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GLOBAL BEHAVIOR FOR SOME CONSERVATIVE NONLINEAR EQUATIONS

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1. Introduction.

In this paper we describe some results on global properties of solutions to conservative nonlinear wave equations. One of the most natural examples of this type of equations is the cubic one-dimensional wave equation with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} - u_{xx} + u^3 = 0, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.1}$$

in $\Omega = (0, \pi)$. It is well known that the initial value problem for (1.1) is well posed in $H_0^1(\Omega) \times L^2(\Omega)$. Moreover, due to the conservation of the energy

$$\mathcal{E}(u, u_t) := \int_\Omega \left\{ \frac{1}{2} u_t(t,x)^2 + \frac{1}{2} u_x(t,x)^2 + \frac{1}{4} u(t,x)^4 \right\} dx,$$

all solutions are global in time and bounded in the energy space $H_0^1(\Omega) \times L^2(\Omega)$ (see for example [16]). In particular, it makes sense to study their asymptotic behavior as $t \to \infty$.

Equation (1.1) is an infinite dimensional Hamiltonian system. For finite dimensional Hamiltonian systems, Poincaré's recurrence theorem says that almost all solutions are recurrent. For completely integrable Hamiltonian systems, a classical result of Liouville (see for example Arnold [1]) asserts that most solutions are quasi-periodic. For the (linear) wave equation $u_{tt} - \Delta u = 0$ with Dirichlet boundary conditions in a bounded domain, all solutions are almost periodic. It is natural to wonder to what extent this property persists with the addition of a nonlinear term. A well-known result of Rabinowitz [21] (see also [2]) says that the equation (1.1) has nontrivial time-periodic solutions. However, there are very few results concerning the global properties of a generic solution (see [6,7]).

In [8], we presented the results of numerical computations which we used to test some conjectures on the global behavior of solutions. In particular, we tested the recurrence property, the relative compactness of the trajectories in the energy space and, in connection with the last property, the possible existence of nontrivial solutions converging weakly to 0 as $t \to \infty$. These numerical experiments suggest an almost periodic behavior, but up to now there is no definite result in this direction. Note also that the almost periodic behavior is supported by a few results concerning other nonlinear conservative equations. In particular, Lax [18] showed the existence of a large class of quasi-periodic solutions to the KdV equation, and Cabannes and Haraux [4] showed the existence of a large family of almost periodic solutions to the equation of a string with fixed ends vibrating against a straigh fixed obstacle.

In [9,10,11], we studied the equation

$$\begin{cases} u_{tt} - u_{xx} + u \left(\int_\Omega u(t,x)^2 \, dx \right) = 0, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.2}$$

which can be considered as an approximation of the equation (1.1). The initial value problem is well posed in $H_0^1(\Omega) \times L^2(\Omega)$, and there is conservation of the energy

$$\mathcal{E}(u, u_t) := \int_\Omega \left\{ \frac{1}{2} u_t(t,x)^2 + \frac{1}{2} u_x(t,x)^2 \right\} dx + \frac{1}{4} \left(\int_\Omega u(t,x)^2 \, dx \right)^2.$$

(See [10].) In particular, all solutions are globally defined and bounded in the energy space. Moreover, equation (1.2) has some interesting properties that are not shared by (1.1). In particular, if one expands the solution in a Fourier sine series, the coefficients u_j satisfy the following infinite system of ODE's

$$u_j'' + \lambda_j u_j + \left(\sum_{i \geq 1} u_i^2 \right) u_j = 0, \tag{1.4}$$

where the λ_j's are the eigenvalues of the Laplacian in $H_0^1(\Omega)$, i.e. $\lambda_j = j^2$. The energy for (1.4) has the form

$$E(U,V) := \frac{1}{2}\sum_{j\geq 1} v_j^2 + \frac{1}{2}\sum_{j\geq 1}\lambda_j u_j^2 + \frac{1}{4}\left(\sum_{j\geq 1} u_j^2\right)^2,$$

where $U = (u_j)_{j\geq 1}$ and $V = (v_j)_{j\geq 1}$. Furthermore, the solutions to the central force motion

$$u_{tt} + u\left(\int_\Omega u(t,x)^2\,dx\right) = 0,$$

are quasi-periodic with two basic frequencies, so that (1.2) corresponds to the superposition of two operators which both produce almost periodic solutions.

