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TRAVELLING WAVES IN A HYPERBOLIC SYSTEM MODELLING TRAFFIC FLOW

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1. Introduction

We will discuss here the behavior of smooth travelling wave solutions for a hyperbolic system of equations introduced independently by Lighthill and Whitham [5] and Richards [9] to model traffic flow on long crowded highways. Simple as it looks, this model seems nevertheless to describe fairly well the essential qualitative features of traffic flow [2], [10]. A particular feature of great interest to all who deal with traffic, from drivers themselves to traffic engineers, is the behavior of *shock waves*. These are nonlinear, compressible waves carrying a rapid increase in the flow density, propagating at a speed slower than that of the cars behind them. If this increase is too abrupt, these waves can be very harmful, since drivers hitting the wave will have little time to react to the rapidly varying local conditions. Skillful drivers aim high as they drive to continuously check the traffic conditions ahead and behind. This behavior introduces *diffusion* into the system. Diffusion dissipates sharp profiles — the more diffusion we have, the less likely it is to come across a harmful wave. How dangerous a rapidly varying wave turns out to be depends upon how long it takes the drivers to react. Thus, being attentive and able to react fast are undoubtedly very important attributes to good driving — indeed, they may be all that it takes to make our highways a lot safer. It is reassuring to see these conclusions coming out easily from the mathematical model discussed below. For convenience of the reader, we present in Section 2 a short derivation of the equations involved, following [5], [9], [10]. These are a pair of coupled partial differential equations of hyperbolic type expressing the conservation of cars and the way drivers react to local flow conditions. We introduce a parameter κ to measure how much the traffic conditions ahead and behind the drivers get to influence their decisions. This is how diffusion comes in. Also, there is a time scale in the problem corresponding to how long drivers take to react. This is the *relaxation time* τ introduced in Section 2. One consequence of diffusion is the existence of smooth shock waves. In Section 3, we review a result of Liu [6] which shows that these travelling waves are nonlinearly stable. Finally, in Section 4 we derive new results which establish the decay rate of disturbances.

2. The Traffic Flow Equations

For convenience, we will give a brief derivation of the Lighthill-Whitham-Richards traffic flow equations, referring the reader to [2], [5], [9], [10] for further details. We let $\rho(x,t)$ stand for the car density, i.e., the local number of cars per unit length of the road, at time t. Also, we let $u(x,t)$ denote the flow speed, i.e., the speed of the car which reaches position x at time t. On a stretch of highway with no entries or exits, the number of cars is conserved, so that we can write

$$\rho_t + (\rho u)_x = 0\,, \tag{1}$$

which is our first equation. It remains to derive an equation for the second flow variable, $u(x,t)$. It seems fair to assume that drivers on a highway continuously relax their speed towards an optimum value, the highest speed at which they can safely drive under the local traffic conditions. Let $u_*(x,t)$ be this speed; thus the drivers accelerate or decelerate their cars so that $u_*(x,t)$ is achieved, or

$$u_t + u\,u_x = -\frac{u - u_*}{\tau} \tag{2}$$

where τ is the scale representing the relaxation time. Typically, τ will depend upon ρ, but there is no essential change if we think of it as a constant. We have to say how u_* is determined. First, u_* should depend upon ρ, say $u_* = U_*(\rho)$, with $U_*(\rho)$ decreasing as ρ increases from zero to its maximum value $\rho_{\max}$, when the cars are bumper to bumper. We thus assume that

$$U_*'(\rho) < 0 \ , \quad V_*''(\rho) < 0 \ \text{ for } 0 \leq \rho \leq \rho_{\max} \ , \tag{3}$$

where $V_*(\rho) \equiv \rho U_*(\rho)$. A second effect modeled by u_* is precisely the one that introduces diffusion in our system: u_* should be smaller than the value $U_*(\rho)$ dictated by ρ alone if ρ is increasing at that point, so that the driver compensates for the more dangerous traffic conditions ahead; similarly, he can aford a larger value for u_* if the number of cars is smaller ahead. It is natural then to take

$$u_* = U_*(\rho) - \kappa \frac{\rho_x}{\rho} \tag{4}$$

where κ is a positive constant measuring how much the drivers weigh the traffic conditions ahead of them (and behind). A quick look at equation (4) shows that κ has the dimension $\mathrm{L}^2\mathrm{T}^{-1}$, like a diffusion coefficient. We can argue that κ *is* indeed a diffusion coefficient in the following way. According to (2), u tends to relax to u_*, so that, to a first approximation, we can view equation (1) as

$$\rho_t + (\rho U_*(\rho) - \kappa \rho_x)_x = 0 \quad ,$$

or

$$\rho_t + V_*(\rho)_x = \kappa \rho_{xx} \ . \tag{5}$$

a non-linearly advected heat equation with diffusion coefficient given by κ. Thus, we should not be surprised to see small disturbances to equilibrium states (ρ_0, u_0) propagating with speed $\lambda_* = V_*'(\rho_0)$ and diffusing out as they go. Equation (5) is closely related to the hyperbolic equation [4]

$$\rho_t + V_*(\rho)_x = 0 \tag{6}$$

We refer the reader to Liu [6], [7] for a discussion of the role of (5),(6) in the study of equations (1), (2).

We note that we can rewrite (1), (2) as

$$\rho_t + v_x = 0 \tag{7a}$$

$$v_t + \left(\frac{v^2}{\rho} + p(\rho)\right)_x = \frac{V_*(\rho) - v}{\tau(\rho)} \tag{7b}$$

where

$$v \equiv \rho u \tag{8}$$

gives the momentum per unit length, $p'(\rho) = \kappa/\tau(\rho)$ is the pressure derivative, and $V_*(\rho) = \rho U_*(\rho)$. A quick calculation gives the characteristic speeds

$$\lambda_1(\rho, u) = u - \sqrt{p'(\rho)}\ , \quad \lambda_2(\rho, u) = u + \sqrt{p'(\rho)}\ , \tag{9}$$

so that the system (7) is strictly hyperbolic. Our interest here will be on *smooth traveling-wave solutions of* (1) *propagating at subcharacteristic speeds*, see (12) below. The existence of such waves is established in [6]; this is related to admissible shock discontinuities [4], [8] for the equation (6). If $(\rho, v)(x,t) = (\varphi, \psi)(x - \sigma t)$ is a travelling wave for (1) connecting constant states $(\rho_\pm, v_\pm) = (\varphi, \psi)(\pm\infty)$, we must have

$$v_\pm = V_*(\rho_\pm) \tag{10}$$

since the only constant state solutions of (7) are equilibrium states. From (7a), we then get

$$\sigma \cdot (\rho_+ - \rho_-) = V_*(\rho_+) - V_*(\rho_-)\ , \tag{11}$$

so that (ρ_-, ρ_+) satisfies the jump condition for the equilibrium equation (6); if this discontinuity is admissible and its speed σ differs from $\lambda_1(\rho, v)$ and $\lambda_2(\rho, v)$ along the line $\sigma\rho - v = \sigma\rho_\pm - V_*(\rho_\pm)$ for all ρ between ρ_- and ρ_+, then the existence of (φ, ψ) can be shown [6], and σ is subcharacteristic everywhere along the wave, i.e.,

$$\lambda_1(\,(\varphi, \psi)(\xi)\,) < \sigma < \lambda_2(\,(\varphi, \psi)(\xi)\,) \tag{12}$$

for all ξ (including $\xi = \pm\infty$), see [6] for details.

