

COMBUSTION FRONTS IN PETROLEUM RESERVOIRS

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Abstract

We determine the planar traveling wave solutions for a nonstrictly hyperbolic system of two conservation laws modeling combustion in-situ in petroleum reservoirs. Thus, the study of the analogue of the Chapman-Jouguet combustion theory in gas dynamics is initiated for two-phase flow in porous media.

1. Introduction

We determine nonlinear waves in a two-phase flow model, which represent combustion of oil with oxygen in a porous medium. Understanding these waves is important to maximize the oil recovery in methods such as combustion in-situ [1], widely used to extract heavy oil in deep petroleum reservoirs.

It is difficult to extract oil from the porous rock where it lies. One of the goals of petroleum engineering is to develop methods for improving oil recovery. Most of the methods employed consist in injecting some fluid, such as water, gas, steam, polymer, etc, in some producing wells to displace oil and recover it from other wells. Because these methods are expensive, optimal strategies for their usage are determined by means of computer simulations. To validate such simulations, it is important to compare the numerical results with the nonlinear waves appearing in the flow.

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If a complete mathematical understanding of the nonlinear waves occurring in the flow is attained, such a knowledge can be used as a basis for the computer implementation of certain extremely accurate numerical methods for the simulation of flow with fronts [2]. The usage of such methods should be particularly fruitful for the simulation of combustion in-situ, because they do not spread the heat contents of the burning front to the enclosing computational blocks. This usage leads the way to realistic simulation of the chemical reactions in the burning front, which are extremely sensitive to temperature.

S. E. Buckley and M. C. Leverett [3] described the mechanism responsible for the displacement of oil in sandstone. The mechanism is shock formation. They considered the displacing fluid to be water or gas, but they did not take into account temperature variations.

Higher temperatures reduce the oil viscosity, facilitating its flow. F. G. Fayers [4] introduced a model for the flow of water and oil, taking into account temperature variations. The Riemann problem for this model (i.e., the Cauchy problem with initial data which is constant by parts) was solved in [5] and [6] independently. For simplicity, the state equations for the internal energy of the water, oil and rock were assumed to be linear functions of temperature. Combustion was not considered in Fayers's model.

The classical theory of combustion in gases was introduced at the beginning of this century by Chapman and Jouguet [7]; one of its basic assumptions is that the combustion process is instantaneous. Also, the total energy of the gas phase is given as the sum of the internal energy of the gas with the binding energy of the unburnt gas part.

In this work, we assume that the total energy of the gas follows the same rule as that of Chapman-Jouguet's theory. The resulting model may be employed to describe the method of combustion in-situ, employed for oil recovery in petroleum reservoirs. This method consists in injecting air or oxygen to maintain a burning zone in part of the reservoir, causing the temperature to rise, and therefore the oil viscosity to reduce, which greatly facilitates the flow of oil.

The model for combustion of oil and gaseous hydrocarbons with oxygen is presented in Section 2. In Section 3 we seek solutions which are traveling waves with profiles connecting an unburnt state ahead with a burnt state behind. Such solutions are represented by orbits of a certain vector field associated to a parabolic system of partial differential equations. We show that the burnt state, the unburnt state and the wave propagation speed are related by a Rankine-Hugoniot condition. We determine the Rankine-Hugoniot curve. For each given unburnt state, this curve consists of burnt states for all possible propagation speeds.

We show that there are burnt states in the Rankine-Hugoniot curve which are not connected by traveling waves to the corresponding unburnt wave. Thus, there are shocks which are physically inadmissible. This behavior is not dissimilar from that of combustion in gases. To select physically meaningful shocks, it is necessary to find the phase portrait of the vector field, as it is seen in Section 3. This is done in Section 4. Conclusions are presented in Section 5.

2. The Model

The system of equations governing the flow we consider here reflects the conservation of mass of each component, as well as the conservation of total energy. It also reflects Darcy's law of force. We work in a cylindrical volume element, see Figure 1.

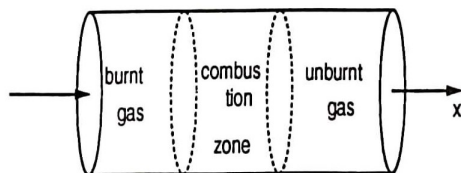


Fig. 1: Element of volume on the porous medium.

One assumes that this volume element contains a finite width combustion

zone. In the current work, compressibility and volumetric changes associated with the combustion process are neglected. This simplifying hypothesis will be removed in forthcoming work. However, heat conduction, as well as the capillary pressure present in displacement of multiphase fluid in porous media are all taken into account. They contribute through parabolic terms whose forms determine the traveling wave solution. We assume that there is a single chemical reaction to write an equation for the chemical reaction rate. Although petroleum reservoir exist in three dimensional space, in the current work we restrict our attention to reservoirs with variations occurring in only one spatial dimension.

Throughout this section, the indices g and o refer to the gaseous and oleic phases, respectively, and r to the porous rock phase.