Equation (1.2) has solutions with only a finite number of nonzero Fourier modes. More precisely, if $u(0)$ and $u_t(0)$ are in the space spanned by k eigenfunctions, then $u(t)$ remains in that space for all $t \in \mathbf{R}$. In particular, any solution to a finite sub-system of (1.4) is an actual solution of (1.2). Solutions with only one excited Fourier mode ℓ are called simple modes. They are solutions of the form $u(t,x) = f(t)\varphi_\ell(x)$, where f solves the equation $f'' + \lambda_\ell f + f^3 = 0$, and in particular u is time periodic. Earlier in our research, we studied numerically solutions with two Fourier modes j, ℓ, i.e. solutions of the system

$$\begin{cases} u'' + u + (u^2+v^2)u = 0, \\ v'' + kv + (u^2+v^2)v = 0, \end{cases} \tag{1.5}$$

with $k = \lambda_\ell/\lambda_j = \ell^2/j^2$, and we observed nonrecurrent solutions. More precisely, we found solutions such that $v(t) \to 0$ as $t \to \infty$. Indeed, we were later able to rigorously establish the existence of nonrecurrent solutions.

It turns out that the system (1.4) is completely integrable, due to the existence of an infinite family of conservation laws $(E_j)_{j\geq 1}$ given by

$$E_j(U,V) = v_j^2 + \lambda_j u_j^2 + \frac{1}{2}u_j^2\sum_{i\geq 1} u_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{(u_i v_j - v_i u_j)^2}{\lambda_i - \lambda_j}. \tag{1.6}$$

By using the conservation laws (1.6), one can describe the global behavior for a wide class of solutions to (1.2).

Some of the properties of equation (1.2) persist for certain non-integrable equations, such as

$$\begin{cases} u_{tt} - u_{xx} + u\left(\int_\Omega u(t,x)^2\,dx\right)^\alpha = 0, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.7}$$

with $\alpha \neq 1$, and the nonlinear string equation

$$\begin{cases} u_{tt} - \left(a + b\int_\Omega u_x^2(t,x)\,dx\right)u_{xx} = 0, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.8}$$

with $a \geq 0$, $b > 0$.

The paper is organized as follows. In Section 2, we describe the quasi-periodicity results for the equation (1.2). Section 3 is devoted to the nonrecurrence results for the same equation, and Section 4 is devoted to the nonrecurrence results for the non-integrable equations (1.7) and (1.8). Finally, Section 5 is devoted to further comments.

2. Quasi-periodicity Results for the Modified Cubic Equation.

We observe that for equations which, like equation (1.4), have solutions with only a finite number of nonzero Fourier modes, we can apply Poincaré's recurrence theorem to any finite (say N dimensional) sub-system, so that for almost all (with respect to the Lebesgue measure in $\mathbf{R}^{2N}$) initial values, the solutions are recurrent. It follows that for the infinite dimensional system, there is a large number (in fact a dense subset in a relevant space) of initial values giving rise to recurrent solutions (see Corollary 5.2 in [10]). We recall that a bounded, continuous function u from $\mathbf{R}$ to some metric space Z is recurrent if for every $t \in \mathbf{R}$ there exists two sequences $\alpha_n \to +\infty$ and $\beta_n \to -\infty$ such that

$$u(t) = \lim_{n\to\infty} u(\alpha_n) = \lim_{n\to\infty} u(\beta_n).$$

In particular, almost periodic functions are recurrent.

Before stating the main quasi-periodicity result of this section, we introduce some notation. We set

$$\mathcal{H} = \{u \in H_0^1(\Omega);\ u \text{ has finitely many nonzero Fourier components}\},$$

so that $\mathcal{H}$ is a dense subset of both $H_0^1(\Omega)$ and $L^2(\Omega)$. For $u, v \in \mathcal{H}$ and $j \geq 1$, we set

$$\mathcal{E}_j(u,v) = E_j\left((u_i)_{i\geq 1}, (v_i)_{i\geq 1}\right),$$

where u_i is the i^{th} Fourier coefficient of u and where E_j is defined by (1.6). We note that $\mathcal{E}_j(u,v) = 0$ except for a finite number of j's. Our main result of this section is the following (see Theorem 6.2 in [10]).