3. Travelling Wave Solutions

In this section, we present a brief outline of the nonlinear stability analysis for travelling waves given by Liu [6]. The basic result discussed here is summarized in Theorem 3.1 below. Let $(\varphi,\psi)(x-\sigma t)$ be a smooth travelling wave of (7) with speed σ, and $(\rho,v)(x,t)$ be the solution of (7) corresponding to a slight perturbation of the initial profile for (φ,ψ), i.e.,

$$(\rho,v)(x,0)=(\varphi,\psi)(x)+(\overline{\rho},\overline{v})(x)\,, \tag{13}$$

with $(\overline{\rho},\overline{v})(\pm\infty)=0$, $(\varphi,\psi)(\pm\infty)=(\rho_\pm,v_\pm)$, $v_\pm=V_*(u_\pm)$. More precisely, we assume that the initial disturbance $(\overline{\rho},\overline{v})$ satisfies

$$\overline{\rho}\in H^2(\mathbf{R})\cap L^1(\mathbf{R}) \quad , \quad \overline{v}\in H^2(\mathbf{R}) \tag{14a}$$

and

$$\int_{-\infty}^{0}\left|\int_{-\infty}^{x}\overline{\rho}(y)\,dy\right|^2 dx+\int_{0}^{+\infty}\left|\int_{x}^{+\infty}\overline{\rho}(y)\,dy\right|^2 dx<\infty \tag{14b}$$

Because (7a) is a conservation law, we have

$$\int_{-\infty}^{+\infty}(\rho(x,t)-\varphi(x-\sigma t))\,dx=\int_{-\infty}^{+\infty}\overline{\rho}(x)\,dx \quad ,$$

so that we cannot expect $(\rho,v)(x,t)$ to tend to $(\varphi,\psi)(x-\sigma t)$ as $t\to\infty$ unless the initial disturbance $\overline{\rho}(x)$ has zero integral on the line [1]. However, observing that, for fixed x_0,

$$\int_{-\infty}^{+\infty}(\varphi(x+x_0)-\varphi(x))\,dx=x_0\cdot(\rho_+-\rho_-) \quad ,$$

we see that

$$\int_{-\infty}^{+\infty}(\rho(x,t)-\varphi(x+x_0-\sigma t))\,dx=\int_{-\infty}^{+\infty}\overline{\rho}(x)\,dx-x_0\cdot(\rho_+-\rho_-)=0$$

provided we take x_0 to be

$$x_0=\frac{1}{\rho_+-\rho_-}\int_{-\infty}^{+\infty}\overline{\rho}(x)\,dx\ . \tag{15}$$

That is, for such x_0, we can expect to show that

$$(\rho,v)(x,t)\to(\varphi,\psi)(x+x_0-\sigma t)\quad \text{as}\ \ t\to\infty\,; \tag{16}$$

thus, the original wave (φ, ψ) is shifted by the amount x_0 given by (15) due to the disturbance. This is in vivid contrast with the corresponding behavior of expansion waves, which interact very weakly with the disturbances [6]. Also, it turns out that, in the case of travelling waves, disturbances might decay quite fast as $t \to \infty$, provided they are sufficiently localized in space. In Section 4, we show that this is indeed the case; for now, we will simply outline the basic steps in proving (16), following [6] It is convenient at this point to introduce the following notation: for given $a \in \mathbf{R}$, let $\mathcal{S}_a$ be the shift operator $(\mathcal{S}_a\phi)(x) = \phi(x+a)$, ϕ an arbitrary function defined on the real line. We have then to study the behavior of $(\rho, v)(\cdot, t) - (\mathcal{S}_{x_0-\sigma t}\varphi, \mathcal{S}_{x_0-\sigma t}\psi)$ as $t \to \infty$. From (14a), (14b), we can easily check that

$$(\rho, v)(\cdot, 0) - (\mathcal{S}_{x_0}\varphi, \mathcal{S}_{x_0}\psi) \in H^2(\mathbf{R}) \tag{17a}$$

and

$$\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{x} \Big(\rho(y,0) - \mathcal{S}_{x_0}\varphi(y)\Big)\, dy \right|^2 dx < \infty . \tag{17b}$$

When these quantities (17a), (17b) are small, we can derive energy estimates related to (16), leading to the following result [6]:

Theorem 3.1 (Liu, 1987). *Consider a travelling wave solution $(\varphi, \psi)(x-\sigma t)$ for the hyperbolic system (7), with $(\varphi, \psi)(\pm\infty) = (\rho_\pm, v_\pm)$, $v_\pm = V_*(\rho_\pm)$, propagating at a subcharacteristic speed σ, i.e., $\lambda_1((\varphi, \psi)(\xi)) < \sigma < \lambda_2((\varphi, \psi)(\xi))$ for all $-\infty \leq \xi \leq +\infty$, where λ_1 and λ_2 denote the characteristic values for (7).*

Then there exist positive constants δ, ε such that the following is true whenever $|\rho_+ - \rho_-| \leq \delta$:

Let $(\overline{\rho}, \overline{v})$ satisfy (14a) and (14b), and let $(\rho, v)(x,t)$ be the solution of (7) corresponding to the initial data (13), i.e.,

$$\rho(x,0) = \varphi(x) + \overline{\rho}(x)$$

$$v(x,0) = \psi(x) + \overline{v}(x)$$

Let

$$\begin{aligned} z(x,t) &= \int_{-\infty}^{x} \left(\rho(y,t) - \mathcal{S}_{x_0}\varphi(y - \sigma t) \right) dy \\ w(x,t) &= v(x,t) - \mathcal{S}_{x_0}\psi(x - \sigma t) \quad , \end{aligned}$$

where x_0 *is given by* (15).

