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SOME SINGULAR LIMIT PROBLEMS IN CONSERVATION LAWS

Gui-Qiang Chen*

Abstract

We are concerned with singular limits for conservation laws with dissipation and/or relaxation, particularly zero dissipation and relaxation limits. Such limits are of interest in many physical situations, including fluid with vanishing viscosity, gas flow near thermoequilibrium, kinetic theory with small mean free path, viscoelasticity with vanishing memory; and in numerical analysis such as the convergence and stability of shock capturing methods. In this paper we discuss some recent efforts made by the author and collaborators in this direction and analyze some important results obtained from these efforts.

1. Introduction

As is well known, many systems arising from applied sciences take the following conservation form

$$\partial_t u + \nabla \cdot f(u) = 0, \quad u \in \mathbf{R}^n, \tag{1.1}$$

for the most ideal cases, where the flux vector function $f(u)$ is smooth. The typical example is the compressible Euler equations for the gas flow in thermoequilibrium [17,87]. For the gas flow affected by the viscosity and heat conductivity [17,87] and by a large variation of the temperature [86,87], corresponding systems generally take the following conservation forms

$$\partial_t U^\epsilon + \nabla \cdot F(U^\epsilon) = \epsilon \nabla \cdot (D(U^\epsilon)\nabla U^\epsilon), \quad U^\epsilon \in \mathbf{R}^n, \tag{1.2}$$

and

$$\partial_t U^\epsilon + \nabla \cdot F(U^\epsilon) + \frac{1}{\epsilon} R(U^\epsilon) = 0, \quad U^\epsilon \in \mathbf{R}^N,\ N \geq n, \tag{1.3}$$

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respectively. For the form (1.2), $D(U)$ is $n \times n$ nonnegative matrix and $F(U)$ is expected to satisfy that $F(U^\epsilon(x,t))$ are close $f(u(x,t))$ in some sense whenever $U^\epsilon(x,t)$ are close to $u(x,t)$. For the case (1.3), the smooth functions $F(U)$ and $R(U)$ satisfy:

(1) There exists an $n \times N$ constant matrix Q with $Rank(Q) = n < N$ such that

$$QR(U) = 0; \tag{1.4}$$

(2) The equilibrium state U such that $R(U) = 0$ can be expressed by

$$U = \mathcal{E}(u), \quad u = QU \in R^n, \tag{1.5}$$

and

$$f(u) = QF(\mathcal{E}(u)). \tag{1.6}$$

The challenging **mathematical problem** is to understand

(1) Limiting behavior of $U^\epsilon(x,t)$ and $F(U^\epsilon(x,t))$ in appropriate sense of topology;

(2) Relationship between the limits of these two sequences and corresponding solution $u(x,t)$ and nonlinear flux function $f(u(x,t))$ of (1.1).

Its physical motivation is well known. For the case (1.2), the constant ϵ measures how strong the viscosity and heat conductivity are in fluid under consideration (cf. [17,87]). This problem relates to the relationship between the compressible Navier-Stokes equations and the compressible Euler equations. For the case (1.3), the system describes the nonequilibrium thermodynamical processes and ϵ measures how far the nonequilibrium state is away from the equilibrium (cf. [86,87]). If one solves such a problem, one understands the stability of the corresponding equilibrium state.

In numerical analysis, it is important to explore numerical regularizations to ensure good numerical approximations to hyperbolic conservation laws without using the apriori structure of solutions, so called shock capturing methods. This problem has motivated the formation of most existed shock capturing numerical methods such as the Lax-Friedrichs scheme, the Glimm scheme, the Godunov scheme, the Lax-Wendroff scheme, the upwind schemes, the kinetic schemes, and relaxing schemes (cf. [42,31,35,45,67,10,68,39,55,72,46]). Convergence and stability analysis of shock capturing methods is another motivation for this problem.