Theorem 2.1. *Given $(u_0, v_0) \in \mathcal{H}$, let $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega)) \cap C^2(\mathbf{R}, H^{-1}(\Omega))$ be the solution of (1.2) such that $u(0) = u_0$ and $u_t(0) = v_0$. If $\mathcal{E}_j(u_0, v_0) > 0$ for all $j \geq 1$ such that $(u_0)_j^2 + (v_0)_j^2 \neq 0$, then u is quasi-periodic.*

The proof is based on Liouville's theorem. One first shows that the E_j's are pairwise in involution, i.e.

$$[E_j, E_\ell] := \sum_{k\geq 1}\left\{\frac{\partial E_j}{\partial u_k}\frac{\partial E_\ell}{\partial v_k} - \frac{\partial E_j}{\partial v_k}\frac{\partial E_\ell}{\partial u_k}\right\} = 0,$$

for all $j, \ell \geq 1$. The positivity assumption on $\mathcal{E}_j$ implies that on the set $\bigcap_{j\geq 1}\{\mathcal{E}_j(u,v) = \mathcal{E}_j(u_0,v_0)\}$, the gradients of the functionals $\mathcal{E}_j$ are linearly independent for all j's such that $(u_0)_j^2 + (v_0)_j^2 \neq 0$.

Corollary 2.2. *There exists $\varepsilon > 0$ such that if $(u_0, v_0) \in \mathcal{H}$ verifies $\mathcal{E}(u_0, v_0) < \varepsilon$ with $\mathcal{E}$ defined by (1.3) then the solution $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega)) \cap C^2(\mathbf{R}, H^{-1}(\Omega))$ of (1.2) such that $u(0) = u_0$ and $u_t(0) = v_0$ is quasi-periodic.*

Corollary 2.3. *Let $(u_0, v_0) \in \mathcal{H}$ and $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega)) \cap C^2(\mathbf{R}, H^{-1}(\Omega))$ be the solution of (1.2) such that $u(0) = u_0$ and $u_t(0) = v_0$. If u_0 and v_0 are linearly dependent, then u is quasi-periodic. In particular, if $u_0 = 0$ or if $v_0 = 0$, then u is quasi-periodic*

Indeed, both the small energy assumption in Corollary 2.2 and the linear dependence assumption in Corollary 2.3 imply the positivity condition of Theorem 2.1. (See Theorems 6.3 and 6.9 in [10].)

Remark 2.4. The above results show the existence of a wide class of quasi-periodic solutions of equation (1.2). The conservation laws can also be used to show some results that are valid for all solutions of (1.2). For example, if $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega)) \cap C^2(\mathbf{R}, H^{-1}(\Omega))$ is a solution of (1.2), then

$$\bigcup_{t \in \mathbf{R}} (u(t), u_t(t))$$

is a relatively compact subset of $H_0^1(\Omega) \times L^2(\Omega)$ (see Theorem 4.5 in [10]). This implies that no nontrivial solution of (1.2) can converge weakly to 0 as $t \to \infty$. (See Theorem 4.1 and Remark 4.2 in [10].)

Remark 2.5. These results can be extended to more general equations of the form

$$u'' + Lu + ||u||^2 u = 0,$$

in a Hilbert space X, where $L \geq 0$ is a self-adjoint operator with a complete set of eigenvectors, under various assumptions on the eigenvalues. In particular, the analogue of Corollary 2.3 holds without any assumption on the eigenvalues (see Theorem 6.9 in [10]).

Remark 2.6. For the system (1.5), one can give a more detailed description of the solutions. (1.5) is the Hamiltonian system associated with the energy

$$E(u, v, u', v') = \frac{1}{2}u'^2 + \frac{1}{2}v'^2 + \frac{1}{2}u^2 + \frac{k}{2}v^2 + \frac{1}{4}(u^2 + v^2)^2. \tag{2.1}$$

Furthermore, (1.5) has a second conservation law $F(u, v, u', v')$ given by

$$F(u, v, u', v') = -\frac{(uv' - u'v)^2}{2(k-1)} + v'^2 + kv^2 + \frac{1}{2}v^2(u^2 + v^2). \tag{2.2}$$

If $k > 1$ and if (u, v) is a solution of (1.5) such that $F(u, v, u', v') \neq 0$, then (u, v) is quasi-periodic with at most two basic frequencies. To prove this, it suffices to show the independence condition on ∇E and ∇F. In fact, given a set $A = \{E = a\} \cap \{F = b\}$ with $b \neq 0$, one shows that either ∇E and ∇F are linearly dependent at every point of A, or else A is the union of two curves; so that in the last case all solutions of (1.5) in A are periodic. (See Theorem 2.1 in [11].)