Then

(ρ, v) *is defined for all* $t \geq 0$ *and*

$$\begin{aligned} \| \rho(\cdot,t) - \mathcal{S}_{x_0 - \sigma t}\varphi \|_{H^2} + \| v(\cdot,t) - \mathcal{S}_{x_0 - \sigma t}\psi \|_{H^2} &= \\ = O(1) \left(\| z(\cdot,0) \|_{H^3} + \| w(\cdot,0) \|_{H^2} \right) \end{aligned}$$

and

$$\| \rho(\cdot,t) - \mathcal{S}_{x_0 - \sigma t}\varphi \|_{H^1} + \| v(\cdot,t) - \mathcal{S}_{x_0 - \sigma t}\psi \|_{H^1} \to 0 \quad \text{as} \quad t \to \infty$$

whenever

$$\| z(\cdot,0) \|_{H^3} + \| w(\cdot,0) \|_{H^2} \leq \varepsilon \, .$$

It will be useful to review the basic steps in deriving Theorem 3.1 above. To simplify the notation, let us write (φ, ψ) for $(\mathcal{S}_{x_0}\varphi, \mathcal{S}_{x_0}\psi)$, that is, we translate $(\varphi, \psi)(x)$ to $(\varphi, \psi)(x + x_0)$ so that we assume, for simplicity,

$$\int_{-\infty}^{+\infty} (\, \rho(x,t) - \varphi(x - \sigma t) \,) \; dx = 0 \, . \tag{18}$$

Now, both (ρ, v) and (φ, ψ) satisfy (7); by taking the difference of the two systems of equations and integrating the first equation with respect to x, we obtain

$$z_t + w = 0 \tag{19a}$$

$$w_t + (\, g(z_x + \varphi, w + \psi) - g(\varphi, \psi) \,)_x = h(z_x + \varphi, w + \psi) - h(\varphi, \psi) \quad , \tag{19b}$$

where

$$h(\rho, v) = \frac{V_*(\rho) - v}{\tau(\rho)} \tag{20}$$

is the relaxation term in (7), and

$$g(\rho, v) = \frac{v^2}{\rho} + p(\rho) \, . \tag{21}$$

To simplify the notation, whenever a function is evaluated at (φ, ψ), we will omit the argument; thus, $g_\rho \equiv g_\rho(\varphi, \psi)$, $g_x \equiv g(\varphi, \psi)_x$, and so on. Plugging (19a) into (19b), we get, rearranging a few terms,

$$z_{tt} + (\lambda_1 + \lambda_2) z_{xt} + \lambda_1 \lambda_2 z_{xx} - h_v \cdot (z_t + \mu z_x) = \tilde{\mathcal{R}}(x,t)\,, \tag{22}$$

where

$$\begin{aligned} \tilde{\mathcal{R}}(x,t) \;\equiv\; & -g_{vx} z_t \;+\; \mathcal{Q}_1(f)(h_v - g_{vx}) \;+\; g_{\rho x} z_x \\ & -\; \mathcal{Q}_1(h) \;+\; \mathcal{Q}_1(g)_x \;-\; \mathcal{Q}_1(f)_t \;-\; g_v \mathcal{Q}_1(f)_x\,, \end{aligned} \tag{23}$$

$$f(\rho, v) = v \tag{24}$$

and

$$\mu \;\equiv\; -\,h_\rho\, h_v^{-1}\,, \tag{25}$$

where, as noted above, λ_1, λ_2, μ, h_v, etc., are all evaluated at (φ, ψ). Here, for any $\mathcal{F}(\rho, v)$, $\mathcal{Q}_1(\mathcal{F})$ denotes the difference

$$\begin{aligned} \mathcal{Q}_1(\mathcal{F}) \;&\equiv\; \mathcal{F}(z_x + \varphi, w + \psi) - \mathcal{F}(\varphi, \psi) - \mathcal{F}_u(\varphi, \psi) z_x - \mathcal{F}_v(\varphi, \psi)\, w \\ &=\; O(1)\,(\,z_x^2 + w^2\,) \end{aligned} \tag{26}$$

We observe that the right-hand side $\tilde{\mathcal{R}}$ in (22) involves only high powers of terms which we expect to be small, so that it will probably have little effect in the overall analysis. All first powers are written on the left-hand side in (22), which contains a first-order term with speed μ given in (25); this is the *dynamic* characteristic speed governing the propagation of weak disturbances over the travelling wave. We recall that the *equilibrium* speed $\lambda_*(\rho)$ given in (6) can be written as $\mu(\rho, V_*(\rho))$, so that for a non-equilibrium regime (ρ, v) like the travelling wave (φ, ψ) they will be different in general. As a result of the nonlinearities present in the system (7), this characteristic speed μ changes along the wave. A simple computation shows that

$$\mu_x \;=\; (\,f_*''(\rho_{\text{avg}}) \;+\; O(|\rho_+ - \rho_-|)\,)\,\varphi_x \tag{27}$$

where $\rho_{\text{avg}} \equiv \frac{1}{2}(\rho_+ + \rho_-)$. The simplest nonlinearity occurs when μ changes monotonically; we have assumed, for simplicity, that $-V_*(\rho)$ is convex, see (3) above. This is in agreement with actual observations of traffic flow [10]. It then follows that we have $\rho_- > \rho_+$, with φ_x negative at every point; hence, for sufficiently weak travelling waves (φ, ψ), i.e., $|\rho_+ - \rho_-|$ small, (27) gives

$$\mu_x < 0 \tag{28}$$

and

$$M^{-1}(|\varphi_x| + |\psi_x|) \leq |\mu_x| \leq M|\varphi_x| \quad \text{for all} \quad x, t \tag{29}$$

for some positive constant M. Also, for weak waves (φ, ψ),

$$|\mu - \sigma| = O(|\rho_+ - \rho_-|) \ll 1 \tag{30}$$

and, observing that $\varphi_x = O(|\rho_+ - \rho_-|)$,

$$|\varphi_x| + |\psi_x| + |\mu_x| = O(|\rho_+ - \rho_-|) \ll 1 . \tag{31}$$

It is convenient to perform the analysis in a reference frame sitting on the travelling wave. Thus, we introduce the new variables (ξ, t), $\xi = x - \sigma t$, in terms of which (22) reads as

$$\begin{aligned} z_{tt} + (\lambda_1 + \lambda_2 - 2\sigma)z_{\xi t} &+ (\sigma - \lambda_1)(\sigma - \lambda_2)z_{\xi\xi} \\ &+ h_v(\sigma - \mu)z_\xi - h_v z_t = \mathcal{R}(\xi, t) \end{aligned} \tag{32}$$

where

$$\begin{aligned} \mathcal{R}(\xi, t) = & -g_{v\xi}(z_t - \sigma z_\xi) + \mathcal{Q}_1(f)(h_v - g_{v\xi}) + g_{\rho\xi} z_\xi + \\ & \mathcal{Q}_1(g)_\xi + \sigma \mathcal{Q}_1(f)_\xi - \mathcal{Q}_1(h) - \mathcal{Q}_1(f)_t - g_v \mathcal{Q}_1(f)_\xi . \end{aligned} \tag{33}$$

As noted in [6], the stability of the travelling wave is a consequence of three basic facts: its compressibility, expressed by (28) or, in the new variables (ξ, t), by the inequality $\mu_\xi < 0$; the fact that the wave travels at subcharacteristic speeds, see (12), which makes $(\sigma - \lambda_1)(\lambda_2 - \sigma)$ bounded below from zero; and the relaxation effects introduced by the rate term h in (7b). In fact, under the assumptions outlined above, Liu [6] was able to derive, after considerable work, the energy estimate