The flow is described by state quantities depending on (x, t) . They are denoted as follows. The saturation of phase l in the fluid, $l = g, o$, i.e., the fraction of the porous volume occupied by phase l , is denoted by $s_l = s_l(x, t)$; the temperature, which is assumed to be the same for gas, oil and rock at each point (x, t) , is given by $\theta = \theta(x, t)$; and the burnt volume fraction of the gaseous phase is $\epsilon = \epsilon(x, t)$.

The relative permeability of phase l , denoted by k_l , is a dimensionless function of saturation s_l , measured in the laboratory; it is a phase dependent quantity such that Kk_l is the porous medium capability of allowing the flow of phase l . Here K is the absolute permeability of the rock, which measures the porous medium capability of allowing fluid flow.

The capillary pressure $p_c \equiv p_g - p_o = p_c(s_g)$ is obtained empirically as a given decreasing function of s_g , where p_l , $l = g, o$, is the pressure in the phase l .

The other relevant quantities are denoted by: ρ_l , $l = g, o, r$, the density of phase l ; ϕ , the rock porosity or fraction of total volume occupied by the fluid phases; v_l , the seepage velocity of phase l ; and μ_l , the viscosity of phase l , i.e, its intrinsic resistance to motion. The dependence of oil viscosity on temperature will be taken into account.

The equations governing the flow are the following. First, the *conservation of mass of burnt gas* is

$$\frac{\partial}{\partial t}(\phi \rho_g s_g \epsilon) = -\frac{\partial}{\partial x}(\rho_g v_g \epsilon) + \phi \rho_g s_g Q, \quad (2.1)$$

where the first term represents accumulation of burnt gas mass. The second term represents transport of the burnt gas mass, and the third term represents the rate of gain of burnt gas mass due to chemical reaction. In this third term, $Q = Q(\theta, \epsilon)$ is the volumetric fraction of burnt gas generated in unit time.

Similarly, the *conservation of mass of unburnt gas* is

$$\frac{\partial}{\partial t}(\phi \rho_g s_g (1 - \epsilon)) = -\frac{\partial}{\partial x}(\rho_g v_g (1 - \epsilon)) - \phi \rho_g s_g Q, \quad (2.2)$$

the third term represents the rate of loss of unburnt gas mass due to the chemical reaction.

The *conservation of oil mass* is

$$\frac{\partial}{\partial t}(\phi \rho_o s_o) = -\frac{\partial}{\partial x}(\rho_o v_o). \quad (2.3)$$

In this equation, the oil mass loss due to the combustion process is neglected. Thus, the equation for total gas mass resulting from adding (2.1) and (2.2) has no source term. Oil mass loss should be considered in more realistic models.

Finally, denoting by E_l , $l = g, o, r$, the total energy of phase l , the equation of *conservation of energy* is

$$\begin{aligned} \frac{\partial}{\partial t}(\phi \rho_g s_g E_g + \phi \rho_o s_o E_o + \rho_r E_r) = \\ -\frac{\partial}{\partial x}(\rho_g v_g E_g + \rho_o v_o E_o) + \frac{\partial}{\partial x}(k \frac{\partial \theta}{\partial x}), \end{aligned} \quad (2.4)$$

where the first term represents energy accumulation in the gas, oil and rock. The second term represents energy transport by convection in the gas and oil. The last term of this equation stands for heat conduction, and k is an effective thermal conductivity of the composite material formed by the several fluid and rock components.

From Darcy's force law in porous media, the volumetric flow of phase l is given by

$$v_l = -K\lambda_l \frac{\partial p_l}{\partial x}, \quad l = g, o. \quad (2.5)$$

where $\lambda_l \equiv k_l/\mu_l$ is the mobility of the phase l . We remark that kinetic energy terms are neglected in Darcy's law, because accelerations have negligible effects in usual petroleum reservoir flow conditions.

Substituting (2.5) in equations (2.1), (2.2), (2.3) and (2.4) we have, respectively,

$$\frac{\partial}{\partial t}(\phi\rho_g s_g \epsilon) = \frac{\partial}{\partial x}(K\rho_g \epsilon \lambda_g \frac{\partial p_g}{\partial x}) + \phi\rho_g s_g Q \quad (2.6)$$

$$\frac{\partial}{\partial t}(\phi\rho_g s_g (1 - \epsilon)) = \frac{\partial}{\partial x}(K\rho_g (1 - \epsilon) \lambda_g \frac{\partial p_g}{\partial x}) - \phi\rho_g s_g Q \quad (2.7)$$

$$\frac{\partial}{\partial t}(\phi\rho_o s_o) = \frac{\partial}{\partial x}(K\rho_o \lambda_o \frac{\partial p_o}{\partial x}) \quad (2.8)$$

$$\begin{aligned} \frac{\partial}{\partial t}(\phi\rho_g s_g E_g + \phi\rho_o s_o E_o + \rho_r E_r) = \\ \frac{\partial}{\partial x}(K\rho_g \lambda_g E_g \frac{\partial p_g}{\partial x} + K\rho_o \lambda_o E_o \frac{\partial p_o}{\partial x}) + \frac{\partial}{\partial x}(k \frac{\partial \theta}{\partial x}). \end{aligned} \quad (2.9)$$

