Matemática Contemporânea, Vol. 8, 239–268
http://doi.org/10.21711/231766361995/rmc812

A POST-PROCESSING TECHNIQUE TO APPROXIMATE THE VELOCITY FIELD IN MISCIBLE DISPLACEMENT SIMULATIONS

Sandra M. C. Malta **Abimael F. D. Loula**
Eduardo L. M. Garcia

Abstract

Finite element methods are used to solve a nonlinear system of partial differential equations which models incompressible miscible displacement of one fluid by another in a porous medium. Our main purpose is to analyse the influence of the velocity approximation on the calculation of the concentration. A sequentially implicit time discretization is defined. The pressure is approximated by a classical Galerkin's method and then accurate velocities of the mixture are obtained via a post-processing technique which involves the conservation of the mass and Darcy's law. To solve the concentration equation we use a streamline upwind Petrov-Galerkin (**SUPG**) method. Rates of convergence for pressure, velocity field and concentration are exhibited. Numerical results are presented to confirm the rates of convergence predicted for pressure and velocity approximations. To show the influence of the velocity on the concentration an oil recovery process is simulated in a five spot geometry.

1. Introduction

We study finite element approximations for a system of nonlinear partial differential equations governing incompressible miscible displacement in two dimensional porous media. The mathematical model consists of an elliptic system coming from the conservation of mass and Darcy's law and a degenerate parabolic equation expressing the conservation of the injected fluid (concentration equation). The concentration of the injected fluid in the mixture is the variable of main interest. However, special attention must be paid to the velocity field, which is responsible for the transport of the mixture. Let Ω be a bounded domain in the plane $\mathbf{R}^2$ with smooth boundary $\partial\Omega$ and $T > 0$ a fixed number. Our differential system, under appropriate physical assumptions, is given by [10-11,19]

$$\text{div } \mathbf{u} = f \quad \text{on} \quad \Omega \times (0,T), \tag{1.1}$$

$$\mathbf{u} = -\frac{k(x)}{\mu(c)}\nabla p \quad \text{on} \quad \Omega \times (0,T), \tag{1.2}$$

with the boundary condition

$$\mathbf{u} \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.3}$$

and

$$\phi\frac{\partial c}{\partial t} + \text{div }(c\mathbf{u}) - \text{div }(D\nabla c) = \hat{c}f \quad \text{on} \quad \Omega \times (0,T), \tag{1.4}$$

with the boundary and initial condition

$$D\nabla c \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.5}$$

$$c(x,0) = c_0(x) \quad \text{on} \quad \Omega, \tag{1.6}$$

where p and $\mathbf{u}$ are the pressure and Darcy's velocity of the mixture, $\phi = \phi(x)$ and $k = k(x)$, the porosity and permeability of the medium, respectively, $\nu = (\nu_1, \nu_2)$ denotes the exterior normal to $\partial\Omega$, f denotes source and sink terms. The diffusion-dispersion tensor D will be considered as in [10], i.e.,

$$D = D(\mathbf{u}) = d_m\mathbf{I} + |\mathbf{u}|\{d_l E(\mathbf{u}) + d_t E^{\perp}(\mathbf{u})\}, \tag{1.7}$$

with

$$\begin{aligned} E(\mathbf{u}) &= \tfrac{1}{|\mathbf{u}|^2}\mathbf{u} \otimes \mathbf{u}, \\ E^{\perp}(\mathbf{u}) &= \mathbf{I} - E(\mathbf{u}), \end{aligned} \tag{1.8}$$

where d_m, d_l and d_t are, respectively, molecular diffusion, longitudinal and transverse dispersion coefficients. Normally dispersion is physically more important than the molecular diffusion; also, d_l is usually considerably larger than d_t, and we shall make this assumption in our analysis. Since $p(x,t)$ is determined up to an arbitrary additive constant, we normalize it by imposing the condition

$$\int_{\Omega} p(x,t)dx = 0, \quad t \in (0,T). \tag{1.9}$$

Finally, we note that in Equation (1.2) $\mu = \mu(c)$ is the local viscosity of the mixture which depends on the concentration c. Such dependence is very important in the emergence of viscous fingering, displacement efficiency, and ultimate oil recovery. In the literature the most widely used form to represent μ is

$$\mu(c) = \mu(0)\left[1 - c + M^{\frac{1}{4}}c\right]^{-4}, \quad c \in [0,1], \tag{1.10}$$

where $M = \mu(0)/\mu(1)$ is the mobility ratio. It has been observed that for $M > 1$ displacement fronts may become physically unstable [21]. In this work we will only consider the case $M \geq 1$. The other case ($M < 1$) is not so relevant from the numerical point of view [15-16].

It is important to note that velocity and not the pressure appears explicitly in Equation (1.4). Consequently, that is the motivation for using an efficient numerical method to approximate this field, which is the main purpose of the present work.

A brief outline the paper follows. In §2 after introducing the basic notation and assumptions a continuous-space approximate problem is defined by using a sequentially implict time discretization. In §3 three different finite element formulations for the elliptic equations are present. One is based on Galerkin's method for the pressure, another on a mixed which approximates pressure and velocity simultaneously and finally a post-processing technique. In §4 the **SUPG** method is applied to approximate the concentration. In §5 the post-processing approach is combined with a technique of subtraction of singularity to treat the case of point sources and sinks. Numerical results are presented in section §6 confirming the rates of convergence of velocity and pressure approximation in a particular situation with known solution and illustrating the influence of the velocity on the concentration approximation for different mobility ratios.

