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REMARKS ON THE DEGENERATE VISCO-ELASTIC MODEL OF KIRCHHOFF-CARRIER FOR VIBRATIONS OF STRINGS

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Introduction.

The visco-elastic model proposed by Kirchhoff [12], Carrier [5] for vibrations of stretched strings, fixed at the ends, is given by:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_o}{\rho a} + \frac{E}{2L\rho} \int_0^L \left(\frac{\partial u}{\partial x}(x,s) \right)^2 ds \right) \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial t} = 0. \qquad (*)$$

Note that $0 \leq x \leq L$, where L is the length of the string; $t > 0$ is the time; $u(x,t)$ is the vertical displacement of the point x of the string at time t; ρ the density of the material of the string; a area of the cross section of the string; P_o the initial tension and E the Young modulus of the material. It is supposed that the string vibrates inside a medium with viscosity given by $-\gamma \frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial t}$, $\gamma \geq 0$.

We formulate a mixed problem, motivated by (*), as follows. Let us consider a bounded open set Ω of $\mathbf{R}^n$, with regular boundary Γ. By Q we represent the cylinder $\Omega \times]0,T[$, $T > 0$. Given the functions $f = f(x,t)$, $x \in \Omega$, $t \in]0,T[$, $u_o = u_o(x)$ and $u_1 = u_1(x)$, find $u\colon Q \to \mathbf{R}$ satisfying the conditions:

$$\left| \begin{array}{l} \frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2 \, dx\right) \Delta u - \gamma \, \Delta \frac{\partial u}{\partial t} = f \quad \text{on} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma = \Gamma \times]0,T[\\ u(x,0) = u_o(x), \ \ \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{on} \quad \Omega, \end{array} \right. \qquad (**)$$

with $\gamma > 0$.

Model (*) is the case $n = 1$ in (**), when

$$M(\lambda) = \frac{P_o}{\rho a} + \frac{E\lambda}{2L\rho} \qquad \lambda \geq 0.$$

In (**) $M(\lambda)$ is a numerical function defined for $\lambda \geq 0$. There are different types of questions about the mixed problem (**), depending on the conditions assumed on $M(\lambda)$, γ, u_o and u_1. In this work we are interested in the case $M(\lambda) \geq 0$, that is, $M(\lambda)$ has zeros, which correspond to the case $P_o = 0$: this means, physically, that vibrations are observed without initial tension. With this hypothesis on $M(\lambda)$ the problem (**) for the inviscid case $\gamma = 0$ was studied by Nishida [21], $M(\lambda) = \lambda$, $n = 1$; Ebihara-Medeiros-Milla Miranda [9] studied the case when $M(\lambda)$ behaves as a polynomial, using Ebihara penalty method, cf. also Yamada [27]; Arosio-Garivaldi [2]; Crippa [5]; D'Ancona-Spagnolo [6]. For analytic initial conditions u_o, u_1, see Arosio-Spagnolo [1].

In the degenerate visco-elastic case $M(\lambda) \geq 0$ for all $\lambda \geq 0$ and $\gamma > 0$, there exists global solution in t, as in Nishihara [23], Matos-Pereira [16], Muñoz Rivera [19], Nishihara-Yamada [22], Medeiros-Miranda [17]. In Nishihara-Yamada [22] the viscosity is $-2\gamma \frac{\partial u}{\partial t}$.

The non-degenerate inviscid case when $\gamma = 0$ was studied by Bernstein [3], Dickey [7], Pohozhaev [24], Lions [14], Rivera [25], Menzala [18], Matos [15].

In the present work, we consider the degenerate case $M(\lambda) = \lambda^\alpha$, $\alpha \geq 1$ with viscosity $\gamma > 0$ as Yamada [24] and Nishihara-Yamada [26]. We exploit the monotonicity of the operator $\left(\int_\Omega |\nabla u(x,t)|^2 \, dx\right)^\alpha (-\Delta u)$, $\alpha \geq 1$, obtaining, by a direct method, weak solutions for (**) with weak hypothesis on u_o, u_1 cf. Theorem 1 in this work. We also obtain the asymptotic behavior for the solutions, depending on $\alpha \geq 1$.

We knowledge Ducival D. Pereira and M. Milla Miranda for several constructive remarks on this work.

The plan of this article is the following. Section 1 is an introduction about notations and terminology. In Section 2 we prove existence of weak solutions for (**) and in Section 3 we prove the asymptotic behavior for these solutions.

1. Notation and Terminology

Let V and H be two real Hilbert spaces with scalar product and norm given, respectively, by $((\ ,\ ))$, $||\cdot||$ and $(\ ,\ )$, $|\cdot|$. We suppose that $V \subset H$ is dense and the embedding of V in H is continuous. We represent by A the operator defined by the triplet $\{V, H, ((\ ,\ ))\}$. We know that the operator A has a domain $D(A)$ dense in V and that A is self adjoint and positive. As a consequence of this method of definition of A we have $((u,v)) = (Au, v)$ for all $v \in V$. As a consequence of the spectral theorem is well defined the square root of A, that is, $A^{\frac{1}{2}}$ and $D(A^{\frac{1}{2}}) = V$. It follows that for all $u \in D(A)$ we have $a(u) = (A^{\frac{1}{2}} u, A^{\frac{1}{2}} u) = |A^{\frac{1}{2}} u|^2 = ||u||^2$. We suppose that the embeding of V in H is compact. We represent by $\langle\ ,\ \rangle$ the duality pairing between a Banach space and its dual.