As we already mentioned, the capillary pressure is a given decreasing function of s_g , which measures the pressure difference between oil and gas

$$p_c \equiv p_c(s_g) = p_o - p_g,$$

and hence

$$\frac{\partial p_c}{\partial x} = \frac{\partial p_o}{\partial x} - \frac{\partial p_g}{\partial x}. \quad (2.10)$$

The total flow of the fluid is given by

$$v_T = v_g + v_o,$$

and hence

$$v_T = -K\lambda_g \frac{\partial p_g}{\partial x} - K\lambda_o \frac{\partial p_o}{\partial x}. \quad (2.11)$$

From (2.10) and (2.11), we have

$$\frac{\partial p_g}{\partial x} = -f_o \frac{\partial p_c}{\partial x} - \frac{v_T}{K(\lambda_g + \lambda_o)} \quad (2.12)$$

$$\frac{\partial p_o}{\partial x} = -f_g \frac{\partial p_c}{\partial x} - \frac{v_T}{K(\lambda_g + \lambda_o)}, \quad (2.13)$$

where f_l , $l = g, o$, is called the “fractional flow function” of phase l . This function depends on s_l and θ ; it is defined by

$$f_l(s_l, \theta) = \lambda_l / (\lambda_g + \lambda_o).$$

Substituting (2.12) and (2.13) in (2.6)-(2.9) we have, respectively,

$$\frac{\partial}{\partial t}(\phi \rho_g s_g \epsilon) + \frac{\partial}{\partial x}(\rho_g \epsilon f_g v_T) = -\frac{\partial}{\partial x}(K \rho_g \epsilon \lambda_o f_g \frac{\partial p_c}{\partial x}) + \phi \rho_g s_g Q \quad (2.14)$$

$$\frac{\partial}{\partial t}(\phi \rho_g s_g (1 - \epsilon)) + \frac{\partial}{\partial x}(\rho_g (1 - \epsilon) f_g v_T) = -\frac{\partial}{\partial x}(K \rho_g (1 - \epsilon) \lambda_o f_g \frac{\partial p_c}{\partial x}) - \phi \rho_g s_g Q \quad (2.15)$$

$$\frac{\partial}{\partial t}(\phi \rho_o s_o) + \frac{\partial}{\partial x}(\rho_o f_o v_T) = \frac{\partial}{\partial x}(K \rho_o \lambda_o f_g \frac{\partial p_c}{\partial x}) \quad (2.16)$$

$$\begin{aligned} \frac{\partial}{\partial t}(\phi \rho_g s_g E_g + \phi \rho_o s_o E_o + \rho_r E_r) + \frac{\partial}{\partial x}((\rho_g E_g f_g + \rho_o E_o f_o) v_T) = \\ -\frac{\partial}{\partial x}(K(\rho_g E_g - \rho_o E_o) \lambda_o f_g \frac{\partial p_c}{\partial x}) + \frac{\partial}{\partial x}(k \frac{\partial \theta}{\partial x}). \end{aligned} \quad (2.17)$$

In the following derivations we will use the fact that $s_o = 1 - s_g$, and $f_o = 1 - f_g$.

From now on, we will assume for simplicity that the fluids are incompressible, and that there are no volume or pressure changes due to the chemical reaction. Thus, ρ_o and ρ_g are constant, independent of temperature and pressure. It follows that Equations (2.14), (2.15) and (2.16) are equivalent to the following three equations

$$\frac{\partial s_g}{\partial t} + \frac{v_T}{\phi} \frac{\partial f_g}{\partial x} = -\frac{K}{\phi} \frac{\partial}{\partial x}(\lambda_o f_g \frac{\partial p_c}{\partial x}) \quad (2.18)$$

$$\frac{\partial}{\partial t}(\epsilon s_g) + \frac{v_T}{\phi} \frac{\partial}{\partial x}(\epsilon f_g) = -\frac{K}{\phi} \frac{\partial}{\partial x}(\epsilon \lambda_o f_g \frac{\partial p_c}{\partial x}) + s_g Q \quad (2.19)$$

$$\frac{\partial v_T}{\partial x} = 0. \quad (2.20)$$

From (2.20), we see that v_T depends on time only. For simplicity we assume it to be a constant in this work.

The thermodynamical equations of state we will employ in the current work are

$$\rho_g E_g = C_g \theta - \epsilon q \quad (2.21)$$

$$\rho_o E_o = C_o \theta \quad (2.22)$$

$$\rho_r E_r = C_r \theta, \quad (2.23)$$

where C_l , $l = g, o, r$, are the thermal capacity of component l for unit volume, and q is the heat released by the combustion per unit volume. In this work, we will make the simplifying assumptions that q and C_l , $l = g, o, r$, are constant, and that the effective thermal conductivity k is also constant.

