Delta$. Hence, for all $h \in C(\mathbb{R}^2)$ we have (2.9).

Now, by the same proof of the theorem in section 1 of [8], we obtain that for all $\phi \in C_0^\infty((0, L) \times [0, \infty))$ we have that (2.16) holds for $\nu_{x,t}$. To prove that $\nu_{x,t}$ satisfies (1.5) in definition 1.1 we make use of the following lemma.

Lemma 3.1. *Given $\zeta \in C_0^1((0, T))$ we have*

$$|\int_0^T (\mathbf{f}(\mathbf{u}^\varepsilon(x, t)) - \mathbf{f}(\mathbf{u}^\varepsilon(y, t)))\zeta(t) dt| \leq C \text{Var}(\zeta)(|x - y| + \varepsilon), \quad (3.9)$$

for a.e. $0 < x, y < L$, and

$$|\int_0^T (\mathbf{f}(\mathbf{u}^\varepsilon(x, t)) - \mathbf{e})\zeta(t) dt| \leq C \text{Var}(\zeta)(x + \varepsilon), \quad (3.10)$$

for a.e. $0 < x < L$ and some constant $C > 0$ independent of x, T .

Proof. The inequality (3.10) is clearly a consequence of (3.9). Since, for each $t > 0$, $\mathbf{u}^\varepsilon(x, t)$ is constant for $(k - \frac{1}{2})\Delta x \leq x < (k + \frac{1}{2})\Delta x$, $k \in \mathbb{N}$, $0 \leq k \leq M$, we can assume that $x = k_1\Delta x$ and $y = k_2\Delta x$, with $0 \leq k_1 < k_2 \leq M$. We can also assume $T = N\Delta t$, because of the form of the right-hand side of (3.9). Then, we have

$$\begin{aligned} & |\int_0^{N\Delta t} (\mathbf{f}(\mathbf{u}^\varepsilon(k_1\Delta x, t)) - \mathbf{f}(\mathbf{u}^\varepsilon(k_2\Delta x, t)))\zeta(t) dt| \\ & \leq |\sum_{n=0}^{N-1} (\mathbf{f}(\mathbf{u}^{k_1,n}) - \mathbf{f}(\mathbf{u}^{k_2,n}))\zeta(n\Delta t)\Delta t| + C_1 \text{Var}(\zeta)\Delta t \\ & = |\sum_{n=0}^{N-1} \zeta(n\Delta t) \sum_{k=k_1+1}^{k_2} (\mathbf{f}(\mathbf{u}^{k,n}) - \mathbf{f}(\mathbf{u}^{k-1,n}))\Delta t| + C_1 \text{Var}(\zeta)\Delta t \\ & = |\sum_{n=0}^{N-1} \zeta(n\Delta t) \sum_{k=k_1+1}^{k_2} (\mathbf{u}^{k,n+1} - \mathbf{u}^{k,n})\Delta x| + C_1 \text{Var}(\zeta)\Delta t \\ & \leq |\sum_{k=k_1+1}^{k_2} \sum_{n=1}^N \mathbf{u}^{k,n}(\zeta((n-1)\Delta t) - \zeta(n\Delta t))\Delta x| + C_2 \text{Var}(\zeta)\Delta t \\ & \leq C \text{Var}(\zeta)((k_2 - k_1)\Delta x + \varepsilon), \end{aligned}$$

and this gives (3.9). The lemma is then proved. \square

Now, (3.9) allows us to obtain a subsequence of $\mathbf{u}^\varepsilon(x, t)$, such that, for a.e. $x \in (0, L)$, $\mathbf{f}(\mathbf{u}^\varepsilon(x, \cdot))$ converges weakly as a function of t (we get first the convergence for a countable dense subset of $(0, L)$ and then use (3.9) to extend the convergence to a.e. $x \in (0, L)$). This weak limit must then coincide a.e. in

$(0, L) \times [0, \infty)$ with $\langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle$. In particular, (3.10) gives us

$$\left| \int_0^T (\langle \nu_{x,t}, \mathbf{f}(\mathbf{u}) \rangle - \mathbf{e}) \zeta(t) dt \right| \leq C \text{Var}(\zeta)x, \quad (3.11)$$

for a.e. $x \in (0, L)$ and some constant $C > 0$ independent of x, T . We now use (2.16) and (3.11) to conclude that $\nu_{x,t}$ satisfies (1.5), exactly as we did in the end of the last section. So, $\nu_{x,t}$ constructed by the finite difference scheme given by (3.2)–(3.4), with condition (3.1), is a m-v solution of (1.1)–(1.3).