The problem (**) with $M(\lambda) = \lambda^\alpha$, $\alpha \geq 1$ in this framework consists in finding a function $u\colon [0,T[\to H$ such that:

$$\left| \begin{array}{l} u'' + |A^{\frac{1}{2}} u|^{2\alpha} Au + Au' = f \\ u(0) = u_o, \quad u'(0) = u_1 \end{array} \right. \tag{1}$$

Note that we choose $\gamma = 1$ and that we represent by u' the derivative $\frac{du}{dt}$.

Let us consider the real valued function

$$\psi(\lambda) = \frac{1}{2}\int_0^\lambda s^\alpha\, ds, \qquad \alpha \geq 1, \quad \lambda > 0.$$

Then $\psi''(\lambda) = \frac{\alpha}{2}\lambda^{\alpha-1} \geq 0$, which says that $\psi(\lambda)$ is convex.

For $u \in V$, let us consider the real valued function

$$\phi(u) = \psi(a(u)) \qquad \text{for all} \qquad u \in V$$

where $a(u) = |A^{\frac{1}{2}} u|^2$ as above. The function $\phi(u)$ is convex on V, because ψ is convex. The Gateaux derivative of $\phi(u)$, represented by $\partial\phi(u)$, is an object of V', dual of V, defined by:

$$\langle \partial\phi(u), v\rangle = \frac{d}{d\lambda}\phi(u + \lambda v)\mid_{\lambda=0},$$

for all $v \in V$. We have:

$$\begin{aligned} \tfrac{d}{d\lambda}\,\phi(u+\lambda v) &= \tfrac{d}{d\lambda}\,\psi(a(u+\lambda v)) = \tfrac{1}{2}\,a(u+\lambda v)^{\alpha}\,\tfrac{d}{d\lambda}\,a(u+\lambda v) = \\ &= a(u+\lambda v)^{\alpha}\,a(u+\lambda v, v). \end{aligned}$$

Then, when $\lambda = 0$, we have:

$$\langle \partial\phi(u), v\rangle = a(u)^{\alpha}\,a(u,v) \qquad \text{for all} \qquad v \in V.$$

This implies that

$$\partial\phi(u) = |A^{\frac{1}{2}}\,u|^{2\alpha}\,Au.$$

From the general theory, cf. Brezis [4] or Gomes [10] we know that $\partial\phi(u)$ is a mapping from V into its dual V', which is monotonic, that is,

$$\langle \partial\phi(u) - \partial\phi(v), u - v\rangle \geq 0 \qquad \text{for all} \qquad u, v \in V.$$

We can prove that $\partial\phi(u) = |A^{\frac{1}{2}}\,u|^{2\alpha}\,Au$ takes bounded sets of V to bounded set of V'.

The real valued function $\lambda \to \langle \partial\phi(u+\lambda v), w\rangle$ is continuous for all $u, v, w \in V$. Because of this reason we say that $\partial\phi(u)$ is hemicontinuous.

Therefore, we write the equation (1) as

$$\left| \begin{array}{l} u'' + \partial\phi(u) + Au' = f \\ u(0) = u_o, \quad u'(0) = u_1 \end{array} \right. \tag{2}$$

In the Section 2 we prove existence of weak solutions of (2) and in Section 3 we study the asymptotic behavior of these solutions.

2. Existence of Weak Solutions

In this section we shall prove existence for the initial value problem (2), in the case of weak hypothesis on u_o, u_1. Our method is direct: we do not need approximate u_o, u_1 by regular functions and take limits of the corresponding regular solution. The monotonicity of $\partial\phi(u)$ allows us to work directly with weak hypothesis about u_o, u_1. This is clear in the proof of Theorem 1.

Theorem 1. *Suppose* $M(\lambda) = \lambda^\alpha$, $\alpha \geq 1$, $\lambda \geq 0$,

$$u_o \in V, \qquad u_1 \in H \qquad \text{and} \qquad f \in L^2(0,T;H). \tag{3}$$

Then there exists a unique vector valued function $u: [0,T] \to H$, *such that:*

$$u \in L^\infty(0,T;V) \tag{4}$$

$$u' \in L^\infty(0,T;H) \cap L^2(0,T;V) \tag{5}$$

$$u'' \in L^2(0,T;V') \tag{6}$$

$$u'' + \partial\phi(u) + Au' = f \quad \text{in} \quad L^2(0,T;V' \tag{7}$$

$$u(0) = u_o, \qquad u'(0) = u_1 \tag{8}$$

Proof: We will prove the theorem by means of the Faedo-Galerkin method. Let us consider $(w_j)_{j\in\mathbf{N}}$ and $(\lambda_j)_{j\in\mathbf{N}}$ the solutions of the spectral problem:

$$((w_j, w)) = \lambda_j(w_j, v) \quad \text{for all} \quad v \in V. \tag{9}$$

We know that the set of finite linear combinations of the eigenvectors $(w_j)_{j\in\mathbf{N}}$ is dense in V. We assume that $(w_j)_{j\in\mathbf{N}}$ is orthonormalized in H.

Let us represent by $V_m = [w_1, w_2, \ldots, w_m]$ the finite dimensional subspace V_m of V generated by the m-first eigenvectors $w_1, w_2, \ldots, w_m$. We propose the following approximate problem:

$$\left| \begin{array}{l} \text{Find} \quad u_m(t) \in V_m \quad \text{such that:} \\ (u''_m(t), w) + (\partial\phi(u_m(t)), w) + ((u'_m(t), w)) = (f, w) \\ \text{for all} \quad w \in V_m. \\ u_m(0) = u_{om} \to u_o \quad \text{in} \quad V \\ u'_m(0) = u_{1m} \to u_1 \quad \text{in} \quad H \end{array} \right. \tag{10}$$

Note that if $u_m(t) \in V_m$, then

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j. \tag{11}$$

With this notation, it follows that (10) is a system of nonlinear ordinary differential equations in the unknown functions $g_{jm}(t)$, given by (11). This system has local solution in $[0, t_m[$, $t_m < T$. To obtain a global solution on $[0, T[$, we need a priori estimates for $u_m(t)$ and $u'_m(t)$.