Substituting (2.21), (2.22) and (2.23) in (2.17), we obtain the following equation for the conservation of energy

$$\begin{aligned} & \frac{\partial}{\partial t}(((C_g - C_o)s_g + C_o + C_r/\phi)\theta - \epsilon q s_g) + \\ & \frac{v_T}{\phi} \frac{\partial}{\partial x}(((C_g - C_o)f_g + C_o)\theta - \epsilon q f_g) = \\ & -\frac{K}{\phi} \frac{\partial}{\partial x}(((C_g - C_o)\theta - \epsilon q)\lambda_o f_g \frac{\partial p_c}{\partial x}) + \frac{k}{\phi} \frac{\partial^2 \theta}{\partial x^2}. \end{aligned} \quad (2.24)$$

We will use the notations s and f in lieu of s_g and f_g . Assuming that v_T is not too small, we rescale time by $t \rightarrow (v_T/\phi)t$, and the equations (2.18), (2.24), and (2.19) become, respectively,

$$(2.5) \quad \frac{\partial s}{\partial t} + \frac{\partial f}{\partial x} = -\frac{K}{v_T} \frac{\partial}{\partial x} (\lambda_o f \frac{\partial p_c}{\partial x}) \quad (2.25)$$

$$(2.6) \quad \begin{aligned} \frac{\partial}{\partial t} ((s + \alpha)\theta - \eta \epsilon s) + \frac{\partial}{\partial x} ((f + \beta)\theta - \eta \epsilon f) = \\ -\frac{K}{v_T} \frac{\partial}{\partial x} ((\theta - \eta \epsilon) \lambda_o f \frac{\partial p_c}{\partial x}) + \frac{\gamma}{v_T} \frac{\partial^2 \theta}{\partial x^2} \end{aligned} \quad (2.26)$$

$$(2.7) \quad \frac{\partial}{\partial t} (\epsilon s) + \frac{\partial}{\partial x} (\epsilon f) = -\frac{K}{v_T} \frac{\partial}{\partial x} (\epsilon \lambda_o f \frac{\partial p_c}{\partial x}) + \frac{\phi}{v_T} s Q, \quad (2.27)$$

where α , β , η and γ are constants defined as

$$(2.8) \quad \alpha = \frac{C_o + C_g/\phi}{C_g - C_o}, \quad \beta = \frac{C_o}{C_g - C_o}, \quad \eta = \frac{q}{C_g - C_o}, \quad \gamma = \frac{k}{C_g - C_o}.$$

3. Combustion waves

Given the unburnt state $(s_R, \theta_R, \epsilon_R = 0)$ and the burnt state $(s_L, \theta_L, \epsilon_L = 1)$, we want to know if there is a traveling wave solution of the system (2.25)-(2.27) in the form

$$(3.1) \quad s(x, t) = s(x - \sigma t) = s(z)$$

$$(3.2) \quad \theta(x, t) = \theta(x - \sigma t) = \theta(z)$$

$$(3.3) \quad \epsilon(x, t) = \epsilon(x - \sigma t) = \epsilon(z),$$

with

$$(3.4) \quad \lim_{z \rightarrow -\infty} (s(z), \theta(z), \epsilon(z)) = (s_L, \theta_L, \epsilon_L = 1)$$

and

$$(3.5) \quad \lim_{z \rightarrow -\infty} (s'(z), \theta'(z), \epsilon'(z)) = (0, 0, 0)$$

$$\lim_{z \rightarrow \infty} (s(z), \theta(z), \epsilon(z)) = (s_R, \theta_R, \epsilon_R = 0), \quad (3.6)$$

$$\lim_{z \rightarrow \infty} (s'(z), \theta'(z), \epsilon'(z)) = (0, 0, 0), \quad (3.7)$$

where $z = x - \sigma t$, $z \in R$, and σ is the propagation speed of the wave.

Proposition 3.1. *In order for (3.1)-(3.3) to represent a traveling wave solution of system (2.25)-(2.27), it is necessary that this solution be an orbit of the dynamical system*

$$\frac{ds}{dz} = \frac{\gamma_1 + \sigma s - f(s, \theta)}{H(s, \theta)} \quad (3.8)$$

$$\frac{d\theta}{dz} = \frac{v_T}{\gamma} ((\gamma_1 - \sigma\alpha + \beta)\theta - \eta\gamma_1\epsilon - \gamma_2) \quad (3.9)$$

$$\frac{d\epsilon}{dz} = \frac{\phi}{v_T\gamma_1} sQ(\theta, \epsilon), \quad (3.10)$$

satisfying (3.5)-(3.7). Here $H(s, \theta) = (K/v_T)\lambda_o f(s, \theta)(dp_c/ds)$ is a strictly negative function, $\gamma_1 = f_R - \sigma s_R$ and $\gamma_2 = (f_R + \beta - \sigma(s_R + \alpha))\theta_R$, $f_R = f(s_R, \theta_R)$ are constant. In this case σ and the states $(s_R, \theta_R, \epsilon_R = 0)$ and $(s_L, \theta_L, \epsilon_L = 1)$ are related by the shock condition

$$\sigma = \frac{f_L - f_R}{s_L - s_R} = \frac{(f_R + \beta)\theta_R + \eta f_R - (f_R + \beta)\theta_L}{(s_R + \alpha)\theta_R + \eta s_R - (s_R + \alpha)\theta_L}. \quad (3.11)$$