We conclude this section with the description of another scheme which also gives us a m-v solution to problem (1.1)–(1.3). This time we discretize only the space variable. So, we set

$$\mathbf{u}^0(t) = \mathbf{e}, \quad (3.12)$$

$$\mathbf{u}^k(0) = \mathbf{u}_0(k\Delta x), \quad (3.13)$$

$k \in \mathbf{N}$, $1 \leq k \leq M$, and

$$\frac{d}{dt} \mathbf{u}^k = -\frac{1}{\Delta x} (\mathbf{f}(\mathbf{u}^k(t)) - \mathbf{f}(\mathbf{u}^{k-1}(t))), \quad k \in \mathbf{N}, \quad 1 \leq k \leq M. \quad (3.14)$$

The equation (3.14) defines a system of ordinary differential equations in \mathbf{R}^{2M} . So, we have a initial value problem given by (3.14), (3.12) and (3.13). Set

$$\Omega = \prod_{k=1}^M \Delta,$$

and let X denote the vector field defined in Ω given by the right-hand side of (3.14). It is easy to see that X satisfies the hypotheses of Picard's theorem on the local existence and uniqueness of solutions to initial value problems for systems of ordinary differential equations. We will show that we can extend this local solution to a global one $(\mathbf{u}^k(t))_{k=1}^M$ with $\mathbf{u}^k(t) \in \Delta$, for all $t > 0$, $k = 1, \dots, M$.

Let ω be a point in $\partial\Omega$, with $\omega = (\mathbf{u}^k)_{k=1}^M$, $\mathbf{u}^k \in \Delta$. Then, there exists $k \in \{1, \dots, M\}$ such that $\mathbf{u}^k \in \partial K$. That is, for this k we have one of the following alternatives:

$$u_i^k = 0, \quad \text{for } i = 1 \quad \text{or } 2, \quad (3.15)$$

or

$$u_1^k + u_2^k = 1. \quad (3.16)$$

We will have shown to be possible to extend indefinitely any local solution of the problem in question, with initial value in Δ , if we verify that

$$\langle X(\omega), N(\omega) \rangle \leq 0, \quad (3.17)$$

for all $\omega \in \partial\Omega$, where $N(\omega)$ is the outward, unit normal vector to $\partial\Omega$, defined in ω . But, (3.17) is a consequence of the fact that

$$\langle X(\omega), N(\omega) \rangle = -X_i^k(\omega) = -\frac{1}{\Delta x} f_i(u^{k-1}),$$

if (3.15) holds, for $i = 1$ or 2 , and

$$\langle X(\omega), N(\omega) \rangle = X_1^k(\omega) + X_2^k(\omega) = -\frac{1}{\Delta x} (1 - f_1(u^{k-1}) - f_2(u^{k-1})),$$

if (3.16) holds, where $\omega = (u^k)_{k=1}^M$, with $u^k \in \Delta$. So, we can extend the local solution to a global solution, $(u^k(t))_{k=1}^M$, of (3.14), (3.12), (3.13).

Again, we define for $\varepsilon = \Delta x$

$$u^\varepsilon(x, t) = u^k(t), \quad \text{if } (k - \frac{1}{2})\Delta x \leq x < (k + \frac{1}{2})\Delta x, \quad 0 \leq k \leq M.$$

By Tartar's theorem on the existence of Young measures we obtain a family of probability measures $\nu_{x,t}$, with $\text{Supp } \nu_{x,t} \in \Delta$, satisfying (2.9) for all $h \in C(\mathbf{R}^2)$. We prove that $\nu_{x,t}$ is a m-v solution to (1.1)–(1.3) exactly as we did for the Young measure obtained from the solutions of the scheme (3.2)–(3.4).