Estimate i: Choose $w = 2u'_m(t)$ in (10). Note that

$$2(\partial\phi(u_m(t)), u'_m(t)) = 2|A^{\frac{1}{2}} u_m(t)|^{2\alpha} a(u_m(t), u'_m(t)) = \frac{1}{\alpha+1}\frac{d}{dt}|A^{\frac{1}{2}} u_m(t)|^{2\alpha+2}.$$

We obtain:

$$\frac{d}{dt}\left(|u'_m(t)|^2 + \frac{1}{\alpha+1}|A^{\frac{1}{2}} u_m(t)|^{2\alpha+2}\right) + 2|A^{\frac{1}{2}} u'_m(t)|^2 = 2(f(t), u'_m(t)).$$

Integrating from 0 to $t < t_m$ and applying the Gronwall inequality, we obtain:

$$|u'_m(t)|^2 + |A^{\frac{1}{2}} u_m(t)|^{2\alpha+2} + \int_0^t ||u'_m(s)||^2 \, ds < C \tag{12}$$

for all t in $[0, T[$, after extending the solution from $[0, t_m[$ to $[0, T]$.

From (12) we obtain:

$$|u'_m(T)|^2 + \int_0^T ||u'_m(s)||^2 \, ds < C. \tag{13}$$

Whence,

$$u'_m \quad \text{is bounded in} \quad L^2(0, T; V). \tag{14}$$

We have that $|A^{\frac{1}{2}} u_m(t)|^{2\alpha+2} = ||u_m(t)||^{2\alpha+2}$ is bounded in $[0, T]$ by (12). This implies that $u_m(t)$ belongs to a bounded set of V, when $t \in [0, T]$. Note that $\partial\phi(u)$ takes bounded set of V into bounded sets of V'. Then we obtain:

$$||\partial\phi(u_m(t))||_{V'} < C_1 \quad \text{in} \quad [0, T]. \tag{15}$$

From (12), (13) and (15) it follows that:

$$\left|\begin{array}{l} u_m \quad \text{is bounded in} \quad L^\infty(0, T; V) \\ u'_m \quad \text{is bounded in} \quad L^\infty(0, T; H) \cap L^2(0, T; V) \\ \partial\phi(u_m) \quad \text{is bounded in} \quad L^\infty(0, T; V') \\ u'_m(T) \quad \text{is bounded in} \quad H \\ u_m(T) \quad \text{is bounded in} \quad V \end{array}\right. \tag{16}$$

Estimate ii. Now we estimate the second derivative $u''_m(t)$ using a projection method, cf. Lions [14].

Let $P_m: H \to V_m \subset V$ be the orthogonal projection operator defined by:

$$h \in H \longrightarrow P_m h = \sum_{j=1}^{m} (h, w_j) w_j .$$

It follows that P_m is bounded in $\mathcal{L}(V, V)$, the space of continuous linear mappings from V to V. The adjoint P_m^* is bounded in $\mathcal{L}(V', V')$. For all $w \in V_m$ we have $P_m w = w$. Then from the approximate equation, we can write:

$$(u_m''(t), P_m w) + (\partial\phi(u_m(t), P_m w) + (Au_m'(t), P_m w) = (f, P_m w)$$

whence,

$$(u_m''(t), w) + (P_m^* \partial\phi(u_m(t)), w) + (P_m^* A u_m'(t), w) = (P^* f, w)$$

for all $w \in V_m$. It follows that

$$u_m''(t) = -P_m^* \partial\phi(u_m(t)) - P_m^* A u_m'(t) + P^* f$$

in V_m. We obtain,

$$u_m'' \quad \text{is bounded in} \quad L^2(0, T; V'). \tag{17}$$

From (16) and (17) we obtain a subsequence (u_μ) of (u_m), such that:

$$\left| \begin{array}{l} u_\mu \rightharpoonup u \quad \text{weak star} \quad L^\infty(0, T; V) \\ u_\mu' \rightharpoonup u' \quad \text{weak star} \quad L^\infty(0, T; H) \\ u_\mu' \rightharpoonup u' \quad \text{weak} \quad L^2(0, T; V) \\ u_\mu'' \rightharpoonup u'' \quad \text{weak} \quad L^2(0, T; V') \\ u_\mu(T) \rightharpoonup u(T) \quad \text{weak} \quad V \\ u_\mu'(T) \rightharpoonup u'(T) \quad \text{weak} \quad H \\ \partial\phi(u_\mu) \rightharpoonup \chi \quad \text{weak star} \quad L^\infty(0, T; V') \end{array} \right. \tag{18}$$

Remark 1. Note that from (18) it follows that $u \in C_w^o([0,T]; V) \cap C^o([0,T]; H)$ and $u' \in C_w^o([0,T]; V) \cap C^o([0,T]; V')$. Then it makes sense to evaluate $u(0)$ and $u'(0)$. Also from (18) we have $u(0) = u_o$ and $u'(0) = u_1$.