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left(\textstyle\sum_{j=0}^{3} |\nabla^j z|^2(x,t) + \sum_{j=0}^{2} |\nabla^j w|^2(x,t) \right) dx \; + \\ &\quad + \int_0^T \int_{-\infty}^{+\infty} \left(|\mu_x| z^2 + \textstyle\sum_{j=1}^{3} |\nabla^j z|^2 + \sum_{j=0}^{2} |\nabla^j w|^2 \right) dx\, dt \; = \\ &\quad = O(1) \left(\| z(\cdot, 0) \|_{H^3}^2 + \| w(\cdot, 0) \|_{H^2}^2 \right) \end{aligned} \tag{34}$$

provided we take $\| z(\cdot,0) \|^2_{H^3} + \| w(\cdot,0) \|^2_{H^2}$ sufficiently small, see [6] for the detailed derivation. We simply remark that (34) immediately gives

$$\sum_{j=0}^{2} \| \nabla^j z(\cdot,t) \|_{L^\infty} + \sum_{j=0}^{1} \| \nabla^j w(\cdot,t) \|_{L^\infty} \to 0 \text{ as } t \to +\infty \,, \tag{35}$$

but, due to the fact that $\mu_x \to 0$ as $x \to \pm\infty$, no decay rate can be directly inferred from (34). This will be done by a different, though related, analysis in the next section.

4. Decay Rate Analysis

We will now improve the stability results discussed in Section 3 above by deriving decay estimates for the disturbances (z,w) over the travelling wave (φ,ψ) introduced by the initial data (13). It turns out that these time decay rates depend upon how fast the initial disturbances die off on the real line. Thus, besides the conditions (14a), (14b) of Theorem 3.1, we assume that $(\overline{\rho},\overline{v})$ is such that

$$\int_{-\infty}^{+\infty} \left(1 + |x|^N\right) \left(|\overline{\rho}(x)|^2 + |\overline{v}(x)|^2 \right) dx \tag{36a}$$

and

$$\begin{aligned} &\textstyle\int_{-\infty}^{0} \left(1 + |x|^N\right) \left| \int_{-\infty}^{x} \overline{\rho}(y)\, dy \right|^2 dx \; + \\ &+ \textstyle\int_{0}^{+\infty} \left(1 + |x|^N\right) \left| \int_{x}^{+\infty} \overline{\rho}(y)\, dy \right|^2 dx \end{aligned} \tag{36b}$$

are both finite for some positive integer N. This amounts to requiring a certain algebraic decay of $(\overline{\rho}(x), \overline{v}(x))$ as $x \to \pm\infty$, which in turn allows us to show

Theorem 4.1. *Under all the hypotheses of Theorem 3.1, assume furthermore that for the initial disturbance $(\overline{\rho},\overline{v})$ both* (36a) *and* (36b) *are finite for some positive integer N.*
Then there exist positive constants δ, ε such that

$$\| (\rho,v)(\cdot,t) - (\mathcal{S}_{x_0-\sigma t}\varphi, \mathcal{S}_{x_0-\sigma t}\psi) \|_{H^2} = O(1)(1+t)^{-N/2}$$

whenever

$$|\rho_+ - \rho_-| \le \delta \quad \textit{and} \quad \| z(\cdot,t) \|_{H^3} + \| w(\cdot,t) \|_{H^2} \le \varepsilon \,.$$

Before we prove Theorem 4.1, we will present the basic reasoning behind the argument. The key step is to strip off the (complex) dynamics for the disturbances described by equations (32), (33) to its essential terms. We do this in the following way. To a first approximation, we can think of equation (32) as

$$z_t + (\mu - \sigma) z_\xi = 0 \ , \tag{37}$$

where $\mu = \mu(\varphi(\xi), \psi(\xi))$ is a function of ξ only and σ is a constant, the propagation speed of the travelling wave (φ, ψ), see (11), (25). A simple computation gives [6]

$$\mu(\xi) \ - \ \sigma \ = \ \frac{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}{h\, h_v} \, h_\xi \ ; \tag{38}$$

since $h(\varphi(\xi), \psi(\xi))$ must vanish at the equilibrium states at $\xi = \pm\infty$, we then get

$$\mu(\xi) \ - \ \sigma \ = \ 0 \quad \text{for some } \xi_0 \ \in \ \mathbf{R} \, . \tag{39}$$

Thus, recalling (28), at large times most of the information for solutions of (37) come from points ξ far away from ξ_0 on the initial line. This suggests considering a decay factor of the form $t^\alpha |\xi - \xi_0|^\beta$, involving both t and ξ, in deriving energy inequalities with decay for (32), or, for regularity reasons, $(1+t)^\alpha < \xi - \xi_0 >^\beta$, where

$$< \xi > \equiv \sqrt{1 + \xi^2} \ , \tag{40}$$

α and β being non-negative constants which are at our disposal. In what follows here we assume that we have, in view of (34),

$$\| \, z(\cdot, t) \, \|_{H^3} \ + \| \, w(\cdot, t) \, \|_{H^2} \ \leq \varepsilon \quad \text{for all}\, t \geq 0 \ , \tag{41}$$

where $\varepsilon \ll 1$, and similarly,

$$| \, \rho_+ - \rho_- \, | = \delta \quad , \ \delta \ \text{small} \ . \tag{42}$$

Using (28) and (39), we can show

$$
\begin{aligned}
-\int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta} h_v \cdot (\mu - \sigma) z z_\xi \, d\xi \, dt \; &= \\
= \tfrac{1}{4}\int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta} |h_v| \cdot |\mu_\xi| \, z^2 \, d\xi & \\
+ \tfrac{1}{2}\int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta} h_{v\xi} \cdot (\mu - \sigma) \, z^2 \, d\xi \; + & \\
+ \tfrac{1}{2}\int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta - 1} \mathcal{M}(\xi) \;\; z^2 \, d\xi &
\end{aligned}
\tag{43}
$$

where

$$
\mathcal{M}(\xi) \;\equiv\; \tfrac{1}{2} |h_v| \cdot |\mu_\xi| < \xi - \xi_0 > \, + \, \beta \frac{|\xi - \xi_0|}{< \xi - \xi_0 >} |\mu - \sigma| \; . \tag{44}
$$