3. Nonrecurrent solutions for the modified cubic equation.

In this section, we consider the system (1.5), which is a particular case of the system (1.4), hence of equation (1.2). In other words, a solution (u, v) of (1.5) gives an exact solution of (1.2) of the form

$$ju(jt)\varphi_j(x) + jv(jt)\varphi_\ell(x),$$

if $k = \ell^2/j^2$ and $(\varphi_j)_{j\geq 1}$ are the normalized eigenfunctions of $-\partial_{xx}$ in $H_0^1(\Omega)$. Throughout this section, we assume $k > 1$. The system (1.5) has the two conservation laws (2.1) and (2.2), and it follows from Remark 2.6 above that all solutions such that $F(u, v, u', v') \neq 0$ are quasi-periodic. Therefore, the possible nonrecurrent solutions are those for which $F(u, v, u', v') = 0$. Our main result is the following (see Theorems 2.2 and 2.3 in [11]).

Theorem 3.1. *There exists $E_0 > 0$ such that if (u, v) is a solution of system (1.5) with $F(u, v, u', v') = 0$ and $E(u, v, u', v') \geq E_0$, then v and v' tend exponentially to 0 as $t \to +\infty$ and as $t \to -\infty$. Moreover, $u(t)$ is exponentially asymptotic as $t \to +\infty$ to one of the solutions of $w'' + w + w^3 = 0$ such that $\frac{1}{2}w'^2 + \frac{1}{2}w^2 + \frac{1}{4}w^4 = E(u, v, u', v')$, and u' is asymptotic to w'. Furthermore, similar conclusions hold as $t \to -\infty$ with w replaced by $w(\cdot + \tau)$ for some $\tau \in \mathbf{R}$.*

Remark 3.2. Note that Theorem 3.1 concerns a three parameter family of solutions. For most of these solutions, we have $v \not\equiv 0$, so that these solutions are in general nonrecurrent (and in particular not quasi-periodic). On the other hand, it follows from Poincaré's recurrence theorem that the set of nonrecurrent states for (1.5) has measure zero in $\mathbf{R}^4$. Therefore, the counterexample of Theorem 3.1 is in some sense optimal.

Remark 3.3. With the notation of Theorem 3.1, $w(\cdot)$ and $w(\cdot + \tau)$ have the same orbit in phase space. Therefore, Theorem 3.1 shows the existence of homoclinic orbits of (1.5).

Sketch of the proof of Theorem 3.1. We observe that if (u, v) is a solution of (1.5), then both u and v must vanish on every interval of length π. Given a solution (u, v) as in the statement of the theorem, if $v(t_0) = 0$ and $v \not\equiv 0$, then it follows from the property $F = 0$ that $u(t_0)^2 = 2(k-1)$ and that $u'(t_0)^2 + v'(t_0)^2 = 2E(u, v, u', v') - 2k(k-1) > 0$. Therefore, up to a translation, and because of the symmetries of the system, we may assume that $u(0) = \sqrt{2(k-1)}$, $v(0) = 0$, $v'(0) > 0$.

Setting

$$a = 2E(u, v, u', v') - 2k(k-1) > 0, \tag{3.1}$$

we define the mapping $\mathbf{T} : (0, \pi) \to (0, \pi)$ as follows. For $0 < \theta < \pi$, let (f, g) be the solution of system (1.5) such that $f(0) = \sqrt{2(k-1)}$, $g(0) = 0$, $f'(0) = a \cos\theta$ and $g'(0) = a \sin\theta$, and let τ be the first positive zero of g. It follows from what precedes that there exists $\varphi \in (0, \pi)$ such that $f'(\tau) = -a\cos\varphi$ and $g'(\tau) = -a\sin\varphi$. We define $\mathbf{T}(\theta) = \varphi$. It follows easily that $\mathbf{T}$ is a bijective, bicontinuous and monotone map $(0, \pi) \to (0, \pi)$. Moreover, we claim that $\mathbf{T}$ is increasing and that if a is large enough, then