Limits of Approximate Solutions

From (18) line two, the subsequence (u_μ') converges to u' weakly in $L^2(0, T; H)$. This means that

$$\int_0^T (u_\mu'(t), w(t)) dt \longrightarrow \int_0^T (u'(t), w(t)) dt$$

for all $w \in L^2(0,T;H)$. If $w(t) = \theta(t)v$, for $v \in H$ and $\theta \in \mathcal{D}(0,T)$, the above convergence of integrals implies:

$$(u'_\mu(t), v) \rightharpoonup (u'(t), v) \quad \text{in} \quad \mathcal{D}'(0,T).$$

Whence,

$$\frac{d}{dt}(u'_\mu(t), v) \rightharpoonup \frac{d}{dt}(u'(t), v) \quad \text{in} \quad \mathcal{D}'(0,T), \tag{19}$$

for all $v \in H$.

From $(18)_7$ we have:

$$\int_0^T (\partial\phi(u_\mu), w) dt \longrightarrow \int_0^T \langle \chi, w \rangle \, dt \tag{20}$$

for all $w \in L^1(0,T;V)$, whose strong dual is identified with $L^\infty(0,T;V')$.

From $(18)_3$ it follows that:

$$\int_0^T ((u'_\mu(t), w)) dt \longrightarrow \int_0^T ((u'(t), w)) dt \tag{21}$$

for all $w \in L^2(0,T;V)$.

If we consider (u_μ) in the approximate equation (10) and let μ goes to infinity, we obtain:

$$\frac{d}{dt}(u'(t), v) + \langle \chi(t), v \rangle + ((u'(t), v)) = (f, v) \tag{22}$$

in the sense of $\mathcal{D}'(0,T)$, for all $v \in V$.

If u is solution in the sense of (22), we can prove that u satisfies:

$$u'' + \chi + Au' = f \quad \text{in} \quad L^2(0,T;V').$$

Convergence of the Non Linear Term

To complete the proof of Theorem 1, we need to prove that $\chi = \partial\phi(u)$. In fact, we prove this fact by means of the monotonicity of $\partial\phi(u)$. We have for all $v \in L^\infty(0,T;V)$:

$$X_\mu = \int_0^T \langle \partial\phi(u_\mu) - \partial\phi(v), u_\mu - v \rangle \, dt \geq 0$$

for all μ. Then:

$$0 \leq X_\mu = \int_0^T (\partial\phi(u_\mu), u_\mu)dt - \int_0^T (\partial\phi(u_\mu), v)dt - \int_0^T \langle \partial\phi(v), u_\mu - v\rangle\, dt. \quad (23)$$

The only difficulty lies in limits in the first integral at the right hand side of (23). In fact, in the approximate equation (10) with $m = \mu$, $w = u_\mu$, integrating in $]0, T[$, we obtain:

$$\begin{aligned} &\textstyle\int_0^T (u''_\mu, u_\mu)dt + \int_0^T (\partial\phi(u_\mu), u_\mu)dt + \\ &+ \textstyle\int_0^T ((u'_\mu, u_\mu))dt = \int_0^T (f, u_\mu)dt. \end{aligned} \quad (24)$$

Note that

$$\int_0^T (u''_\mu, u_\mu)dt = (u'_\mu(T), u_\mu(T)) - (u'_\mu(0), u_\mu(0)) - \int_0^T |u'_\mu(t)|^2\, dt \quad (25)$$

and

$$\int_0^T ((u'_\mu, u_\mu))dt = \frac{1}{2}||u_\mu(T)||^2 - \frac{1}{2}||u_\mu(0)||^2. \quad (26)$$

Substituting (25) and (26) in (23) we obtain:

$$\begin{aligned} 0 \ \leq\ & (u'_\mu(0), u_\mu(0)) - (u'_\mu(T), u_\mu(T)) + \\ +\ & \textstyle\int_0^T |u'_\mu(t)|^2\, dt + \frac{1}{2}||u_\mu(0)||^2 - \frac{1}{2}||u_\mu(T)||^2 + \int_0^T (f, u_\mu)dt - \\ -\ & \textstyle\int_0^T (\partial\phi(u_\mu), v)dt - \int_0^T \langle \partial\phi(v), u_\mu - v\rangle\, dt. \end{aligned} \quad (27)$$

Taking the superior limit in (27) we have:

$$\begin{aligned} &0 \leq \overline{\lim_{\mu\to\infty}}(u'_\mu(0), u_\mu(0)) + \overline{\lim_{\mu\to\infty}}[-(u'_\mu(T), u_\mu(T))] + \\ &+ \overline{\lim_{\mu\to\infty}} \int_0^T |u'_\mu(t)|^2 dt + \frac{1}{2}\overline{\lim_{\mu\to\infty}}||u_\mu(0)||^2 + \overline{\lim_{\mu\to\infty}}[-||u_\mu(T)||^2] + \\ &+ \overline{\lim_{\mu\to\infty}} \int_0^T (f, u_\mu)dt + \overline{\lim_{\mu\to\infty}}\Big[\int_0^T (\partial\phi(u_\mu), v)dt\Big] + \\ &+ \overline{\lim_{\mu\to\infty}}\Big[-\int_0^T \langle \partial\phi(v), u_\mu - v\rangle\, dt\Big]. \end{aligned} \quad (28)$$

There is some difficulty in the term

$$\overline{\lim}_{\mu\to\infty} \int_0^T |u'_\mu(t)|^2\, dt. \quad (29)$$

It happens that the viscosity term Au' in the equation gives an extra estimate in order to get strong convergence of (u'_μ) in $L^2(0, T; H)$ and this estimate solve the difficulty in (29).