It is important to note that we have

$$
\mathcal{M}(\xi) \;\geq\; m \cdot \beta \quad \text{for all } \xi \in \mathbf{R} \tag{45}
$$

for some positive constant $m = m(\delta)$, which will be used below. We now get to the crucial step of the whole analysis. Taking $C \gg 1$, multiplying (32) by $(1+T)^{\alpha} < \xi - \xi_0 >^{\beta} \, (\, z/C \, + \, z_t \,)$ and integrating the result over $\mathbf{R}$ x $[0, T]$, one can show, following [11], that, for ε and δ small,

$$
\begin{aligned}
(1+T)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T) \right) d\xi & \\
+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 \right) d\xi \, dt & \\
+ \beta \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} < \xi >^{\beta - 1} z^2 \; d\xi \, dt \; &= \\
= \; O(1) \left\{ \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z^2(\xi, 0) + z_\xi^2(\xi, 0) + z_t^2(\xi, 0) \right) d\xi \right. & \\
+ \, \alpha \int_0^T (1+t)^{\alpha - 1} \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z^2 + z_\xi^2 + z_t^2 \right) d\xi \, dt & \\
\left. + \, \beta \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 \right) d\xi \, dt \; \right\} &
\end{aligned}
\tag{46}
$$

This weighted energy inequality will be the key point in the decay analysis below. More general estimates are derived in [11], but for completeness we will sketch the basic steps behind (46). Multiplying (32) by $(1+t)^{\alpha} < \xi - \xi_0 >^{\beta} z$

and integrating the result over $\mathbf{R} \times [0,T]$, we get, after a few integrations by parts,

$$
\begin{aligned}
&\tfrac{1}{2}(1+T)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta |h_v|\, z^2(\xi,T)\, d\xi \\
&- \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta h_v \cdot (\mu - \sigma)\, z\, z_\xi \; d\xi\, dt \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} |\lambda_1 - \sigma| \cdot |\lambda_2 - \sigma| < \xi - \xi_0 >^\beta z_\xi^2 \; d\xi\, dt \; = \\
&= \; - \; (1+T)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z(\xi,T)\, z_t(\xi,T)\; dx \\
&+ \tfrac{1}{2} \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta |h_v|\, z^2(\xi,0)\, d\xi \; + \\
&+ \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z(\xi,0) z_t(\xi,0)\, d\xi \\
&+ \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z z_t \; d\xi\, dt \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z_t^2\, d\xi\, dt \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta (\lambda_1 + \lambda_2 - 2\sigma) z_\xi z_t\, d\xi\, dt \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta (\lambda_1 + \lambda_2)_\xi\; z z_t\, d\xi\, dt \\
&+ \tfrac{1}{2}\alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta |h_v| z^2\, d\xi\, dt \; + \\
&\beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta-2} (\xi - \xi_0)\, (\lambda_1 + \lambda_2 - 2\sigma)\, z\, z_t \; d\xi\, dt \\
&+ \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta-2} (\xi - \xi_0)\, (\lambda_1 - \sigma)(\lambda_2 - \sigma)\, z\, z_\xi\, d\xi\, dt \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \; ((\lambda_1 - \sigma)(\lambda_2 - \sigma))_\xi\; z\, z_\xi\, d\xi\, dt \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \mathcal{R}\; (\xi,t)\, d\xi\, dt
\end{aligned} \tag{47}
$$

where we have used (12), (28). ¿From (12), (29), (45), (43), (47), we get

$$
\begin{aligned}
&(1+T)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z^2(\xi,T)\, d\xi \; + \\
&\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z_\xi^2 \, d\xi\, dt \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta |\mu_\xi| z^2 \, d\xi\, dt \; + \\
&+ \; m \cdot \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta-1} z^2 \, d\xi\, dt \; = \\
&= \; O(1) \Big\{ \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \left(z^2(\xi,0) + z_t^2(\xi,0) \right) d\xi \\
&+ \; (1+T)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z_t^2(\xi,T)\, d\xi \\
&+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z_t^2 \, d\xi\, dt \\
&+ \; \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \left(z^2 + z_t^2 \right) d\xi\, dt \\
&+ \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta-1} \left(|z_\xi| + |z_t| \right) |z| \; d\xi\, dt \\
&+ \left| \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z\, \mathcal{R}(\xi,t)\, d\xi\, dt \right| \Big\}
\end{aligned}
\tag{48}
$$

We must now estimate the last two terms on the right-hand side of (48). We have, recalling (45) above, that for a given $\varepsilon > 0$,

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta-1} \left(|z_\xi| + |z_t| \right) |z| \, d\xi \\
&\leq \varepsilon m \int_{-\infty}^{+\infty} < \xi - \xi_0 >^{\beta-1} z^2 \, d\xi \; + \\
&\varepsilon \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \left(z_\xi^2 + z_t^2 \right) \; d\xi \; + \; \mathcal{K}(\delta) \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 \right) \; d\xi
\end{aligned}
\tag{49}
$$

where $\mathcal{K}(\delta)$ is a positive constant depending on δ, see (42). As to the last term in

(48), we can show, using (19a), (26), (29), (41), (42), after a few computations,

$$\begin{aligned}
&\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta z\, \mathcal{R}(\xi,t)\, d\xi\, dt \;= \\
&= O(1)\,(\sqrt{\delta} + \varepsilon) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 \right) d\xi\, dt \\
&\quad + O(\varepsilon)\,(1+T)^\alpha \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \left(z^2(\xi,T) + z_\xi^2(\xi,T) + z_t^2(\xi,T) \right) d\xi \\
&\quad + O(\varepsilon) \int_{-\infty}^{+\infty} < \xi - \xi_0 >^\beta \left(z^2(\xi,0) + z_\xi^2(\xi,0) + z_t^2(\xi,0) \right) d\xi
\end{aligned} \tag{50}$$

Taking (49), (50) into (48), we obtain, for ε and δ small,

$$\begin{aligned}
&(1+T)^\alpha \int_{-\infty}^{+\infty} < \xi >^\beta z^2(\xi,T)\, d\xi + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^\beta z_\xi^2\, d\xi\, dt \\
&\quad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^\beta |\mu_\xi|\, z^2\, d\xi\, dt \\
&\quad + m \cdot \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^{\beta-1} z^2\, d\xi\, dt \;= \\
&= O(1) \Big\{ \int_{-\infty}^{+\infty} < \xi >^\beta \left(z^2(\xi,0) + z_\xi^2(\xi,0) + z_t^2(\xi,0) \right) d\xi \\
&\quad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} < \xi >^\beta \left(z^2 + z_t^2 \right) d\xi\, dt \\
&\quad + \mathcal{K}(\delta)\, \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 \right) d\xi\, dt \;+ \\
&\quad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^\beta z_t^2\, d\xi\, dt \\
&\quad + (1+T)^\alpha \int_{-\infty}^{+\infty} < \xi >^\beta \left(z_t^2(\xi,T) + \varepsilon z_\xi^2(\xi,T) \right) d\xi \Big\}\,.
\end{aligned} \tag{51}$$