$$\mathbf{T}(\theta) < \theta \text{ for all } 0 < \theta < \pi, \tag{3.2}$$

$$0 < \mathbf{T}'(0) = 2 - \mathbf{T}'(\pi) < 1. \tag{3.3}$$

If $(t_n)_{n \geq 0}$ denotes the increasing sequence of nonnegative zeroes of v, then $v'(t_n) = (-1)^n a \sin(\mathbf{T}^n(\theta_0))$, where $\theta_0 \in (0, \pi)$ is defined by $v'(0) = a \sin\theta_0$. Assuming (3.2) and (3.3), it follows that there exists $\eta \in (0, 1)$ such that $|v'(t_n)| \leq C\eta^n$. It is not difficult to deduce the exponential decay of v, then the exponential convergence of u as $t \to +\infty$. The behavior as $t \to -\infty$ is obtain similarly by considering the decreasing sequence of nonpositive zeroes of v.

The proof of (3.2) and (3.3) is made in two steps. One proves that $\mathbf{T}$ is increasing and satisfies (3.3) by letting $\theta \downarrow 0$ and studying the linearized system

$$\begin{cases} w'' + w + w^3 = 0, \\ z'' + kz + w^2 z = 0. \end{cases}$$

(3.2) follows from (3.3) and the property $\mathbf{T}(\theta) \neq \theta$ for all $0 < \theta < \pi$. This last fact is proved by studying the limiting system

$$\begin{cases} w'' + w^3 = 0, \\ z'' + w^2 z = 0, \end{cases}$$

obtained by letting $a \to \infty$, with a suitable rescaling.

Remark 3.4. The large energy requirement in Theorem 3.1 is needed for the proof of (3.2) which is made by contradiction, letting the energy go to infinity. We made some numerical experiments that indicate that the large energy requirement is possibly unnecessary. Note, however, that if (u, v) is a solution of (1.5) such that $F(u, v, u', v') = 0$ and $v \not\equiv 0$, then necessarily $E(u, v, u', v') > 2k(k-1)$ by (3.1). On the other hand, property (3.3) is proved without any condition on a (i.e. on the energy), and this property has an interesting consequence. Let $E_0 > 2k(k-1)$ and w be a solution of the equation

$$w'' + w + w^3 = 0,$$

such that

$$\frac{1}{2}w'^2 + \frac{1}{2}w^2 + \frac{1}{4}w^4 = E_0.$$

w is periodic and unique up to a translation, so that without loss of generality we may assume that $w(0) = 0$ and $w'(0) > 0$. Using property (3.3) and the argument of the proof of Theorem 3.1, one can show that there exists $0 < \theta_0 < \pi$ and $\alpha > 0$ with the following property. If (u, v) is the solution of (1.5) such that $u(0) = b$, $v(0) = 0$, $u'(0) = a\cos\theta$ and $v'(0) = a\sin\theta$ and if $0 < \theta \leq \theta_0$, then there exist σ and C such that $v(t)^2 + v'(t)^2 \leq Ce^{-\alpha t}$ and $(u(t+\sigma) - w(t))^2 + (u'(t+\sigma) - w'(t))^2 \leq Ce^{-\alpha t}$ for $t \geq 0$. (See Corollary 5.4 in [11].) In particular, the solution (u, v) is nonrecurrent. Note, however, that this is a local result, as opposed to the global result of Theorem 3.1. Changing t to $-t$, one can interpret this result by saying that the periodic solution $(w, 0)$ has a nontrivial unstable manifold.

By using (3.3), one can show that all recurrent solutions of (1.5) are quasi-periodic. (See Proposition 2.4 in [11].)