We have by estimates (16) and (17):

$$u'_\mu \quad \text{and} \quad u''_\mu \quad \text{bounded, respectively, in} \quad L^2(0,T;V) \quad \text{and} \quad L^2(0,T;V'). \tag{30}$$

Since $V \subset H \subset V'$ is continuous, dense and $V \subset H$ is compact, it follows by (30) and by Aubin-Lions compactness theorem [13], that there exists a subsequence, still denoted by (u_μ), such that $u'_\mu \to u'$ strongly in $L^2(0,T;H)$, that is, in (29) we have:

$$\lim_{\mu\to\infty} \int_0^T |u'_\mu(t)|^2\, dt = \int_0^T |u'(t)|^2\, dt. \tag{31}$$

Remark 2. Note that we have

$$\lim(u'_\mu(0), u_\mu(0)) = (u'(0), u(0)).$$

We also have $\lim ||u_\mu(0)||^2 = ||u(0)||^2$.

Remark 3. We have that $u_\mu(T)$ bounded in V, $V \subset H$ is compact. Then there exists a subsequence $(u_\mu(T))$ such that $\lim u_\mu(T) = \xi = u(T)$, strongly in H. We have $\lim u'_m(T) = u'(T)$ weakly in H. Then,

$$\lim(u'_\mu(T), u_\mu(T)) = (u'(T), u(T)).$$

Remark 4. By $(18)_5$ we have $\lim u_\mu(T) = u(T)$ weakly in V. It follows that $||u(T)||^2 \leq \underline{\lim}\, ||u_\mu(T)||^2$. Then, $-||u(T)|| \geq -\underline{\lim}||u_\mu(T)||^2 = \overline{\lim}(-||u_\mu(T)||^2)$.

It follows from Remarks 1, 2, 3, 4 and from the inequality (2828), for $\mu \to \infty$, that

$$\begin{aligned}&(u'(0), u(0)) - (u'(T), u(T)) + \textstyle\int_0^T |u'(t)|^2\, dt + \frac{1}{2}||u(0)||^2 - \\ &-\tfrac{1}{2}||u(T)||^2 + \textstyle\int_0^T (f,u)dt - \int_0^T \langle \chi, v\rangle\, dt - \int_0^T \langle \partial\phi(v), u-v\rangle\, dt \geq 0.\end{aligned} \tag{32}$$

Multiplying the approximate equation by θ in $C^1([0,T];R)$ and integrating on $]0,T[$, we get:

$$\begin{aligned}&(u'_\mu(T), w\theta(T)) - (u'_\mu(0), w\theta(0)) - \textstyle\int_0^T (u'_\mu, w\theta')dt + \\ &+ \textstyle\int_0^T \langle \partial\phi(u_\mu), w\theta\rangle\, dt + \int_0^T ((u'_\mu, w\theta))dt = \int_0^T (f, w\theta)dt.\end{aligned}$$

Let $\mu \to \infty$:

$$\begin{aligned}&(u'(T), w\theta(T)) - (u'(0), w\theta(0)) - \textstyle\int_0^T (u', w\theta')dt + \\ &+ \textstyle\int_0^T (\chi, w\theta)dt + \int_0^T ((u', w\theta))dt = \int_0^T (f, w\theta)dt.\end{aligned} \tag{33}$$

The identity (33) is true for all $w \in V$ and $\theta \in C^1([0,T];\mathbf{R})$. Then it is true for all functions in G instead of $w\theta$, where G is defined by:

$$G = \{\psi \in L^2(0,T;V); \psi' \in L^2(0,T;H)\}.$$

Note that the solution $u \in G$, so that we have from (33):

$$\begin{array}{l}(u'(T),u(T)) - (u'(0),u(0)) - \int_0^T |u'(t)|^2\,dt + \\ + \int_0^T \langle \chi, u\rangle\,dt + \frac{1}{2}||u(T)||^2 - \frac{1}{2}||u(0)||^2 = \int_0^T (f,u)dt.\end{array} \tag{34}$$

By (32) and (34) we obtain:

$$\int_0^T \langle \chi - \partial\phi(v), u - v\rangle\,dt \geq 0.$$

If $v = u + \lambda w$, $w \in L^\infty(0,T;V)$, $\lambda > 0$ we obtain:

$$\int_0^T \langle \chi - \partial\phi(u+\lambda w), w\rangle\,dt \geq 0.$$

By the hemicontinuity of $\partial\phi(u)$ we get:

$$\int_0^T \langle \chi - \partial\phi(u), w\rangle\,dt \geq 0,$$

for all $w \in L^\infty(0,T;V)$. Taking $-w$ we conclude that

$$\int_0^T \langle \chi - \partial\phi(u), w\rangle\,dt = 0 \quad \text{for all} \quad w \in L^\infty(0,T;V)$$

what implies $\chi = \partial\phi(u)$. Then u satisfies:

$$\frac{d}{dt}(u'(t),v) + \langle \partial\phi(u), v\rangle + ((u'(t),v)) = (f(t),v)$$

in the sense of $\mathcal{D}'(0,T)$, for all $v \in V$.

□

To obtain uniqueness we employ the usual method of energy.

3. Asymptotic Behavior

Let us consider $f = 0$ in the equation (1). Then the corollary below follows from Theorem 1.