4. Weak Asymptotic Steadiness.

Here we study the asymptotic behavior of the m-v solutions of (1.1)–(1.3). We will be interested in proving the following result.

Theorem 4.1. *Let $\nu_{x,t}$ be a m-v solution of (1.1)–(1.3) and set*

$$\langle \mu_T^x, \cdot \rangle = \frac{1}{T} \int_0^T \langle \nu_{x,t}, \cdot \rangle dt.$$

Then, for a.e. $x \in [0, L]$, $\mu_T^x \rightarrow \delta_e$, as $T \rightarrow \infty$, where δ_e is the Dirac measure concentrated at e . Further, the rates of convergence of the limits

$$\lim_{T \rightarrow \infty} \langle \mu_T^x, u_1 \rangle = 1, \quad \lim_{T \rightarrow \infty} \langle \mu_T^x, u_2 \rangle = 0, \quad (4.1)$$

for a.e. $x \in [0, 1]$, can always be estimated using properties of the map f and the fact that the rate of convergence of the limit

$$\lim_{T \rightarrow \infty} \langle \mu_T^x, f \rangle = e$$

is $O(T^{-1})$.

The theorem 4.1 will follow from proposition 1.2, more precisely, from (1.6), and some lemmas about weak convergence of probability measures to Dirac measures that we now pass to state. The proofs of these results are very simple and given in [4].

Lemma 4.2. Consider a one-parameter family of probability Borel measures defined on the interval $[0, 1]$, $\mu_T \in \mathbf{P}([0, 1])$, $0 < T < \infty$. Let $\sigma: [0, 1] \rightarrow [0, 1]$ be a Borel function satisfying: for some $\varepsilon_0 > 0$, $\sigma([0, 1 - \varepsilon_0]) \cap \sigma([1 - \varepsilon_0, 1]) = \emptyset$ and $\sigma|_{[1 - \varepsilon_0, 1]}$ is injective increasing, with $\sigma(1) = 1$. Assume

$$\lim_{T \rightarrow \infty} \langle \mu_T, \sigma \rangle = 1. \quad (4.2)$$

Then $\mu_T \rightarrow \delta_{\{1\}}$, where $\delta_{\{1\}}$ is the Dirac measure on $[0, 1]$, concentrated at 1. Suppose, further, that (4.2) converges at a rate $O(T^{-\alpha})$, $\alpha > 0$. Then, for the rate of convergence of the limit

$$\lim_{T \rightarrow \infty} \langle \mu_T, id \rangle = 1, \quad (4.3)$$

where id denotes the identity map on $[0, 1]$, $s \mapsto s$, we have the following:

- (1) The rate of convergence of (4.3) can always be estimated by the order of the envelope of a family of functions $\rho^\varepsilon(T)$ of the form

$$\rho^\varepsilon(T) = \varepsilon + \frac{C}{1 - \sigma(1 - \varepsilon)} T^{-\alpha},$$

for some $C > 0$.

- (2) If we can choose ε_0 sufficiently small so that $\sigma|_{[1-\varepsilon_0, 1]}$ is continuously differentiable and $\sigma' > 0$ in $[1-\varepsilon_0, 1]$, then (4.3) is realized with, at least, the same rate of convergence $O(T^{-\alpha})$ as (4.2).
- (3) If $\rho : [0, 1] \rightarrow [0, 1]$ is convex with $\rho(0) = 0$ and $\rho \circ \sigma(s) \leq s$, $s \in [0, 1]$, then (4.3) is attained at a rate at least $O(1 - \rho(1 - O(T^{-\alpha})))$.

The analogous result for probability measures on $[0, 1]$ converging weakly to a Dirac measure concentrated at 0, is the following.