4. Nonrecurrent solutions in the non-integrable case.

The proof of the existence of nonrecurrent solutions for the system (1.5) (see Theorem 3.1) depends on the rather special property that the system admits a second conservation law. This leaves open the question as to whether or not the result itself is due to the completely integrable character of the system (1.5). To answer this questions, we consider the equation (1.7) with $\alpha > 0$. By considering a two mode solution we are led, as above, to the following system of two ODE's

$$\begin{cases} u'' + u + (u^2 + v^2)^\alpha u = 0, \\ v'' + kv + (u^2 + v^2)^\alpha v = 0, \end{cases} \tag{4.1}$$

where $k > 1$. (4.1) is the Hamiltonian system associated with the Hamiltonian

$$E(u, v, u', v') = \frac{1}{2}u'^2 + \frac{1}{2}v'^2 + \frac{1}{2}u^2 + \frac{k}{2}v^2 + \frac{1}{2(\alpha+1)}(u^2 + v^2)^{\alpha+1}. \tag{4.2}$$

If $\alpha \neq 1$, then a result of Grotta Ragazzo [15] suggests that (4.1) is not completely integrable. (In fact, the non-integrability is proved for a slightly different system under the assumption that α is an integer, but the argument suggests that the system is not integrable even for noninteger α's.) However, the following result shows that system (4.1), hence equation (1.7), has nonrecurrent solutions. (See Theorem 1.1 in [12].)

Theorem 4.1. *There exists $C > 0$ such that for every $E_0 \geq C$, there exists a two dimensional submanifold $\mathcal{M}$ of the (three dimensional) manifold $\{E(u, v, u', v') = E_0\}$ with the following property. If $(u_0, v_0, u_0', v_0') \in \mathcal{M}$, and if (u, v) is the solution of (4.1) with initial data (u_0, v_0, u_0', v_0'), then v and v' converge exponentially to 0 as $t \to \infty$, and there exists a solution w of the equation $w'' + w + |w|^{2\alpha}w = 0$ with energy $\frac{1}{2}w'^2 + \frac{1}{2}w^2 + \frac{1}{2(\alpha+1)}|w|^{2(\alpha+1)} = E_0$ such that $u - w$ and $u' - w'$ converge exponentially to 0 as $t \to \infty$.*

Remark 4.2. The manifold $\{E(u, v, u', v') = E_0\} \cap \{v = v' = 0\}$ is one dimensional; and so, for most solutions of (1.1) with initial values in $\mathcal{M}$, we have $v \not\equiv 0$. It is clear that such solutions are nonrecurrent, hence not quasi-periodic. Note that Theorem 4.1 is a local result which does not describe the behavior of (u, v) as $t \to -\infty$.

Remark 4.3. Since solutions of $w'' + w + |w|^{2\alpha} w = 0$ with a given energy are unique up to a translation, changing t to $-t$ in Theorem 4.1 implies that the periodic solutions $(w, 0)$ of sufficiently large energy have a nontrivial unstable manifold.

Theorem 4.1 is proved by showing that the Poincaré map associated to a periodic solution $(w, 0)$ of sufficiently large energy has a hyperbolic fixed point. More precisely, $E_0 > 0$ being fixed, we define the mapping $T : \mathcal{U} \to \mathbf{R}^2$, where $\mathcal{U} \subset \mathbf{R}^2$ is a neighborhood of 0, as follows. Let

$$\mathcal{U} = \left\{ (a, b) \in \mathbf{R}^2;\ \frac{b^2}{2} + \frac{ka^2}{2} + \frac{a^{2(\alpha+1)}}{2(\alpha+1)} < E_0 \right\},$$

and for $(a, b) \in \mathcal{U}$ let $u_0' > 0$ be defined by

$$\frac{u_0'^2}{2} + \frac{b^2}{2} + \frac{ka^2}{2} + \frac{a^{2(\alpha+1)}}{2(\alpha+1)} = E_0.$$

Given $(a, b) \in \mathcal{U}$, we consider the solution (u, v) of (4.1) with initial data $(u, v, u', v')(0) = (0, a, u_0', b)$. Since $u(0) = 0$ and $u'(0) > 0$, we have $u(t) > 0$ for small $t > 0$. On the other hand, one sees easily that u must have a zero on $(0, \pi)$. Let τ be the first positive zero of u. We define

$$T(a, b) = -(v(\tau), v'(\tau)).$$

We next define the linear operator $B \in \mathcal{L}(\mathbf{R}^2, \mathbf{R}^2)$ as follows. Let $w_0' = \sqrt{2E_0}$, and let w be the solution of the equation

$$w'' + w + |w|^{2\alpha} w = 0,$$

with initial data $w(0) = 0$ and $w'(0) = w_0'$. Since $w(0) = 0$ and $w'(0) > 0$, we have $w(t) > 0$ for $t > 0$ and small. Let ρ be the first positive zero of w. Given $(a, b) \in \mathbf{R}^2$, let z be the solution of the (linear) equation