From the theory of partial differential equations, this problem involves the relationship among different types of nonlinear differential equations. The problem for the case (1.2) corresponds to that of the singular limit of nonlinear parabolic equations to hyperbolic or mixed hyperbolic-elliptic equations. The problem for the case (1.3) involves the singular limit of nonlinear integral partial differential equations to nonlinear partial differential equations as well as the limiting problem from nonlinear strictly hyperbolic equations to mixed hyperbolic-elliptic equations. For example, the following system

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t p + \Lambda^2 \partial_x v + \frac{p - \tilde{p}(u)}{\epsilon} = 0, \end{cases}$$

is semilinear strictly hyperbolic with eigenvalues $-\Lambda, 0, \Lambda$, which describes the phase transition [69], where the function $\tilde{p}(u)$ satisfies $\tilde{p}'(u) \leq 0$, when $u_1 \leq u \leq u_2$, and $\tilde{p}'(u) > 0$, when $u < u_1$, or $u > u_2 > u_1$. Its local system is

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x \tilde{p}(u) = 0, \end{cases}$$

that is a mixed hyperbolic-elliptic system.

A quasilinear system of conservation laws in one dimension

$$\partial_t U + \partial_x F(U) = 0, \quad U \in \mathbf{R}^N, \tag{1.7}$$

is hyperbolic if the $N \times N$ matrix $\nabla F(U)$ has N real eigenvalues $\lambda_j(U)$ and linearly independent eigenvectors $r_j(U), 1 \leq j \leq N$. Denote

$$\mathcal{D} \equiv \{U : \lambda_i(U) = \lambda_j(U), \quad i \neq j, \, 1 \leq i, j \leq N\},$$

as the degenerate set. If the set $\mathcal{D}$ is empty, then this system is strictly hyperbolic, otherwise it is nonstrictly hyperbolic. Such a set allows a degree of interaction, or nonlinear resonance, among different characteristic modes, which is missing in the strictly hyperbolic case, and causes an analytic difficulty due to the degeneracy. A point $U_* \in \mathcal{D}$ is hyperbolically degenerate if $\nabla F(U_*)$ is diagonalizable, otherwise it is parabolically degenerate.

Recent decades, nonstrictly hyperbolic systems of conservation laws have arisen from many important fields such as continuous mechanics including the vacuum problem, multiphase flows in porous media, MHD, and elasticity. On the other hand, the nonstrict hyperbolicity of systems is quite generic. For example, for three space dimensional systems of conservation laws, Lax [43] indicated that systems with $2(mod\, 4)$ equations must be nonstrictly hyperbolic. The result can be extended to systems with $\pm 2, \pm 3, \pm 4 (mod\, 8)$ equations [27]. Then the plane wave solutions of such systems are governed by one-dimensional hyperbolic systems with $\mathcal{D} \neq \emptyset$.

2. Zero Dissipation Limit

The zero dissipation limit includes the continuous version such as the vanishing viscosity limit for viscous approximate solutions and the discrete version such as the zero mesh length limit for shock capturing numerical approximate solutions.

2.1. Scalar Conservation Laws

Zero dissipation limit, or vanishing viscosity limit, in the continuous version for scalar conservation laws has been extensively studied (cf. see [43,80,46]). The behavior of this limit is simple. The convergence and stability of shock capturing methods with first-order accuracy have been analyzed in a large literature. Some efforts on convergence and entropy consistency of some shock capturing methods with high resolution have been made. For example, see [49,88,66,16] for the Godunov type schemes, [81] for the streamline diffusion finite element methods, and [82,7] for the viscosity spectral methods.

In the paper [12], we are concerned with the convergence of Lax-Wendroff type schemes with high resolution. These schemes include the original Lax-Wendroff scheme proposed by Lax and Wendroff in 1960 [45], and its two step versions–the Richtmyer scheme [72] and the MacCormack scheme [55]. As observed by Harten-Hyman-Lax [36] and Majda-Osher [56,57], the second-order numerical viscosity in this scheme is essential to guarantee that the numerical solutions are nonlinear stable and converge to the physical solutions. For the convex scalar conservation laws with algebraic growth flux function, we proved the convergence of these schemes to the weak solution satisfying appropriate entropy inequalities as the second-order numerical viscosity and the mesh length vanish. The proof is based on detailed L^p estimates of the approximate solutions, H^{-1} compactness estimates of the corresponding entropy dissipation measures, and a compensated compactness framework. These techniques are generalized to study the convergence problem for the nonconvex scalar case and for hyperbolic systems of conservation laws.