Lemma 4.3. Consider a one-parameter family of probability Borel measures defined on the interval $[0, 1]$, $\mu_T \in \mathbf{P}([0, 1])$, $0 < T < \infty$. Let $\tilde{\sigma} : [0, 1] \rightarrow [0, 1]$ be a Borel function satisfying: for some $\varepsilon_0 > 0$, $\tilde{\sigma}([0, \varepsilon_0]) \cap \tilde{\sigma}((\varepsilon_0, 1]) = \emptyset$ and $\tilde{\sigma}|_{[0, \varepsilon_0]}$ is injective increasing, with $\tilde{\sigma}(0) = 0$. Assume

$$\lim_{T \rightarrow \infty} \langle \mu_T, \tilde{\sigma} \rangle = 0. \quad (4.4)$$

Then $\mu_T \rightarrow \delta_{\{0\}}$. Further, if (4.4) converges with a rate $O(T^{-\alpha})$, $\alpha > 0$, then, with respect to the rate of convergence of

$$\lim_{T \rightarrow \infty} \langle \mu_T, t \rangle = 0, \quad (4.5)$$

where t denotes the identity map $t \mapsto t$ on $[0, 1]$, we have the following:

- (1) The rate of convergence of (4.5) can always be estimated by the order of the envelope of a family of functions of the form

$$\rho^\varepsilon(T) = \varepsilon + \frac{C}{\tilde{\sigma}(\varepsilon)} T^{-\alpha},$$

for some $C > 0$.

- (2) If we can choose ε_0 sufficiently small so that $\tilde{\sigma}|_{[0, \varepsilon_0]}$ is continuously differentiable and $\tilde{\sigma}' > 0$ in $[0, \varepsilon_0]$, then (4.5) is achieved with, at least, the same rate of convergence $O(T^{-\alpha})$ which estimate the rate of (4.4).
- (3) If $\tilde{\rho} : [0, 1] \rightarrow [0, 1]$ is a concave function with $\tilde{\rho}(1) = 1$ and $\tilde{\rho} \circ \tilde{\sigma}(t) \geq t$, $t \in [0, \varepsilon_0]$, then (4.5) is realized at a rate at least $O(\tilde{\rho}(O(T^{-\alpha})))$.

We easily see that lemma 4.3 follows from lemma 4.2 by making $s = 1 - t$ (with a corresponding pull-back of the measures μ_T) and defining $\sigma(s) = 1 - \tilde{\sigma}(t)$.

The last lemma is an extension of the above ones for probability measures defined on multidimensional intervals. Before stating it we need to establish some general definitions.

Let $K = \prod_{i=1}^n [0, 1]$ be the n -dimensional unit cube and \mathbf{u} denote a generic element of K . For the time being, let us denote by \mathbf{e} any vertice of K , that is, $\mathbf{e} = (\delta_1, \dots, \delta_n)$ where $\delta_i = 0$ or 1 , $i = 1, \dots, n$. For $\varepsilon_0 > 0$, set $U_{\varepsilon_0} = \prod_{i=1}^n [-\varepsilon_0, \varepsilon_0]$ and denote

$$K(\mathbf{e}; \varepsilon_0) = K \cap (\mathbf{e} + U_{\varepsilon_0}).$$

Definition 4.3. A K -flux of first kind is a continuous map $\mathbf{f} : K \rightarrow K$, $\mathbf{f} = (f_1, \dots, f_n)$ satisfying:

- (i) $f_i(u_1, \dots, u_{i-1}, \delta_i, u_{i+1}, \dots, u_n) = \delta_i$, where $\delta_i = 0$ or 1 , $i = 1, \dots, n$;
- (ii) for a certain $\varepsilon_0 > 0$ and a certain vertice \mathbf{e} of K , \mathbf{f} is a homeomorphism of $K(\mathbf{e}; \varepsilon_0)$ onto its image by \mathbf{f} .

The vertice \mathbf{e} in (ii) of definition 4.3 will be said a vertice of coerciveness for the K -flux \mathbf{f} . We say that the K -flux \mathbf{f} is of class C^k if \mathbf{f} can be extended as a map C^k to a neighborhood of K . We denote by $\mathcal{F}^+(K)$ the set of all K -fluxes of first kind.