Proof: Substituting the traveling wave (3.1)-(3.3) in (2.25), (2.26) and (2.27), we obtain, respectively,

$$-\sigma s + f + \frac{K}{v_T}\lambda_o f \frac{dp_c}{dz} = \gamma_1 \quad (3.12)$$

$$-\sigma(\theta(s + \alpha) - \eta\epsilon s) + \theta(f + \beta) - \eta\epsilon f + \frac{K}{v_T}(\theta - \eta\epsilon)\lambda_o f \frac{dp_c}{dz} - \frac{\gamma}{v_T} \frac{d\theta}{dz} = \gamma_2 \quad (3.13)$$

$$-\sigma \frac{d}{dz}(\epsilon s) + \frac{d}{dz}(\epsilon f) + \frac{K}{v_T} \frac{d}{dz}(\epsilon \lambda_o f \frac{dp_c}{dz}) = \frac{\phi}{v_T} s Q, \quad (3.14)$$

where γ_1 and γ_2 are integration constants. Since

$$\frac{dp_c}{dz} = \frac{dp_c}{ds} \frac{ds}{dz}$$

the equation (3.12) becomes

$$\frac{ds}{dz} = \frac{\gamma_1 + \sigma s - f}{H}. \quad (3.15)$$

Using (3.12), the equation (3.13) reduces to

$$\frac{d\theta}{dz} = \frac{v_T}{\gamma} ((\gamma_1 - \sigma\alpha + \beta)\theta - \eta\gamma_1\epsilon - \gamma_2). \quad (3.16)$$

From (3.14), we have

$$\frac{d\epsilon}{dz} = \frac{\phi}{v_T\gamma_1} s Q. \quad (3.17)$$

From (3.4), (3.5), (3.6) and (3.7), we have

$$\lim_{z \rightarrow \pm\infty} \frac{ds}{dz} = \lim_{z \rightarrow \pm\infty} \frac{d\theta}{dz} = 0.$$

Hence from (3.15), we have

$$\gamma_1 + \sigma s_R - f_R = \gamma_1 + \sigma s_L - f_L = 0,$$

or

$$\sigma = \frac{f_L - f_R}{s_L - s_R}. \quad (3.18)$$

From (3.16), we have

$$(\gamma_1 - \sigma\alpha + \beta)\theta_R - \gamma_2 = 0$$

and

$$(\gamma_1 - \sigma\alpha + \beta)\theta_L - \eta\gamma_1 - \gamma_2 = 0.$$

Eliminating γ_2 from these two equations and using $\gamma_1 = f_R - \sigma s_R$, we obtain

$$\sigma = \frac{(f_R + \beta)\theta_R + \eta f_R - (f_R + \beta)\theta_L}{(s_R + \alpha)\theta_R + \eta s_R - (s_R + \alpha)\theta_L},$$

which completes the proof. \square

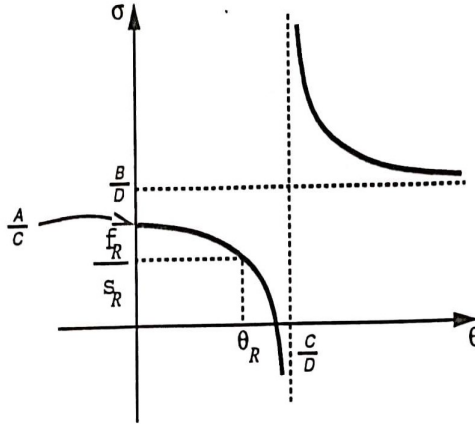


Fig. 2: Speed of the combustion front.

We saw that if there exists a combustion wave traveling with speed σ connecting the states $(s_R, \theta_R, \epsilon_R = 0)$ and $(s_L, \theta_L, \epsilon_L = 1)$, then the shock condition in Proposition 3.1 holds. As we will see, the converse is not true.

Let us fix the unburnt state $(s_R, \theta_R, \epsilon_R = 0)$ and assume that $f_R/s_R < \beta/\alpha$. The case $f_R/s_R \geq \beta/\alpha$ can be treated in a similar way. Notice that the speed σ of the combustion wave depends only on the temperature at the burnt front and on the unburnt state, since

$$\sigma = \frac{A - B\theta}{C - D\theta}, \quad (3.19)$$

where

$$A = (f_R + \beta)\theta_R + \eta f_R$$

$$B = f_R + \beta$$

$$C = (s_R + \alpha)\theta_R + \eta s_R$$

$$D = s_R + \alpha.$$

In this case $d\sigma/d\theta < 0$ and the plot of $\sigma(\theta)$ is shown in Figure 2.

From now on, the capillary pressure function $p_c(s)$ will be assumed C^1 and $dp_c/ds < 0$. Similarly, the fractional flow function $f(s, \theta)$ will be assumed to be C^2 and to be S -shaped in s for each θ , and to decrease in θ , see Figure 3. Thus $f(0, \theta) = 0$, $f(1, \theta) = 1$, $\partial f / \partial s$ vanishes for $s = 0$ and $s = 1$, for each θ , $\partial^2 f / \partial s^2$ is first positive and then negative for s in $(0, 1)$. We also have $\partial f / \partial \theta < 0$ for s in $(0, 1)$. These assumptions are used to model two-phase thermal flow in porous media, with the oil viscosity being a decreasing function of temperature [4, 5].