2.2. 2×2 Hyperbolic Systems

The limiting behavior of the zero dissipation limit for hyperbolic systems of conservation laws is much more complicated.

2.2.1. Strictly Hyperbolic Case

For strictly hyperbolic and genuinely nonlinear systems of conservation laws with C^2 flux functions: $F \in C^2$ and $\nabla\lambda_j(U) \cdot r_j(U) \neq 0, U \in \mathbf{R}^N$, $1 \leq j \leq N$. DiPerna [23] proved that any uniformly bounded viscous solution sequence must be compact in L^∞ no matter whether corresponding approximate initial data sequence is compact or not. This theorem was extended to the linear degenerate case by Serre [78]. An alternative proof of DiPerna's theorem was given by Morawetz [60]. The uniform $L^p, 1 < p \leq \infty$, estimates for viscosity approximate solutions were discussed by Dafermos [19]. Also see [48] for the convergence of uniformly L^p bounded viscous approximate solutions for the system of elasticity.

If the flux function F is not C^2 or the genuine nonlinearity fails, the situation is different. Greenberg and Rascle [34] showed by an example that the loss of C^2 smoothness for the flux function can produce oscillation. For the linearly degenerate case, the initial oscillation will propagate along the linearly degenerate characteristic fields (cf. [4,78,70]). In these cases, one cannot expect the convergence of viscous approximate solutions. The next issue is whether the degenerate set $\mathcal{D}$ affects the limiting behavior of dissipative approximate solutions to conservation laws when the dissipation disappears. To make my points more precisely, I will focus on two of the most important systems in this area. The first one is the system of isentropic gas dynamics, which is of parabolic degeneracy.

2.2.2. Parabolically Degenerate Case

The system of isentropic Euler equations are of the form

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x(\frac{m^2}{\rho} + p(\rho)) = 0, \end{cases} \tag{2.1}$$

where ρ, m, and p are the density, the mass, and the pressure, respectively. For the polytropic gas, $p(\rho) = k^2\rho^\gamma$, where γ is the adiabatic exponent and $1 < \gamma \leq 5/3$ for usual gases. In this case, the flux function is not C^2.

One can check that $\mathcal{D} = \{\rho = 0\}$, which is the vacuum state, and the degeneracy is parabolic. An interesting mathematical issue is whether the dissipation limit is still stable near the vacuum.

Since this system is the most typical one in fluid dynamics, the study of this system has an extensive history dating back to the work of Riemann [73], where a special Cauchy problem, so-called Riemann problem, was solved. Zhang and Guo [89] established an existence theorem of the global solutions to this system with a class of initial data by using the characteristic method. Nishida [63] obtained the first large data existence theorem with locally finite total variation for the case $\gamma = 1$ using Glimm's scheme [31]. Large-data theorems have also been obtained for general $\gamma > 1$ in the case where the initial data with locally finite total variation is restricted to prevent the development of cavities (e.g. [22,47,64]). The difficult point in bounding total variation norm at low densities is that the coupling between characteristic fields increases as the density decreases. This difficulty is a reflection of the fact that the strict hyperbolicity fails at the vacuum: $\mathcal{D} \neq \emptyset$. Using the compensated compactness ideas developed by Tartar and Murat [84,62], DiPerna [24] established a global existence theorem with L^∞ bounded data for $\gamma = 1 + \frac{2}{2m+1}, m \geq 2$ integers, with the aid of the viscosity method.