To make (51) useful, we have to estimate the last two terms on the right-hand side. To this end, we multiply (32) by $(1+t)^\alpha < \xi - \xi_0 >^\beta z_t$ and integrate the result in $\mathbf{R} \times [0,T]$, which gives, after a few computations,

$$
\begin{aligned}
&\tfrac{1}{2}(1+T)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2(\xi,T) + |\lambda_1-\sigma|\cdot|\lambda_2-\sigma|\, z_\xi^2(\xi,T)\right) d\xi \\
&\qquad + \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} |h_v| < \xi >^{\beta} z_t^2 \, d\xi\, dt = \\
&= \tfrac{1}{2}\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2(\xi,0) + |\lambda_1-\sigma|\cdot|\lambda_2-\sigma|\, z_\xi^2(\xi,0)\right) d\xi \\
&\quad - \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} (\lambda_1+\lambda_2-2\sigma)\, z_t\, z_{\xi t}\, d\xi\, dt \\
&\quad + \tfrac{\alpha}{2}\int_0^T (1+t)^{\alpha-1}\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2 + |\lambda_1-\sigma|\cdot|\lambda_2-\sigma| z_\xi^2\right) d\xi\, dt \qquad (52)\\
&\quad + \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} \left(< \xi >^{\beta} (\lambda_1-\sigma)(\lambda_2-\sigma)\right)_\xi z_\xi\, z_t \; d\xi\, dt \; + \\
&\quad + \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} h_v < \xi >^{\beta} (\mu-\sigma)\, z_\xi\, z_t \; d\xi\, dt \\
&\quad + \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} z_t \mathcal{R} \; d\xi\, dt
\end{aligned}
$$

in view of (12). On the other hand, using (19a), (26), (29), (42) it is not hard to get [11]

$$
\begin{aligned}
&\int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} z_t \mathcal{R}(\xi,t) \; d\xi\, dt = \\
&= (\delta+\varepsilon)\, O(1) \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2 + z_\xi^2\right) \; d\xi\, dt \qquad (53)
\end{aligned}
$$

Using (30), (42), (53), we then get from (52) that, for ε and δ small,

$$
\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2(\xi,T) + z_\xi^2(\xi,T)\right) d\xi \\
&\qquad + \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} z_t^2 \; d\xi\, dt = \\
&= O(1) \left\{ \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_\xi^2(\xi,0) + z_t^2(\xi,0)\right) d\xi \right. \\
&\quad + (\varepsilon+\delta) \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta} z_\xi^2 \; d\xi\, dt \qquad (54)\\
&\quad + \beta \int_0^T (1+t)^{\alpha}\int_{-\infty}^{+\infty} < \xi >^{\beta-1} \left(z_t^2 + z_\xi^2\right) \; d\xi\, dt \\
&\quad \left. + \alpha \int_0^T (1+t)^{\alpha-1}\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2 + z_\xi^2\right) d\xi\, dt \right\} .
\end{aligned}
$$

Now,

$$\begin{aligned}\int_{-\infty}^{+\infty} &< \xi >^{\beta-1} \left(z_t^2 + z_\xi^2\right) d\xi \ \le \\ &\le \mathcal{L}(\varepsilon) \int_{-\infty}^{+\infty} \left(z_t^2 + z_\xi^2\right) d\xi \ + \varepsilon \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2 + z_\xi^2\right) d\xi\end{aligned} \tag{55}$$

for some positive constant $\mathcal{L}(\varepsilon)$ which depends on ε, so that (54) can be rewritten as

$$\begin{aligned}(1+T)^\alpha &\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2(\xi,T) + z_\xi^2(\xi,T)\right) d\xi \\ &+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^{\beta} z_t^2 \, d\xi \, dt \ = \\ = O(1) \Big\{ &\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_\xi^2(\xi,0) + z_t^2(\xi,0)\right) d\xi \\ &+ (\varepsilon+\delta) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^{\beta} z_\xi^2 \, d\xi \, dt \\ &+ \mathcal{L}(\varepsilon)\, \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_t^2 + z_\xi^2\right) d\xi \, dt \\ + \alpha &\int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z_t^2 + z_\xi^2\right) d\xi \, dt \ \Big\} .\end{aligned} \tag{56}$$

Finally, choosing $C > 0$ big enough, we multiply (51) by $1/C$ and add the result to (56) to get (46) for ε and δ sufficiently small.

Having derived (46), we are in a good position to proceed. We first rewrite (46) in the following equivalent way, using (19):

$$\begin{aligned}(1+T)^\alpha &\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z^2(\xi,T) + z_\xi^2(\xi,T) + z_t^2(\xi,T) + w^2(\xi,T)\right) d\xi \\ &+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2\right) d\xi \, dt \\ &\quad + \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} < \xi >^{\beta-1} z^2 \, d\xi \, dt \ = \\ = O(1) \Big\{ &\int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z^2(\xi,0) + z_\xi^2(\xi,0) + w^2(\xi,0)\right) d\xi \\ &+ \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} < \xi >^{\beta} \left(z^2 + z_\xi^2 + z_t^2\right) d\xi \, dt \\ &\quad + \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2\right) d\xi \, dt \ \Big\} .\end{aligned} \tag{57}$$

We now proceed in much the same way as in Kawashima and Matsumura [3]. Having assumed that (36a) and (36b) are both finite for a certain integer $N \geq 1$, this immediately gives

$$\int_{-\infty}^{+\infty} < \xi >^N \left(z^2(\xi,0) + z_\xi^2(\xi,0) + w^2(\xi,0) \right) d\xi \; < \infty \,, \tag{58}$$

where $< \xi >$ is defined in (40). Then, taking $\beta = N, \alpha = 0$ in (57), we immediately get

$$\begin{aligned} \int_{-\infty}^{+\infty} & < \xi >^N \left(z^2(\xi,T) + z_\xi^2(\xi,T) + z_t^2(\xi,T) + w^2(\xi,T) \right) d\xi \\ & + \int_0^T \int_{-\infty}^{+\infty} < \xi >^N \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2 \right) d\xi \, dt \\ & + \int_0^T \int_{-\infty}^{+\infty} < \xi >^{N-1} z^2 \; d\xi \, dt \; = \; O(1) \,. \end{aligned} \tag{59}$$

Hence, taking now $\beta = 0, \alpha = 1$, one gets the estimate

$$\begin{aligned} (1+T) & \int_{-\infty}^{+\infty} \left(z^2(\xi,T) + z_\xi^2(\xi,T) + z_t^2(\xi,T) + w^2(\xi,T) \right) d\xi \\ & + \int_0^T (1+t) \int_{-\infty}^{+\infty} \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2 \right) d\xi \, dt \; = \; O(1) \,, \end{aligned} \tag{60}$$

which in turn allows us to consider $\beta = N-1$, $\alpha = 1$ in (57) above, thus improving (60) to

$$\begin{aligned} (1+T) & \int_{-\infty}^{+\infty} < \xi >^{N-1} \left(z^2(\xi,T) + z_\xi^2(\xi,T) + z_t^2(\xi,T) + w^2(\xi,T) \right) d\xi \\ & + \int_0^T (1+t) \int_{-\infty}^{+\infty} < \xi >^{N-1} \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2 \right) d\xi \, dt \\ & + \int_0^T (1+t) \int_{-\infty}^{+\infty} < \xi >^{N-2} z^2 \, d\xi \, dt \; = \; O(1) \,. \end{aligned} \tag{61}$$