Using Figure 2 and the relationship $\sigma = (f_L - f_R) / (s_L - s_R)$, we see that there exist temperatures $\theta = \theta_L > C/D$, for which there are three states satisfying the shock condition for a specified state $I = (s_R, \theta_R, \epsilon_R = 0)$, as shown in Figure 3. In this figure the three states corresponding to I are denoted by IV , V and VI . In Figure 3 the specified state $(s_R, \theta_R, \epsilon_R = 0)$ could also be II or III .

Replacing θ_L by θ in (3.11), and allowing θ to vary, we obtain all burnt states $(s, \theta, \epsilon = 1)$, related to $(s_R, \theta_R, \epsilon_R = 0)$ through the Rankine-Hugoniot equation. These states form the projection on the plane $\epsilon = 1$ of the Hugoniot curve through $(s_R, \theta_R, \epsilon_R = 0)$. See Figure 4, where the three states IV , V and VI are indicated. The dashed part of this projection corresponds to states $(s, \theta, \epsilon = 1)$ which are not connected by traveling waves to $(s_R, \theta_R, \epsilon_R = 0)$. To verify this fact, we have to study the phase portrait of the dynamical system of Proposition 3.1. This will be done in the next section.

4. The phase portrait of the associated dynamical system

The dynamical system (3.8)-(3.10) has the following properties. The first equation depends only on s and θ , the second one depends only on θ and ϵ , and it is linear in these two variables. Thus, we may write

$$\frac{ds}{dz} = X(s, \theta) = \frac{\gamma_1 + \sigma s - f(s, \theta)}{H(s, \theta)} \quad (4.1)$$

$$\frac{d\theta}{dz} = Y(\theta, \epsilon) = \frac{v_T}{\gamma}((\gamma_1 - \sigma\alpha + \beta)\theta - \eta\gamma_1\epsilon - \gamma_2) \quad (4.2)$$

$$\frac{d\epsilon}{dz} = Z(s, \theta, \epsilon) = \frac{\phi}{v_T\gamma_1}sQ(\theta, \epsilon), \quad (4.3)$$

where the constants $\gamma_1 = f_R - \sigma s_R$ and $\gamma_2 = (f_R + \beta - \sigma(s_R + \alpha))\theta_R$ depend only on the unburnt state $(s_R, \theta_R, \epsilon_R = 0)$. We also know that $H(s, \theta) < 0$. Now let us consider a temperature θ_L such that the speed σ of the shock connecting $(s_R, \theta_R, \epsilon_R = 0)$ and $(s_L, \theta_L, \epsilon_L = 1)$ is given by the slope of the straight line in Figure 3.

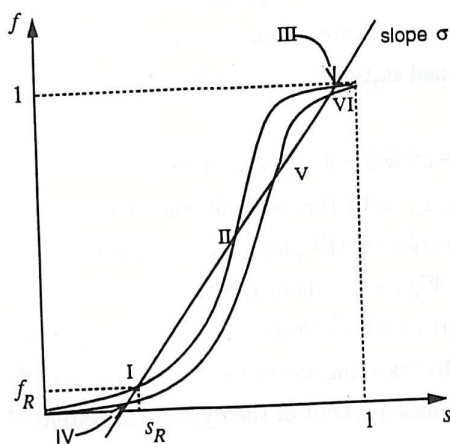


Fig. 3: The states IV, V or VI are connected to the states I, II or III by a shock of speed σ .

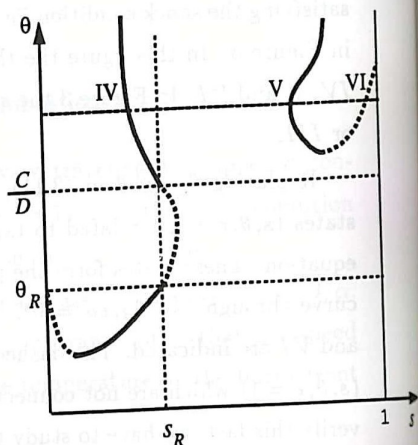


Fig. 4: Projection on the plane $\epsilon = 1$ of the Hugoniot curve through $(s_R, \theta_R, \epsilon_R = 0)$.

In the following we will make the assumption that the volumetric rate of gas burning is given by the function

$$Q(\theta, \epsilon) = \epsilon(1 - \epsilon)\tilde{Q}(\theta),$$

where $\tilde{Q}(\theta) > 0$. This function is artificial because it vanishes at $\epsilon = 0$. More realistic functions Q will be considered in future work. They should not contain the factor ϵ ; rather an Arrhenius-like formula should make the combustion rate

small for low temperatures. However, in such a model the linearized fluxes at the singularities *I*, *II* and *III* have a vanishing eigenvalue; thus the analysis is more difficult than the one presented here. We hope that the current model contains many of the features of the more realistic one.