Corollary 1. *Assume $M(\lambda) = \lambda^\alpha$, $\alpha \geq 1$ for $\lambda \geq 0$,*

$$u_o \in V, \qquad u_1 \in H. \tag{35}$$

Then there exists a function $u: [0,\infty] \to H$ such that:

$$u \in L^\infty(0,\infty;V) \tag{36}$$

$$u' \in L^\infty(0,\infty;H) \cap L^2(0,\infty;V) \tag{37}$$

$$u'' \in L^2(0,\infty;V') \tag{38}$$

$$u'' + \partial\phi(u) + Au' = 0 \quad \text{in} \quad L^2(0,\infty;V') \tag{39}$$

$$u(0) = u_o, \quad u'(0) = u_1 \tag{40}$$

If we take scalar product of (39) with u', we obtain:

$$\frac{1}{2}\frac{d}{dt}|u'(t)|^2 + \frac{1}{2}\frac{1}{\alpha+1}\frac{d}{dt}|A^{\frac{1}{2}}\, u(t)|^{2\alpha+2} + |A^{\frac{1}{2}}\, u'|^2 = 0. \tag{41}$$

If we consider

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2(\alpha+1)}|A^{\frac{1}{2}}\, u(t)|^{2\alpha+2} \tag{42}$$

and call it the energy of the system, we obtain from (41):

$$E'(t) + |A^{\frac{1}{2}}\, u'|^2 = 0, \tag{43}$$

which says that $E'(t) \leq 0$, proving that $E(t)$ is not increasing. This sections is dedicated to study the behavior of the energy $E(t)$ when $t \to \infty$. We use the Nakao method [20], cf. also Zuazua [28] or Haraux-Zuazua [11].

Let us denote $p = 2\alpha + 2$, and $||u||^2 = |A^{\frac{1}{2}}\, u|^2$, which is the norm in V. Then the energy (42) can be written as:

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{p}||u(t)||^p. \text{ bis} \tag{44}$$

By (43), we obtain:

$$E(t) + \int_0^t ||u'(s)||^2 \, ds = E(0), \tag{45}$$

which says that the energy is bounded.

If we integrate (43) from t to $t+1$, we obtain:

$$\int_t^{t+1} ||u'(s)||^2 \, ds = E(t) - E(t+1). \tag{46}$$

By the Poincaré inequality, we obtain:

$$\lambda_1 |v|^2 \leq ||v||^2,$$

norms in H and V, respectively, λ_1 the first eigenvalue of A. Then, from (45) we obtain:

$$\int_t^{t+1} |u'(s)|^2 \leq c[E(t) - E(t+1)] = F(t)^2 \tag{47}$$

where $c = \lambda_1^{-1}$.

Remark 5. The idea of Nakao consists in proving the inequality:

$$\sup_{t \leq s \leq t+1} E(s)^{1+\beta} \leq C_o(E(t) - E(t+1)) \tag{48}$$

for all $t \geq 0$; with $\beta \geq 0$, $C_o > 0$. Once this inequality is proved we obtain:

i) if $\beta = 0$, there exist C_1, δ_1 positive constants, such that

$$E(t) \leq C_1 \, e^{-\delta_1 t} \quad \text{for all} \quad t \geq 1$$

ii) if $\beta > 0$, there exists $C_2 > 0$ such that

$$E(t) \leq C_2(1+t)^{-\frac{1}{\beta}} \quad \text{for all} \quad t \geq 0.$$

Our goal is to prove that the energy (42) or (42 bis) satisfies the inequality (47) with $\beta > 0$. In fact, the first part of energy satisfies (46). We need to analyze the second part as follows.

Let us consider the interval $[t, t+1]$ broken into four equal parts and take $[t, t+\frac{1}{4}]$, $[t+\frac{3}{4}, t+1]$. We have, from (46) and from the mean value theorem for integrals:

$$\frac{1}{4}|u'(t_1)|^2 = \int_t^{t+\frac{1}{4}} |u'(s)|^2 \, ds < F(t)^2 \qquad t < t_1 < t + \frac{1}{4}.$$

Then,

$$|u'(t_1)| < 2\,F(t). \tag{49}$$

By the same argument, there exists t_2 satisfying $t + \frac{3}{4} < t_2 < t+1$, such that:

$$|u(t_2)| < 2\,F(t). \tag{50}$$

The next step is dedicated to control $\frac{1}{p}||u||^p$. Taking inner product of both sides of (7) with u, we obtain:

$$\frac{d}{dt}(u'(t), u(t)) - ||u'(t)||^2 + ||u(t)||^p + ((u'(t), u(t))) = 0.$$

Integrating from t_1 to t_2, points obtained by applying the mean value theorem, we obtain:

$$\begin{aligned} \textstyle\int_{t_1}^{t_2} ||u(s)||^p\,ds \;&=\; -(u'(t_2), u(t_2)) + (u'(t_1), u(t_1)) + \\ &\quad + \textstyle\int_{t_1}^{t_2} |u'(s)|^2\,ds - \int_{t_1}^{t_2}((u'(s), u(s)))ds \end{aligned}$$

or

$$\begin{aligned} \textstyle\int_{t_1}^{t_2} ||u(s)||^p\,ds \;&\leq\; |u'(t_2)|\;|u(t_2)| + |u'(t_1)|\;|u(t_1)| + \\ &+\; \textstyle\int_{t_1}^{t_2} |u'(s)|^2\,ds + \int_{t_1}^{t_2} ||u'(s)||\;||u(s)||ds. \end{aligned}$$

By the Poincaré inequality,

$$\begin{aligned} \textstyle\int_{t_1}^{t_2} ||u(s)||^p\,ds \;&\leq\; C_1|u'(t_2)|\;||u(t_2)|| + C_1|u'(t_1)|\;||u(t_1)|| + \\ &+\; \textstyle\int_{t_1}^{t_2} |u'(s)|^2\,ds + \int_{t_1}^{t_2} ||u'(s)||\;||u(s)||ds. \end{aligned} \tag{51}$$