$$z'' + kz + |w|^{2\alpha} z = 0,$$

with initial data $z(0) = a$ and $z'(0) = b$. We define $B \in \mathcal{L}(\mathbf{R}^2, \mathbf{R}^2)$ by

$$B(a,b) = -(z(\rho), z'(\rho)),$$

for all $(a,b) \in \mathbf{R}^2$. One shows easily that $B = DT(0,0)$ so that, by using a perturbation argument, Theorem 4.1 follows from the property that if E_0 is large enough, then B has eigenvalues λ and λ^{-1} for some $\lambda \in (0,1)$. Since B is associated with the linearized system

$$\begin{cases} w'' + w + |w|^{2\alpha} w = 0, \\ z'' + kz + |w|^{2\alpha} z = 0, \end{cases} \tag{4.3}$$

its eigenvalues for E_0 large are calculated by studying the system

$$\begin{cases} w'' + |w|^{2\alpha} w = 0, \\ z'' + |w|^{2\alpha} z = 0, \end{cases} \tag{4.4}$$

which is a limiting system of (4.3) as $E_0 \to \infty$ after a suitable rescaling. Note, however, that the operator corresponding to the limiting system (4.4) has the double eigenvalue 1, so that the property of B follows from an analysis for large, but finite energy.

Remark 4.4. This method can be adapted to more general systems, for which one still gets to the limiting system (4.4). In particular, if $f(s) = as^\alpha + bs^\beta$ with $a, b > 0$ and $0 < \beta < \alpha$, then the conclusion of Theorem 4.1 holds for the system

$$\begin{cases} u'' + u + f(u^2 + v^2)u = 0, \\ v'' + kv + f(u^2 + v^2)v = 0, \end{cases}$$

which corresponds to the wave equation

$$\begin{cases} u_{tt} - u_{xx} + uf\left(\int_\Omega u(t,x)^2\, dx\right) = 0, \\ u_{|\partial\Omega} = 0. \end{cases}$$

(See Theorem 1.2 in [12], which gives general conditions on the function f so that the same analysis can be carried out.)

A classical model of the nonlinear vibrating string is given by the equation (1.8), where $a \geq 0$, $b > 0$. This model has been extensively studied, for example by Carrier [5], Bernstein [3], Narashimha [20]. See also Dickey [14], Medeiros and Milla Miranda [19] and the references therein. Our main interest in this model is the stability of simple modes, i.e. solutions of the form $u(t,x) = \omega(t)\sin(jx)$, such solutions necessarily being periodic in time. Dickey [14] showed that small amplitude simple modes are stable. We prove that simple modes of sufficiently large energy are unstable, i.e. that they have nontrivial unstable manifolds. This shows in particular the existence of nonrecurrent solutions of (1.8).

Expanding the solution in a Fourier series, we obtain for the Fourier components u_j's the following system of ODE's:

$$u_j'' + j^2\left(a + b\sum_{i=1}^{\infty} i^2 u_i^2\right) u_j = 0. \tag{4.5}$$

It follows that, given any integer $j \geq 1$, equation (1.8) has the particular (simple mode) solution $u(t,x) = \sqrt{\frac{2}{\pi}}\omega(t)\sin(jx)$ where ω solves the equation $\omega'' + j^2(a + bj^2\omega^2)\omega = 0$. To prove that simple modes are always unstable once they have sufficiently large energy, we consider a two-mode solution of (1.8) (or (4.5)), i.e. a solution of the system

$$\begin{cases} u_j'' + j^2(a + bj^2u_j^2 + bk^2u_k^2)u_j = 0, \\ u_k'' + k^2(a + bj^2u_j^2 + bk^2u_k^2)u_k = 0, \end{cases} \tag{4.6}$$

where j, k are integers such that $k > j \geq 1$. After rescaling, we are led to the system

$$\begin{cases} u'' + (\nu + u^2 + v^2)u = 0, \\ v'' + \gamma(\nu + u^2 + v^2)v = 0, \end{cases} \tag{4.7}$$

where $\gamma = k^2/j^2$ and $\nu = a/b$. The system (4.7) is a Hamiltonian system, whose conserved energy E, expressed in terms of u, v, u', v', is given by

$$E(u,v,u',v') = \frac{u'^2}{2} + \frac{v'^2}{2\gamma} + \nu\frac{u^2+v^2}{2} + \frac{(u^2+v^2)^2}{4}.$$

We prove the following result (See Theorem 1.1 in [13]).