Our aim is to prove the following theorem:

Theorem 4.1. *For equilibrium points given as in Figure 3, there are orbits of the vector field (4.1)-(4.3) starting at IV and V which end at I. There is no orbit from VI to I.*

The proof is given in the Lemmas 4.2, 4.3 and 4.4, which follow.

We start by studying the field (X, Y, Z) defined in (4.1)-(4.3). The zero set of X is a cylindrical surface S , whose intersection with the coordinate plane (s, θ) is the curve

$$\gamma_1 + \sigma s - f(s, \theta) = 0.$$

Using the value of γ_1 , we see that this curve is formed by points (s, θ) such that

$$\frac{f(s, \theta) - f_R}{s - s_R} = \sigma.$$

The plot of this curve is given in Figure 5, where $I = (s_R, \theta_R, \epsilon_R = 0)$, *II* and *III* are the states at temperature θ_R indicated in Figure 3. The regions where $X > 0$ are denoted by “+” and those where $X < 0$ by “-”. Note that

$$\gamma_1 = s_R \left(\frac{f_R}{s_R} - \sigma \right) < 0.$$

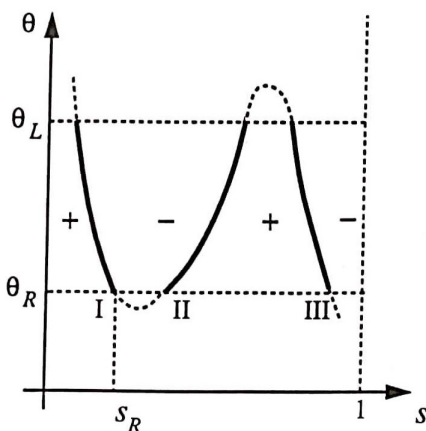


Fig. 5: Intersection of the surface $X = 0$ with the plane (s, θ) .

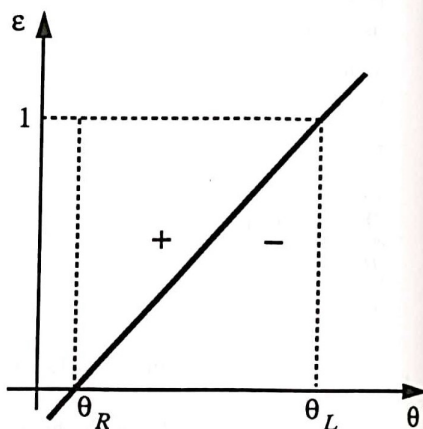


Fig. 6: Intersection of the surface $Y = 0$ with the plane (θ, ϵ) .

The Y component of the field is linear in θ and ϵ , hence its zero set is a plane P . The intersection of P with the coordinate plane (θ, ϵ) is the following straight line (see Figure 6):

$$(\gamma_1 - \sigma\alpha + \beta)\theta - \eta\gamma_1\epsilon - \gamma_2 = 0,$$

or

$$\gamma_2\left(\frac{\theta}{\theta_R} - 1\right) - \eta\gamma_1\epsilon = 0.$$

We indicate by “+” the region where $Y > 0$ and by “-” the region where $Y < 0$. Notice that

$$\gamma_2 = \theta_R(s_R + \alpha)\left(\frac{f_R + \beta}{s_R + \alpha} - \sigma\right) > 0.$$

The Z component of the field vanishes at $\epsilon = 0$ and $\epsilon = 1$, hence these planes are field invariants. Figure 7, in the space (s, θ, ϵ) , shows the surface $S = S_1 \cup S_2 \cup S_3$ where X vanishes, the plane P shows where Y vanishes, and the planes $\epsilon = 0$ and $\epsilon = 1$ show where Z vanishes. This figure also shows that

the field has exactly six equilibrium points, the states I, II, \dots, VI . Since on each of these equilibrium points $\gamma_1 + \sigma s - f = 0$ holds, we have

$$X_s = \frac{\sigma - f_s}{H}$$

$$X_\theta = -\frac{f_\theta}{H}$$

$$X_\epsilon = Z_s = Z_\theta = 0$$

and

$$Z_\epsilon = \frac{1 - 2\epsilon}{\gamma_1} s Q.$$

Hence $Z_\epsilon > 0$ at $\epsilon = \epsilon_L = 1$ and $Z_\epsilon < 0$ at $\epsilon = \epsilon_R = 0$.

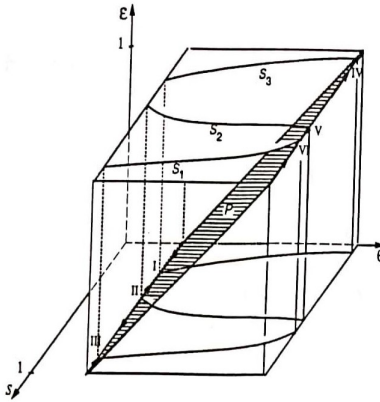


Fig. 7: Surface $S = S_1 \cup S_2 \cup S_3$ where $X = 0$, plane P where $Y = 0$, and the planes $\epsilon = 0$ and $\epsilon = 1$ where $Z = 0$.