In the papers [20], we first succeeded in proving the convergence of the Lax-Friedrichs scheme for this system in the case $\gamma = 3/2$ with the aid of compensated compactness. Then we established a compensated compactness framework for approximate solutions to the Euler equations for the general case $1 < \gamma \leq 5/3$ (cf. [2,3]). Uniform boundedness of the approximate solutions coupling with H^{-1} compactness of corresponding entropy dissipation measures implies the compactness of the approximate solutions in L^∞. This means that the degenerate set $\mathcal{D}$ and the loss of C^2 differentiability of the flux function do not affect the compactness of approximate solutions for this case. The proof is based on a careful analysis of weak entropy and a detailed study of regularities of the family of Young measures $\nu_{x,t}$ that is both determined by the approximate solutions and restricted by a commutativity relation derived from an analysis of the weak entropy fields and a basic continuity theorem for the 2×2 determinant in the weak topology. Then this compactness framework is applied to prove the convergence of the Lax-Friedrichs scheme, the Godunov scheme, the viscous approximate solutions, the MUSCL type schemes, the entropy flux-splitting schemes to this system (cf. [3,21,13,10]). Recently, this framework has been applied to the initial-boundary problem for this system [83], the Euler equations coupled with the Poisson equation that model the hydrodynamic behavior of semiconductors [90,59], and the global solutions of the compressible Euler equations with geometrical structure including transonic nozzle flow, cylindrically symmetric flow, spherically symmetric flow, and symmetric rotating flow [8].

2.2.3. Hyperbolically Degenerate Case

The typical example with hyperbolic degeneracy is the gradient quadratic flux system, which is umbilic degenerate, the most singular case:

$$\partial_t U + \partial_x(\nabla C(U)) = 0, \quad U = (u,v)^{\top} \in \mathbf{R}^2, \tag{2.2}$$

where

$$C(U) = \frac{1}{3}au^3 + bu^2v + uv^2, \tag{2.3}$$

and a and b are two real parameters.

Such systems are quite generic in the following sense. For any smooth nonlinear flux function, take its Taylor expansion about the isolated umibilic point. The first three terms including the quadratic terms determine the local behavior of the hyperbolic singularity near the umbilic point. The hyperbolic degeneracy enables us to eliminate the linear term by a coordinate transformation to obtain the systems with a homogeneous quadratic polynomial flux. Such a polynomial flux contains some inessential scaling parameters. There is a nonsingular linear coordinate transformation to transform the above system into (2.2)-(2.3), first studied by Marchesin, Isaacson, Plohr, and Temple, and in a more satisfactory form by Schaeffer and Shearer [75]. From the viewpoint of group theory, such a reduction from six to two parameters is natural: for the six dimensional space of quadratic mappings acted by the four dimensional group $GL(2,\mathbf{R})$, one expects the generic orbit to have codimension two.

The Riemann solutions for such systems were discussed by Isaacson, Marchesin, Paes-Leme, Plohr, Schaeffer, Shearer, Temple, and others (cf. [37,38,76,77]). Two new shock waves, the overcompressive shock wave and the undercompressive shock wave, were discovered that are quite different from gas dynamical shock. The overcompressive shock can be easily understood by the Lax entropy condition [43]. It is known that there is a traveling wave solution connecting the left state and the right state of the undercompressive shock for the artificial viscosity system. Stability of such traveling waves for the overcompressive shock and the undercompressive shock has been studied ([52,53]). The next issues are whether the compactness of the corresponding approximate solutions is affected by the viscosity matrix as the viscosity parameter goes to zero, to understand the sensitivity of the undercompressive shock wave with respect to the viscosity matrix, and whether corresponding global existence theorem of entropy weak solutions can be established.

The global existence of weak solutions to the Cauchy problem for a special case of such quadratic flux systems was solved by Kan [40] using the viscosity method. A different proof was given independently to the same problem by Lu [54].

In the papers [9], P.-T. Kan and I established an L^∞ compactness framework for sequences of approximate solutions to general hyperbolic systems with umbilic degeneracy specially including (2.2)-(2.3). Under this framework, approximate solution sequences, which are apriori bounded in L^∞ and produce correct entropy dissipations, lead to the compactness of the corresponding Riemann invariant sequences. This means that the viscosity matrix does not affect the compactness of the corresponding uniformly bounded Riemann invariant sequences. One of the principal difficulties associated with such systems is the general lack of enough classes of entropy functions that can be verified to satisfy certain weak compactness conditions. This is due to possible singularities of entropy functions near the regions of nonstrict hyperbolicity. The analysis leading to the compactness involves two steps:

In the first step, we constructed regular entropy functions governed by a highly singular entropy equation. There are two main difficulties. The first is that, in general, the coefficients of the entropy equation are multiple-valued functions near the umbilic points in the Riemann invariant coordinates. This difficulty is overcome by a detailed analysis of the singularities of the Riemann function of the entropy equation. This analysis involves a study of a corresponding Euler-Poisson-Darboux equation and requires very complicated estimates and calculations. An appropriate choice of Goursat data leads to a cancellation of singularities and to the realization to regular entropies in the Riemann invariant coordinates. The second difficulty is that the nonlinear corresponding between the physical state coordinates and the Riemann invariant coordinates is, in general, irregular. A regular entropy function in the Riemann invariant coordinates is usually no longer regular in the physical coordinates. We overcame this by a detailed analysis of the correspondence between these two coordinates.

In the second step, we studied the structure of the Young measures associated with the approximate sequences, and proved that the support of the Young measures lies in finite isolated points or separate lines in the Riemann invariant coordinates. This is achieved by a delicate use of Serre's technique [78] and regular entropy functions, constructed in the first step, in the Tartar-Murat commutation equation [84] for the Young measures associated with the approximate solution sequences.

This compactness framework is successfully applied to prove the convergence of the Lax-Friedrichs scheme, the Godunov scheme, and the viscosity method for the quadratic flux systems. Some corresponding existence theorems of global entropy solutions for such systems are established. The compactness is achieved by reducing the support of the corresponding Young measures to a Dirac mass in the physical space.

Finally we also refer the reader to [28,29,74] for several other important nonlinear systems endowed with similar features.

2.3. Hyperbolic Systems with More Than Two Equations

One of the most typical systems with more than two equations is the system of thermoelasticity, which describes the balance laws of mass, momentum, and energy for one-dimensional elastic media. When the initial data have small total variation, a global BV solution to this system can be constructed by the random choice method of Glimm [31]. The issue of existence of solutions with large initial data is a long-standing open problem. Recently, Dafermos and I [5] succeeded in constructing large solutions by the method of "vanishing viscosity" in which the elastic medium is visualized as the zero viscous limit of a family of viscoelastic media with viscosities for a class of constitutive relations. Oscillations in the entropy field may propagate along the linearly degenerate characteristic field but do not affect the compactness of the velocity field or the pressure field in the viscous approximate solutions as the viscosity coefficient goes to zero.

Some related results (cf. [4,79]) have been established for a class of systems, so called the rich systems (cf. [79]).

2.4. Remarks on Multidimensional Cases

The multidimensional hyperbolic conservation laws are much more complicated. An essential feature in the multidimensional case is degeneracy, the loss of strict hyperbolicity and/or genuine nonlinearity (cf. [44,27,1]). Another distinguishing feature in this case is the rollup of the compressible vortex sheet through the nonlinear interaction of the kink mode and the development of the vorticity in the compressible vortex sheet, which is quite different from the one-dimensional case. The study of the behavior of the zero dissipation limit of multidimensional conservation laws is one of the major challenging problems in the field of nonlinear analysis.

One of the most typical problems in this context is the compressible Euler equations with geometrical structure. Physical examples of this systems include transonic nozzle flow, cylindrically symmetric flow, spherically symmetric flow, and symmetric rotating flow. The similarity solutions to the Euler equations with spherical symmetry were studied by Guderley, von Neumann, Taylor, Sedov, and many others (see [17,87]). The existence of global solutions was discussed by Liu [50] provided that the initial data are close to a constant state and are away from the sonic. Some numerical calculations were performed and analyzed by Embid-Goodman-Majda, Glimm-Marshall-Plohr, and Glaz-Liu [25,32,30]. A global entropy weak solution with spherical symmetry was constructed in [58] for the isothermal case $\gamma = 1$. In [8] a shock capturing numerical scheme is introduced to compute the isentropic flows and to construct dissipative approximate solutions. The convergence and consistency of the dissipative approximate solutions generated from this scheme are proved, which indicate that no oscillation produces in this zero dissipation limit. A corresponding existence theory of global solutions is established and then this theory is applied to the transonic flow, the cylindrically symmetric flow, the spherically symmetric flow, and the symmetric rotating flow.