Proceeding in this way, i.e., taking successively in (57) $\beta = N - j$, $\alpha = j$, and then

$\beta = 0$, $\alpha = j + 1$, for $j = 0, 1, 2, ..., N - 1$, we end up with the following decay estimate

$$
\begin{aligned}
&\sum_{j=0}^{N} (1+T)^j \int_{-\infty}^{+\infty} < \xi >^{N-j} \left(z^2(\xi,T) + z_\xi^2(\xi,T) + \right. \\
&\qquad \left. + z_t^2(\xi,T) + w^2(\xi,T) \right) \, d\xi \\
&+ \sum_{j=0}^{N} \int_0^T (1+t)^j \int_{-\infty}^{+\infty} < \xi >^{N-j} \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2 \right) \, d\xi \, dt \ + \\
&+ \sum_{j=0}^{N-1} \int_0^T (1+t)^j \int_{-\infty}^{+\infty} < \xi >^{N-1-j} \ z^2 \ d\xi \, dt \ \leq \ C \, ,
\end{aligned}
\tag{62}
$$

where C is a positive constant depending on the initial data but not on T. In particular, we have

$$
\begin{aligned}
&(1+T)^N \int_{-\infty}^{+\infty} \left(z^2(\xi,T) + z_\xi^2(\xi,T) + z_t^2(\xi,T) + w^2(\xi,T) \right) \, d\xi \\
&+ \int_0^T (1+t)^N \int_{-\infty}^{+\infty} \left(|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2 \right) \, d\xi \, dt \\
&+ \int_0^T (1+t)^{N-1} \int_{-\infty}^{+\infty} z^2 \ d\xi \, dt \ = \ O(1) \, .
\end{aligned}
\tag{63}
$$

We can now proceed in a similar way as in Liu [6]. Differentiating (32) with respect to ξ, multiplying the resulting equation by $(1+t)^\alpha z_\xi$ and integrating over $\mathbf{R}$x$[0,T]$, we get, after some work,

$$
\begin{aligned}
&(1+T)^\alpha \int_{-\infty}^{+\infty} z_\xi^2(\xi,T) \, d\xi \ + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} z_{\xi\xi}^2 \, d\xi \, dt \ = \\
&= \ O(1) \left\{ \int_{-\infty}^{+\infty} \left(z_\xi^2(\xi,0) + z_{\xi t}^2(\xi,0) \right) \, d\xi \ + \right. \\
&\qquad + (1+T)^\alpha \int_{-\infty}^{+\infty} z_{\xi t}^2(\xi,T) \, d\xi \\
&\qquad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_{tt}^2 + z_{\xi t}^2 \right) \, d\xi \, dt \ + \\
&\qquad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_{\xi t}^2 \right) \, d\xi \, dt \\
&\left. + (\varepsilon + \delta) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 \right) \, d\xi \, dt \ \right\} ,
\end{aligned}
\tag{64}
$$

where we have used (41), (42). In a similar way, differentiating (32) with respect to t, multiplying the result by $(1+t)^\alpha z_{tt}$ and integrating over $\mathbf{R}$ x $[0,T]$, we can show, using (41), (42) again,

$$\begin{aligned}
&(1+T)^\alpha \int_{-\infty}^{+\infty} z_{tt}^2(\xi,T)\,d\xi \;+\; \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} z_{tt}^2\;d\xi\,dt \;= \\
&\quad = O(1)\,\Bigg\{ \int_{-\infty}^{+\infty} z_{tt}^2(\xi,0)\;d\xi \;+ \\
&\quad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_{\xi\xi t}^2 + z_{\xi tt}^2 \right)\,d\xi\,dt \\
&+ (\varepsilon+\delta) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 + z_{tt}^2 + z_{\xi\xi\xi}^2 \right)\,d\xi\,dt \\
&\quad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} z_{tt}^2\;d\xi\,dt \Bigg\}\,.
\end{aligned} \tag{65}$$

If we now differentiate (32) with respect to ξ, multiply the result by $(1+t)^\alpha z_{\xi t}$ and integrate over $\mathbf{R} \times [0,T]$, we get, after similar work,

$$\begin{aligned}
&(1+T)^\alpha \int_{-\infty}^{+\infty} z_{\xi t}^2(\xi,T)\;d\xi \;+\; \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} z_{\xi t}^2\;d\xi\,dt \;= \\
&= O(1)\,\Bigg\{ \int_{-\infty}^{+\infty} z_{\xi t}(\xi,0)\,d\xi \;+\; \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} z_{\xi\xi\xi}^2\;d\xi\,dt \\
&\quad + (\varepsilon+\delta) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 \right)\,d\xi\,dt \\
&\quad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} z_{\xi t}^2\;d\xi\,dt \Bigg\}\,.
\end{aligned} \tag{66}$$

Finally, integrating $(32)_{\xi\xi}(1+t)^\alpha z_{\xi\xi}$, $(32)_{\xi\xi}(1+t)^\alpha z_{\xi\xi t}$, $(32)_{\xi t}(1+t)^\alpha z_{\xi tt}$, and $(32)_{tt}(1+t)^\alpha z_{ttt}$ over $\mathbf{R} \times [0,T]$, we get, after a number of straightforward estimations,

$$\begin{aligned}
&(1+T)^\alpha \int_{-\infty}^{+\infty} z_{\xi\xi}^2(\xi,T)\,d\xi \;+\; \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} z_{\xi\xi\xi}^2\;d\xi\,dt \;= \\
&\quad = O(1)\,\Bigg\{ (1+T)^\alpha \int_{-\infty}^{+\infty} z_{\xi\xi t}^2(\xi,T)\,d\xi \\
&\quad + \int_{-\infty}^{+\infty} \left(z_{\xi\xi}^2(\xi,0) + z_{\xi\xi t}^2(\xi,0) \right)\,d\xi \\
&\quad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} z_{\xi\xi t}^2\,d\xi\,dt \;+ \\
&\quad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \left(z_{\xi\xi}^2 + z_{\xi\xi t}^2 \right)\,d\xi\,dt \\
&+ (\varepsilon+\delta) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 \right)\,d\xi\,dt \Bigg\}\,,
\end{aligned} \tag{67}$$