Remark 6. We have:

$$C_1|u'(t_2)|\;||u(t_2)|| + C_1|u'(t_1)|\;||u(t_2)|| \leq 4C_1\,F(t) \sup_{t\leq s\leq t+1} ||u(s)||.$$

Remark 7. Assume $\frac{1}{q} + \frac{1}{p} = 1$, where p is the energy index. We use $ab \leq \frac{1}{q}\,a^q + \frac{1}{p}\,b^p$, for a, b positive. Then,

$$\int_{t_1}^{t_2} ||u'(s)||\;||u(s)||ds \leq \frac{1}{q}\int_{t_1}^{t_2} ||u'(s)||^q\,ds + \frac{1}{p}\int_{t_1}^{t_2} ||u(s)||^p\,ds.$$

By Remarks 6 and 7, we modify (50) obtaining:

$$\begin{aligned} \left(1 - \tfrac{1}{p}\right) \textstyle\int_{t_1}^{t_2} ||u(s)||^p\,ds \;&\leq\; 4C_1\,F(t)\sup_{t\leq s\leq t+1} ||u(s)|| + F^2(t) + \\ &+\; \tfrac{1}{q}\textstyle\int_{t_1}^{t_2} ||u'(s)||^q\,ds. \end{aligned} \tag{52}$$

Since $||u(s)|| \leq C_2\, E(s)^{1/p}$, $C_2 = p^{\frac{1}{p}}$, we obtain, from (51):

$$\begin{aligned}\left(1-\tfrac{1}{p}\right)\textstyle\int_{t_1}^{t_2}||u(s)||^p\,ds \;&\leq\; 4C_3\,F(t)\sup_{t\leq s\leq t+1}E(s)^{1/p}+F^2(t)+\\ &+\;\tfrac{1}{q}\textstyle\int_{t_1}^{t_2}||u'(s)||^q\,ds.\end{aligned} \tag{53}$$

Remark 8. If $\beta q = 2$, consider α such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Then,

$$\int_{t_1}^{t_2}||u'(s)||^q\,ds \leq (t_2-t_1)^{1/\alpha}\left(\int_{t_1}^{t_2}||u'(s)||^{\beta q}\right)^{1/\beta}$$

or

$$\int_{t_1}^{t_2}||u'(s)||^q\,ds \leq \left(\int_{t_1}^{t_2}||u'(s)||^2\,ds\right)^{q/2} \leq \left(\int_{t}^{t+1}||u'(s)||^2\,ds\right)^{q/2}.$$

By the identity (45), the inequality (46) and the Remark 8, we obtain:

$$\int_{t_1}^{t_2}||u'(s)||^q\,ds \leq C_4\,F(t)^q. \tag{54}$$

From (52) and (53) we obtain:

$$\left(1-\frac{1}{p}\right)\int_{t_1}^{t_2}||u(s)||^p\,ds \leq 4C_3\,F(t)\sup_{t\leq s\leq t+1}E(s)^{1/p}+F^2(t)+\frac{1}{q}C_4\,F(t)^q.$$

Let us define:

$$G(t)^2 = 4C_5\,F(t)\sup_{t\leq s\leq t+1}E(s)^{1/p}+C_6\,F(t)^q, \tag{$***$}$$

and we obtain:

$$\int_{t_1}^{t_2}||u(s)||^p \leq G(t)^2+F(t)^2. \tag{55}$$

Integrating both sides of (42 bis) on $]t_1,t_2[$, we obtain

$$\int_{t_1}^{t_2}E(s)ds = \frac{1}{2}\int_{t_1}^{t_2}|u'(s)|^2\,ds+\frac{1}{p}\int_{t_1}^{t_2}||u(s)||^p\,ds. \tag{56}$$

From (46) and (54), substituting in (55), we get:

$$\int_{t_1}^{t_2}E(s)ds \leq \frac{1}{p}G(t)^2+C_7\,F(t)^2. \tag{57}$$

By the mean value theorem for integrals, there exists a point $t_1 \leq t^* \leq t_2$ such that:

$$\int_{t_1}^{t_2}E(s)ds = (t_2-t_1)E(t^*) \geq \frac{1}{2}\,E(t^*).$$

Therefore,

$$E(t^*) \leq \frac{2}{p}\,G(t)^2 + 2C_7\,F(t)^2. \tag{58}$$

For any $\tau_1 < \tau_2$ we obtain:

$$E(\tau_1) \leq E(\tau_2) + \int_{\tau_1}^{\tau_2} ||u'(s)||^2\,ds.$$

Taking $\tau_2 = t^*$, for any $\tau_1 < t^*$, we have:

$$E(\tau_1) \leq E(t^*) + \int_{\tau_1}^{t^*} ||u'(s)||^2\,ds$$

then,

$$\sup_{t<s<t+1} E(s) \leq E(t^*) + \int_t^{t+1} ||u'(s)||^2\,ds.$$

Whence, from (57) and (45), it follows that:

$$\sup_{t<s<t+1} E(s) \leq \frac{2}{p}\,G(t)^2 + 2C_7\,F(t)^2 + \frac{1}{c}\,F(t)^2. \tag{59}$$

Substituting (***) in (58) we obtain:

$$\sup_{t\leq s\leq t+1} E(s) \leq 4C_5\,F(t)\left(\sup_{t\leq s\leq t+1} E(s)\right)^{1/p} + C_6\,F(t)^q + C_8\,F(t)^2.$$