At this stage, it would be interesting to understand the behavior of the zero dissipation limit for the two-dimensional steady systems of compressible Euler equations and for the incompressible Euler equations because of the appearance of the vorticity in the compressible vortex sheet.

3. Zero Relaxation Limit

The zero relaxation limit is much more complicated. In general, this limit is not stable even for the linear case. For example, consider the following relaxing system:

$$\begin{aligned} &\partial_t u + \partial_x v = 0, \\ &\partial_t v + \Lambda^2 \partial_x u + \tfrac{v-\lambda u}{\epsilon} = 0, \end{aligned} \tag{3.1}$$

where λ and $\Lambda > 0$ are constants. From (3.1), we conclude that u satisfies the equation:

$$\partial_t u + \lambda \partial_x u + \epsilon(\partial_{tt} u - \Lambda^2 \partial_{xx} u) = 0. \tag{3.2}$$

This is the first order operator perturbated by the wave operator. For the equation (3.2), it is well known from the linear theory that the limit is stable as ϵ goes to zero if and only if λ and Λ satisfy

$$-\Lambda < \lambda < \Lambda. \tag{3.3}$$

This means that the characteristic speed λ of the local equation

$$\partial_t u + \lambda \partial_x u = 0$$

is interlaced with the characteristic speeds of the relaxing system (3.1). The same condition is true for 2×2 quasilinear hyperbolic systems of conservation laws with stiff relaxation terms to ensure that the local relaxation approximation is dissipative [51].

In the paper [11], Levermore, Liu, and I studied the limiting behavior of general hyperbolic systems of conservation laws with stiff relaxation terms. The local relaxation approximation and its first correction for a general system of hyperbolic conservation laws with the relaxation terms (1.3)-(1.5) are constructed. A general notion of entropy is introduced for such systems, the existence of which ensures the hyperbolicity of the local relaxation approximation and the dissipativity of its first correct. For general 2×2 strictly hyperbolic systems, the existence of the dissipative entropies is implied by a strictly stability criterion that the equilibrium characteristic lies between the frozen characteristics as pointed in (3.3). This stability theory is then applied to study the convergence to the reduced dynamics for the 2×2 case.

In the zero relaxation limit, the solutions of the relaxing systems tend to those of the local relaxation approximation. The limit is highly singular because of shock and initial layers. In [14] this limit is studied for the physical models in elasticity and phase transition. Entropy pairs are constructed to derive the energy estimates, and the compensated compactness method is then applied to control the oscillations. The stability theory allows to extend to general 2×2 systems and more physical systems in mechanics [11]. A natural application of such a stability theory to the construction of relaxing schemes can be found in [39].

Another related limit is the weakly nonlinear limit. This limit shares the feature of the local relaxation approximation that it does not contain the relaxation time. It is based on the observation that the linearization of the local relaxation approximation about an equilibrium gives a simple advection dynamics with the equilibrium characteristic speed. This suggests that for solutions of the relaxing system (1.3)-(1.5), a small perturbation about an equilibrium will be slowly varying in the corresponding moving frame. This limit is derived for 2×2 systems and is justified through energy estimates in [11].

4. Conservation Laws with Memory

One important class of systems in conservation laws is hyperbolic systems of conservation laws with memory: the solution at each state point depends not only on the present value, but on the entire temporal history, so called nonlocal systems. Typical examples of such systems come from viscoelasticity (cf. [71]) for certain viscoelastic materials such as polymers, suspensions, and emulsions which have memory: the stress at each materials point depends not only on the present value of the deformation gradient and/or velocity gradient, but on the entire temporal history of motion. These materials exhibit behavior intermediate between that of an elastic solid and a viscous fluid.

It would be interesting to study the behavior of the zero dissipation limit for conservation laws with memory. For this reason, we refer the reader to Dafermos [18], Nohel-Rogers-Tzavaras [65], and Chen-Dafermos [6]. Another interesting problem is the vanishing memory limit, which corresponds to the zero relaxation limit for some ideal cases. In this connection, we refer the reader to [14,11] for the zero relaxation limit.

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Departament of Mathematics
The University of Chicago
5734 S. University Avenue
Chicago, Illinois 60637
USA