Theorem 4.5. *Assume that $\nu > 0$ and let*

$$\gamma \in ((m+1)(2m+1), (m+1)(2m+3)),$$

for some nonnegative integer m. There exists $C > 0$ such that if $E_0 \geq C$, then there exists a two dimensional submanifold $\mathcal{M}$ of the (three dimensional) manifold $\{E(u,v,u',v') = E_0\}$ with the following property. If $(u_0, v_0, u_0', v_0') \in \mathcal{M}$, and if (u,v) is the solution of (4.7) with initial data (u_0, v_0, u_0', v_0'), then v and v' converge exponentially to 0 as $t \to \infty$, and there exists a solution w of the equation $w'' + \nu w + w^3 = 0$ with energy $\dfrac{w'^2}{2} + \nu\dfrac{w^2}{2} + \dfrac{w^4}{4} = E_0$ such that $u - w$ and $u' - w'$ converge exponentially to 0 as $t \to \infty$.

Note that, given an integer $j \geq 1$, there exists $k > j$ such that $\gamma = \dfrac{k^2}{j^2}$ falls in one of the intervals defined in Theorem 4.5. Thus, since the system (4.6) is time reversible, Theorem 4.5 indeed proves that a simple mode $\sqrt{\dfrac{2}{\pi}}\omega(t)\sin(jx)$ of sufficiently large energy has a nontrivial unstable manifold.

In fact, more can be said. The first two intervals in Theorem 4.5 are $(1,3)$ and $(6,10)$. If we set $k = 3j$, $\forall j \geq 1$, then $\gamma = \dfrac{k^2}{j^2} = 9$ independent of j. Theorem 4.5 thus implies the existence of an energy level above which all simple modes are unstable.

Theorem 4.5 is proved by showing that a periodic solution of system (4.7) with sufficiently large energy and such that $v \equiv 0$ gives rise to a hyperbolic fixed point of the induced Poincaré map. The details of the proof follow the same outline as for Theorem 4.1, however several new ingredients are needed to study the linearization of the Poincaré map. In particular, the limiting form (for high energy) of the linearized Poincaré map is defined using the system

$$\begin{cases} w'' + w^3 = 0, \\ z'' + \gamma w^2 z = 0. \end{cases} \tag{4.8}$$

The real line is partitioned into a sequence of intervals, and the behavior of solutions to (4.8) depends on which interval contains γ; and using Jacobi polynomials, we can specify the intervals exactly.

Remark 4.6. If we assume $a = 0$ in equation (1.8), then we have an extension of Theorem 4.5 without the energy condition. (See Theorem 4.1 in [13].)

Remark 4.7. This method can be adapted to more general systems. In particular, if $f(s) = as^\alpha + bs^\beta$ with $a, b > 0$ and $0 < \beta < \alpha$, then the conclusion of Theorem 4.5 holds for the system

$$\begin{cases} u'' + u + f(u^2+v^2)u = 0, \\ v'' + \gamma(1 + f(u^2+v^2))v = 0, \end{cases}$$

which corresponds to the equation

$$\begin{cases} u_{tt} - a\left(1 + f\left(\int_\Omega u(t,x)^2\,dx\right)\right)u_{xx} = 0, \\ u_{|\partial\Omega} = 0. \end{cases}$$

(See Theorem 4.3 in [13].) Similar results for the system (4.8) were previously obtained by Yoshida [22], by using analytic function theory. However, this method seems to apply only when $f(s) = |s|^{2m}s$ for $m \in \mathbf{N}$.

Remark 4.8. By the same method as above, one can prove that the second order Galerkin approximation of equation (1.1) has also a three parameter family of nonrecurrent solutions.

5. Conclusion.

In view of the above results, one may naturally expect that conservative nonlinear wave equations posess in general nonrecurrent solutions. This is possibly the case for the cubic equation (1.1). Also, the existence of recurrent trajectories which are not periodic is an open problem for nonintegrable cases such as (4.1) and (4.6).

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