$$
\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty}\left(z_{\xi\xi t}^{2}(\xi,T)+z_{\xi\xi\xi}^{2}(\xi,T)\right)\,d\xi\ + \\
&\quad +\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}z_{\xi\xi t}^{2}\ d\xi\,dt\ = \\
&= O(1)\ \Big\{\int_{-\infty}^{+\infty}\left(z_{\xi\xi t}^{2}(\xi,0)+z_{\xi\xi\xi}^{2}(\xi,0)\right)\,d\xi\ + \\
&(\varepsilon+\delta)\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}\left(z_{\xi}^{2}+z_{t}^{2}+z_{\xi\xi}^{2}+z_{\xi t}^{2}+z_{tt}^{2}+z_{\xi\xi\xi}^{2}+z_{\xi\xi t}^{2}+z_{\xi tt}^{2}\right)\,d\xi\,dt \\
&\quad +\ \alpha\int_{0}^{T}(1+t)^{\alpha-1}\int_{-\infty}^{+\infty}\left(z_{\xi\xi\xi}^{2}+z_{\xi\xi t}^{2}\right)\,d\xi\,dt\ \Big\}\,,
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty}\left(z_{\xi tt}^{2}(\xi,T)+z_{\xi\xi t}^{2}(\xi,T)\right)\,d\xi\ + \\
&\quad +\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}z_{\xi tt}^{2}\ d\xi\,dt\ = \\
&= O(1)\ \Big\{\int_{-\infty}^{+\infty}\left(z_{\xi tt}^{2}(\xi,0)+z_{\xi\xi t}^{2}(\xi,0)\right)\,d\xi\ + \\
&(\varepsilon+\delta)\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}\left(z_{\xi}^{2}+z_{t}^{2}+z_{\xi\xi}^{2}+z_{\xi t}^{2}+z_{tt}^{2}+z_{\xi\xi\xi}^{2}+z_{\xi\xi t}^{2}+z_{\xi tt}^{2}\right)\,d\xi\,dt \\
&\quad +\ \alpha\int_{0}^{T}(1+t)^{\alpha-1}\int_{-\infty}^{+\infty}\left(z_{\xi tt}^{2}+z_{\xi\xi t}^{2}\right)\,d\xi\,dt\ \Big\}\,,
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty}\left(z_{ttt}^{2}(\xi,T)+z_{\xi tt}^{2}(\xi,T)\right)\,d\xi\ + \\
&\quad +\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}z_{ttt}^{2}\ d\xi\,dt\ = \\
&+ O(1)\ \Big\{\int_{-\infty}^{+\infty}\left(z_{ttt}^{2}(\xi,0)+z_{\xi tt}^{2}(\xi,0)\right)\,d\xi\ + \\
&(\varepsilon+\delta)\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}\left(z_{\xi}^{2}+z_{t}^{2}+z_{\xi\xi}^{2}+z_{\xi t}^{2}+z_{\xi\xi\xi}^{2}+z_{\xi\xi t}^{2}+z_{\xi tt}^{2}+z_{ttt}^{2}\right)\,d\xi\,dt \\
&\quad +\ \alpha\int_{0}^{T}(1+t)^{\alpha-1}\int_{-\infty}^{+\infty}\left(z_{ttt}^{2}+z_{\xi tt}^{2}\right)\,d\xi\,dt\ \Big\}\,.
\end{aligned}
\tag{70}
$$

If we now consider $(64)C^{-3}+(65)C^{-2}+(66)C^{-2}+(67)C^{-1}+(68)+(69)+$

(70), we get, choosing $C > 0$ big enough,

$$\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty}\sum_{j=2}^{3}|\nabla^{j}z|^{2}(\xi,T)\,d\xi \;+\\
&+\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}\sum_{j=2}^{3}|\nabla^{j}z|^{2}\,d\xi\,dt \;=\\
&= O(1)\left\{\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}z|^{2}(\xi,0)\,d\xi \;+\right.\\
&\qquad+\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}|\nabla z|^{2}\,d\xi\,dt\\
&\left.+\int_{0}^{T}(1+t)^{\alpha-1}\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}|^{2}\,d\xi\,dt\right\}.
\end{aligned}\tag{71}$$

Together with (57), (63), this immediately implies

$$\begin{aligned}
&(1+T)^{N}\int_{-\infty}^{+\infty}\sum_{j=0}^{3}|\nabla^{j}z|^{2}(\xi,T)\,d\xi \;+\\
&+\int_{0}^{T}(1+t)^{N}\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}z|^{2}\,d\xi\,dt \;=\\
&= O(1)\left\{\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}z|^{2}(\xi,0)\,d\xi \;+\right.\\
&\left.+\int_{-\infty}^{+\infty}<\xi>^{N}\sum_{j=0}^{1}|\nabla^{j}z|^{2}(\xi,0)\,d\xi\right\}.
\end{aligned}\tag{72}$$

The corresponding estimates for $\nabla^{j}w$, $j = 0,1,2$ can be obtained from those for $\nabla^{l}z$, $l = 0,1,2,3$ using (19); in a similar way, we can express the time-derivatives of z in the right-hand side of (72) in terms of ξ-derivatives of z and w. We write this final result in terms of the original variables x,t: assuming (58), there exist positive constants ε, C, which are independent of $\overline{\rho},\overline{v}$ given in

(13), such that, for all $T > 0$,

$$
\begin{aligned}
&(1+T)^N \int_{-\infty}^{+\infty} \Big(\textstyle\sum_{j=0}^{3} |\nabla^j z|^2(x,T) \;+\; \sum_{j=0}^{2} |\nabla^j w|^2(x,T) \Big)\, dx \\
&+ \int_0^T (1+t)^N \int_{-\infty}^{+\infty} \Big(|\mu_x|\, z^2 \;+\; \textstyle\sum_{j=1}^{3} |\nabla^j z|^2 \;+\; \sum_{j=0}^{2} |\nabla^j w|^2 \Big)\, dx\, dt \\
&\qquad + \int_0^T (1+t)^{N-1} \int_{-\infty}^{+\infty} z^2 \, dx\, dt \;\le \qquad\qquad (73) \\
&\le\; C \cdot \Big\{ \int_{-\infty}^{+\infty} (1 + |x|^N)\, (z^2 + z_x^2 + w^2)(x,0)\, dx \\
&+ \int_{-\infty}^{+\infty} \Big(\textstyle\sum_{j=0}^{3} \left|\frac{\partial^j z}{\partial x^j}\right|^2 (x,0) \;+\; \sum_{j=0}^{2} \left|\frac{\partial^j w}{\partial x^j}\right|^2 (x,0) \Big)\, dx \Big\}
\end{aligned}
$$

provided

$$\| z(\cdot,0) \|_{H^3} \;+\; \| w(\cdot,0) \|_{H^2} \;\le\; \varepsilon\,, \qquad (74)$$

with $\varepsilon > 0$ small.

In particular, we see that, for any $2 \le p \le +\infty$,

$$\sum_{j=0}^{2} \| \nabla^j z(\cdot,t) \|_{L^p} \;+\; \sum_{j=0}^{1} \| \nabla^j w(\cdot,t) \|_{L^p} \;=\; O(1)\,(1+t)^{-N/2}\,. \qquad (75)$$

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