Given a K -flux of first kind, let us define

$$\bar{f}_i(s) = f_i(\mathbf{e} + (s - \delta_i)\mathbf{e}_i), \quad s \in [0, 1], \quad (4.6)$$

where $\mathbf{e} = (\delta_1, \dots, \delta_n)$, $\delta_i = 0$ or 1 , is a vertice of coerciveness of \mathbf{f} . The \bar{f}_i satisfy: $\bar{f}_i([0, 1 - \varepsilon_0]) \cap \bar{f}_i([1 - \varepsilon_0, 1]) = \emptyset$ and $\bar{f}_i|_{[1 - \varepsilon_0, 1]}$ is strictly increasing, with $\bar{f}_i(1) = 1$.

Proposition 4.4. *Let $K = \prod_{i=1}^n [0, 1]$ and $\mu_T \in \mathbf{P}(K)$, $0 < T < \infty$. Suppose $\mathbf{f} \in \mathcal{F}^+(K)$ with \mathbf{e} a vertice of coerciveness, and assume that*

$$\lim_{T \rightarrow \infty} \langle \mu_T, \mathbf{f} \rangle = \mathbf{e} \quad (4.7)$$

Then, $\mu_T \rightarrow \delta_{\mathbf{e}}$. Further, if the limit (4.7) is attained at a rate of convergence $O(T^{-\alpha})$, for some $0 < \alpha \leq 1$, then we can obtain estimates for the rates of convergence of the limits

$$\lim_{T \rightarrow \infty} \langle \mu_T, \bar{f}_i \rangle = \delta_i, \quad i = 1, \dots, n,$$

by the order of the envelope of a family of functions $\rho^\varepsilon(T)$, of the form

$$\rho^\varepsilon(T) = \varepsilon + C(\varepsilon)T^{-\alpha}, \quad (4.8)$$

where $C(\varepsilon)$ is a function such that $C(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, which depends only on the map \mathbf{f} . In particular, we can obtain estimates for the rates of convergence of the limits

$$\lim_{T \rightarrow \infty} \langle \mu_T, u_i \rangle = \delta_i, \quad i = 1, \dots, n,$$

by using assertions (1), (2), (3) of lemma 4.2, with $\sigma = \bar{f}_i$, if $\delta_i = 1$, or the corresponding assertions in lemma 4.3, with $\tilde{\sigma} = \bar{f}_i$, if $\delta_i = 0$.

Remark: The order of the envelope of a family of functions like (4.8) is often easy to obtain. For instance, if $C(\varepsilon) = \frac{\gamma}{\varepsilon^\gamma}$, $\gamma > 1$, it is easy to see that it is $O(T^{-\frac{\gamma}{\gamma+1}})$.

Now, we observe that we can extend any Δ -flux to a K -flux of first kind, in the case $n = 2$, by defining

$$\mathbf{f}(\mathbf{u}) = (1 - f_2(1 - u_2, 1 - u_1), 1 - f_1(1 - u_2, 1 - u_1))$$

if $\mathbf{u} \in K$ is such that $u_1 + u_2 \geq 1$.

Proof of Theorem 4.1. By (1.6) in proposition 1.2 we have, for a.e. $x \in [0, L]$,

$$\lim_{T \rightarrow \infty} \langle \mu_T^x, \mathbf{f} \rangle = \mathbf{e},$$

with a rate of convergence $O(T^{-1})$. So, by the above observation, we can apply proposition 4.4 to get all the assertions in the statement of theorem 4.1. \square

Remarks:

1. The estimates for the rates of the expected values of the dependent variables can usually be obtained by combining the use of peculiarities of the map \mathbf{f} with the estimate of the order of envelopes of families of functions. For instance, consider the map $\mathbf{f} = (f_1, f_2)$ given by

$$f_1(u_1, u_2) = \frac{u_1^2}{u_1^2 + u_2^2 + (1 - u_1 - u_2)^2},$$

$$f_2(u_1, u_2) = \frac{u_2^2}{u_1^2 + u_2^2 + (1 - u_1 - u_2)^2},$$

which appears in the simple model for three-phase flow in porous media in [6]. Assume that for a family $\mu_T \in \mathbf{P}(\Delta)$, $0 < T < \infty$, we have

$$\lim_{T \rightarrow \infty} \langle \mu_T, \mathbf{f} \rangle = (1, 0),$$

with a rate of convergence $O(T^{-1})$, as it occurs if the μ_T are the time-averages of a m-v solution to the initial boundary value problem modeling the flow of water, gas and oil in a reservoir submitted to the constant injection of pure water.

Then, since

$$C_1 u_2^2 \leq f_2(u_1, u_2) \leq C_2 u_2^2,$$

for some constants $C_1, C_2 > 0$, we easily see that

$$\lim_{T \rightarrow \infty} \langle \mu_T, u_2^2 \rangle = 0,$$

with a rate $O(T^{-1})$. By Jensen's inequality we have

$$\langle \mu_T, u_2^2 \rangle^{\frac{1}{2}} \geq \langle \mu_T, u_2 \rangle \geq 0,$$

which gives

$$\lim_{T \rightarrow \infty} \langle \mu_T, u_2 \rangle = 0,$$

at a rate at least $O(T^{-\frac{1}{2}})$.

Now, by Tchebychev's inequality, we have, for $\varepsilon > 0$ small,

$$0 \leq \langle \mu_T, \chi_{(\varepsilon, 1]}(u_2) \rangle \leq \frac{1}{\varepsilon^2} \langle \mu_T, u_2^2 \rangle.$$

Hence,

$$\lim_{T \rightarrow \infty} \langle \mu_T, \chi_{(\varepsilon, 1]}(u_2) \rangle = 0,$$

at a rate $O(T^{-1})$, and so

$$\lim_{T \rightarrow \infty} \langle \mu_T, \chi_{[0, \varepsilon]}(u_2) \rangle = 1,$$

with the same rate. We, then, get

$$\lim_{T \rightarrow \infty} \langle \mu_T, \chi_{[0, \varepsilon]}(u_2) f_1 \rangle = 1,$$

at a rate $O(T^{-1})$. So, we obtain, for $\bar{f}_1(s) \equiv f_1(s, 0)$,

$$\begin{aligned} \langle \mu_T, \bar{f}_1 \rangle &\geq \langle \mu_T, \chi_{[0, \varepsilon]}(u_2) \bar{f}_1 \rangle \\ &\geq \langle \mu_T, \chi_{[0, \varepsilon]}(u_2) f_1 \rangle - C_2 \varepsilon \\ &\geq 1 - \frac{C_1}{\varepsilon^2} T^{-1} - C_2 \varepsilon. \end{aligned}$$

Thus, we arrive at

$$\lim_{T \rightarrow \infty} \langle \mu_T, \bar{f}_1 \rangle = 1,$$

the rate of convergence being estimated by the order of the envelope of

$$\rho^\varepsilon(T) = \frac{C_1}{\varepsilon^2} T^{-1} + C_2 \varepsilon,$$

which is easily seen to be $O(T^{-\frac{1}{3}})$. Then, the rate of convergence of

$$\lim_{T \rightarrow \infty} \langle \mu_T, u_1 \rangle = 1,$$

can be estimated by $O(1 - \bar{f}_1^{-1}(1 - O(T^{-\frac{1}{3}})))$, which is the same as $O(T^{-\frac{1}{6}})$, since $\bar{f}_1'(1) = 0$ and $\bar{f}_1''(1) < 0$.

2. By lemma 3.1 we see that the asymptotic behavior of the expected values of m-v solutions to the problem (1.1)–(1.3) is already presented by the approximate

solutions given by the finite difference schemes defined in section 3. This fact allows the investigation of sharp estimates for the rates of convergence on the basis of the issues of numerical experiments.

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