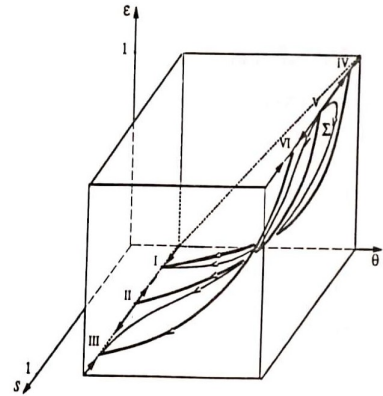


Fig. 8: Unstable manifold Σ for the equilibrium point V .

The Jacobian matrix of the (4.1)-(4.3) field at each of the equilibrium points is given by

$$M = \begin{pmatrix} \frac{\sigma - f_s}{H} & -\frac{f_\theta}{H} & 0 \\ 0 & \frac{v_T \gamma_2}{\gamma_\theta R} & -\frac{v_T \eta \gamma_1}{\gamma} \\ 0 & 0 & Z_\epsilon \end{pmatrix}.$$

The signs of the eigenvalues $(\sigma - f_s)/H$, $v_T\gamma_2/(\gamma\theta_R)$ and Z_ϵ of the matrix M at each of the equilibrium points are given in Table 1.

$I :$	$-$	$-$	$-$
$II :$	$+$	$-$	$-$
$III :$	$-$	$-$	$-$
$IV :$	$-$	$-$	$+$
$V :$	$+$	$-$	$+$
$VI :$	$-$	$-$	$+$

Table 1: Signs of the eigenvalues of the Jacobian M at the equilibrium points.

Lemma 4.2. *The only orbit which departs from $VI = (s_{VI}, \theta_{VI}, \epsilon = 1)$ enters III . Furthermore, this orbit is located underneath P , between the plane $s = s_{VI}$ and the surface S_1 .*

Proof: Let (a, b, c) be the eigenvector of the matrix M at VI associated to the eigenvalue Z_ϵ . Then

$$\frac{\sigma - f_s}{H}a - \frac{f_\theta}{H}b = Z_\epsilon a \quad (4.4)$$

and

$$\frac{v_T\gamma_2}{\gamma\theta_R}b - \frac{v_T\eta\gamma_1}{\gamma}c = Z_\epsilon b. \quad (4.5)$$

If $c = 0$ then $a = b = 0$. Hence $c \neq 0$, $b \neq 0$ and $a \neq 0$. We know that the only orbit which exits VI is tangent to (a, b, c) . From (4.5), we have

$$\frac{c}{b} = \frac{\gamma_2}{\theta_R\eta\gamma_1} - \frac{\gamma Z_\epsilon}{v_T\eta\gamma_1} > \frac{\gamma_2}{\theta_R\eta\gamma_1}. \quad (4.6)$$

Because the last term in (4.6) is the slope of the straight line defining P , we conclude that the orbit is initially located below P . Since no orbit may cross P upwards, the orbit has to lie below P .

On the other hand, from (4.4), we have

$$\frac{b}{a} = \frac{\sigma - f_s}{f_\theta} - \frac{HZ_\epsilon}{f_\theta} < \frac{\sigma - f_s}{f_\theta} < 0,$$

where $(\sigma - f_s)/f_\theta$ is the slope of the curve $\gamma_1 + \sigma s - f(s, \theta) = 0$, which defines S_1 . Then the orbit initially lies in the region between $s = s_{VI}$ and S_1 , and it

must stay there. Now we can easily see that the orbit has to enter *III*, and this completes the proof. \square

Lemma 4.3. *The only orbit exiting $IV = (s_{IV}, \theta_{IV}, \epsilon = 1)$ enters *I*. Furthermore, this orbit is located underneath *P*, between the plane $s = s_{IV}$ and the surface S_3 .*

We skip the proof since it is analogous to that of Lemma 4.3. Finally, we have the last lemma needed to complete Theorem 4.2.

Lemma 4.4. *There is an infinite number of orbits exiting *V* and entering *I*, all located below the plane *P*.*

Proof: As shown in Figure 8, let Σ be the two-dimensional manifold formed by the orbits exiting *V*. Let *T* be the tangent plane to Σ at *V*. The plane *T* is generated by $(1, 0, 0)$ and by the eigenvector associated to the eigenvalue Z_ϵ . Proceeding as in the proof of Lemma 4.2, we see that *T* is located below *P* in the region $0 < \epsilon < 1$. Now, if an orbit exits *V* with direction close enough to $(-1, 0, 0)$, then clearly it has to enter *I*, and this completes the proof. \square

We could make a much more complete analysis of the phase portrait of (4.1)-(4.3), studying the invariant planes $\epsilon = 0$ and $\epsilon = 1$ and the stable manifold of *II*, but such an analysis is not necessary in this work.

5. Conclusions

We have shown that it is possible to find the Rankine-Hugoniot relationship between burnt and unburnt states in a very simple model for combustion of two-phase flow in porous media. We were able to determine which pairs of states correspond to traveling waves. We hope to generalize this analysis to more realistic models in future work.

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