By Hölder inequality with $\frac{1}{p}+\frac{1}{q}=1$, we have:

$$\sup_{t\leq s\leq t+1} E(s) \leq C_9\,F(t)^q + C_8\,F(t)^2$$

or

$$\sup_{t\leq s\leq t+1} E(s) \leq F(t)^q(C_9 + C_8\,F(t)^{2-q}),$$

then

$$\sup_{t\leq s\leq t+1} E(s)^{2/q} \leq C_{10}\,F(t)^2,$$

because the energy is bounded. Note that $F(t)^2 = C(E(t) - E(t+1))$ and $\frac{2}{q} = 1 + \frac{\alpha}{\alpha+1}$, because $p = 2\alpha + 2$. Then, by Nakao [20], (note Remark 5 in this section) we obtain the asymptotic behavior:

$$E(t) \leq C\,(1+t)^{-\frac{\alpha+1}{\alpha}}, \qquad t \geq 0 \tag{60}$$

□

To complete the proof of the asymptotic behavior for weak solutions, we need to obtain the asymptotic estimate (59) for an weak solution, because (59) is true for approximated solutions u_m. Therefore, the next step is dedicated to prove that (59) is still true for an weak solution given by Theorem 1.

In fact, take any t_o in $[0, T[$, fixed. Then, from (59) we obtain

$$\frac{1}{2}|u_m(t_o)|^2 + \frac{1}{p}||u_m(t_o)||^2 \leq C\,(1+t_o)^{-\frac{\alpha+1}{\alpha}}. \tag{61}$$

From the estimates obtained in Section 2, follows:

$$\left| \begin{array}{lll} u_m \rightharpoonup u & \text{weak star} & L^\infty(0,T;V) \\ u'_m \rightharpoonup u' & \text{weak star} & L^\infty(0,T;H) \\ u_m(t_o) \rightharpoonup \xi & \text{weak in} & V \\ u'_m(t_o) \rightharpoonup \eta & \text{weak in} & H \end{array} \right. \tag{62}$$

Remark 9. We have

$$u_m(t) - u_m(s) = \int_t^s u'_m(\sigma)d\sigma.$$

From Section 2, taking H norm of both sides of the last equality it follows:

$$|u_m(t) - u_m(s)| \leq C|t-s|,$$

what proves that (u_m) is equicontinuous in $[0, T]$.

We also have $|u_m(t)| < C$ on $[0, T]$. Then, from Arzela-Ascoli theorem we have $\lim u_m = u$ in $C^o([0,T]; H)$. Consequently $\lim u_m(t_o) = u(t_o)$ in H, that is $\xi = u(t_o)$.

Taking inferior limit in (60), we obtain:

$$\frac{1}{2}|\eta|^2 + \frac{1}{p}||u(t_o)||^p + C\,(1+t_o)^{-\frac{\alpha+1}{\alpha}}. \tag{63}$$

We need to prove that $\eta = u'(t_o)$. In fact, let be $\theta \in C^o([0,T]; \mathbf{R})$ defined by:

$$\theta(t) = \left| \begin{array}{lll} 1 & \text{if} & 0 < t < t_o \\ -\frac{t}{\delta} + \frac{t_o+\delta}{\delta} & \text{if} & t_o \leq t \leq t_o + \delta \\ 0 & \text{if} & t_o + \delta < t \leq T \end{array} \right.$$

Let us consider the subsequence (u_μ) of the Section 2. We obtain from the approximated equation:

$$(u_\mu''(t), w) + (\partial\phi(u_\mu(t)), w) + ((u_\mu(t), w)) = 0 \tag{64}$$

for all $w \in V_m$. Multiply both sides of (63) by θ and integrate on $[0, T]$. We get:

$$\begin{aligned} -(u_\mu(t_o), w) &+ \tfrac{1}{\delta}\textstyle\int_{t_o}^{t_o+\delta}(u_\mu'(t), w)dt + \int_{t_o}^{t_o+\delta}(\delta\phi(u_\mu(t)), w)\theta(t)dt + \\ &+ \textstyle\int_{t_o}^{t_o+\delta}((u_\mu'(t), w))\theta(t)dt = 0. \end{aligned}$$

Taking the limits when $\mu \to \infty$, observing (18) in Section 2, we have:

$$\begin{aligned} -(\eta, w) &+ \tfrac{1}{\delta}\textstyle\int_{t_o}^{t_o+\delta}(u'(t), w)dt + \int_{t_o}^{t_o+\delta}(\delta\phi(u), w)\theta(t)dt + \\ &+ \textstyle\int_{t_o}^{t_o+\delta}((u'(t), w))\delta(t)dt = 0. \end{aligned}$$

Taking the limits in the above equality when $\delta \to 0$, we find $\eta = u'(t_o)$ in V.

Note that the above argument is true for any t in $[0, T[$. In T it is sufficient to consider

$$\theta(t) = \left| \begin{array}{lll} 0 & \text{in} & 0 \le t \le t - \delta, \quad \text{for} \quad \delta > 0 \\ \text{linear in} & T - \delta \le t \le T & \end{array} \right.$$

Consequently, if u is an weak solution of Theorem 1, we have the behavior (60) on $[0, T]$, for all $0 < T < \infty$.

□

We summarize the above argument in the following:

Theorem 2. *The weak solutions u obtained in Theorem 1 has the following asymptotic behavior:*

$$|u'(t)|^2 + \frac{1}{\alpha+1}|A^{\frac{1}{2}}u(t)|^{2\alpha+2} \le C(1+t)^{-\frac{\alpha+1}{\alpha}},$$

for all $t > 0$.

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