2. Preliminaries

2.1. Notations and Definitions

To introduce the basic notation, we define the following Sobolev spaces [1] on Ω

$$L^p(\Omega) = \left\{ f;\ \int_\Omega |f|^p dx < \infty \right\},$$

$$W^{m,p}(\Omega) = \left\{ f\ \in L^p(\Omega);\ \frac{\partial^\alpha f}{\partial x^\alpha} \in L^p(\Omega);\ |\alpha| \leq m \right\}$$

with their respective norms

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{\frac{1}{p}},$$

$$\|f\|_{m,p} = \left(\sum_{|\alpha|\leq m} \int_\Omega |\frac{\partial^\alpha f}{\partial x^\alpha}|^p dx \right)^{\frac{1}{p}}$$

and suitable modifications for $p = \infty$. When $p = 2$, we have $\|f\|_{m,2} = \|f\|_{H^m(\Omega)} = \|f\|_m$. The case $m = 0$, i.e., $H^0(\Omega) = L^2(\Omega)$ has its norm denoted by $\|\cdot\|_0 = \|\cdot\|$, and inner product

$$(f, g) = \int_\Omega fg dx.$$

2.2. Regularity of the Data

Initially we impose very strong hypotheses in the analysis of system (1.1)-(1.10). Some of these hypotheses will be weakened in section 5. The functions $k(x), \phi(x)$ and $f(x,t)$ are assumed bounded and measurable in the cylinder $Q = \Omega \times (0,T)$. This means that we shall tacitly assume that the external sources/sinks are not concentrated at wells, but smoothly distributed over Ω. Mathematically, we have

$$0 < k^* \leq k(x) \leq k_* < \infty, \tag{2.1}$$

$$0 < \phi^* \leq \phi(x) \leq \phi_* < \infty. \tag{2.2}$$

Introducing the function $\lambda(\cdot) : \mathbf{R} \rightarrow \mathbf{R}$ given by:

$$\lambda(c) := k(x)/\mu(c), \tag{2.3}$$

which is well-defined since $\mu(c)^{-1}$ is a polynomial function and can be extended to all real set $\mathbf{R}$, considering the bounds (2.1)-(2.2) and definition (1.10), we easily see that

$$k^*/\mu(0) < \lambda(c) < k_* M/\mu(0). \tag{2.4}$$

Furthermore, we can prove that $\lambda(\cdot)$ is a Lipschitz function in c, with constant equal to $L > 0$ and strictly positive. The initial condition $c_0(x)$, in Equation (1.6), is assumed to have enough regularity. The dispersion-diffusion tensor $D(\mathbf{u})$ is given by Equation (1.7)-(1.8) with $d_m > 0$ and $d_l \geq d_t > 0$, such that for each $\xi, \eta \in \mathbf{R}^2$ we have

$$D(\mathbf{u})\xi \cdot \xi \geq (d_m + d_t|\mathbf{u}|)|\xi|^2 \tag{2.5}$$

and

$$D(\mathbf{u})\xi \cdot \eta \leq (d_m + d_l|\mathbf{u}|)|\xi||\eta|. \tag{2.6}$$

Under the above hypotheses Feng in [11] has shown existence and uniqueness of solution, in some weak sense, for system (1.1)-(1.10).

2.3. A Sequentially Implicit Time Discretization

Let $N \in \mathbf{Z}$, $\Delta t = T/N$, $t^\sigma = \sigma \Delta t$, $\sigma \in \mathbf{R}$ and $\partial_t \psi^n = (\psi^{n+1} - \psi^n)/\Delta t$, where ψ^n is an approximation to $\psi(x, t^n)$. Considering that

$$\text{div}\,(\mathbf{u}c) = \mathbf{u} \cdot \nabla c + c\,\text{div}\,\mathbf{u} = \mathbf{u} \cdot \nabla c + fc, \tag{2.8}$$

a sequentially implicit method is defined by: For $n = 0, 1, 2, \ldots$ given $c^0 = c_0(x)$, find $\mathbf{u}^n, p^n$ and c^{n+1} satisfying

$$\text{div}\,\mathbf{u}^n = f^n, \quad \text{on} \quad \Omega, \tag{2.9}$$

$$\mathbf{u}^n = -\lambda(c^n)\nabla p^n \quad \text{on} \quad \Omega, \tag{2.10}$$

$$\int_\Omega p^n dx = 0, \tag{2.11}$$

$$\begin{aligned} &\phi\tfrac{c^{n+1} - c^n}{\Delta t} + \mathbf{u}^n \cdot \nabla c^{n+1} - \text{div}\,(D(\mathbf{u}^n)\nabla c^{n+1}) + c^{n+1} f^{n+1} = \\ &= \hat{c}^{n+1} f^{n+1} \quad \text{on} \quad \Omega. \end{aligned} \tag{2.12}$$

This sequentially implicit method combines the advantages of explicit and fully implicit methods [7,15-17], that is:

i) the original system becomes partially uncoupled;

ii) stability is provided by the implicit approximation of the concentration.

To simplify the notation, in the following sections, we will suppress the dependence of the variables on n. The analysis corresponding to the time discretization will appear in [17].

3. Finite Element Approximations for Pressure and Velocity

Substituting (2.10) into (2.9) and using the no-flow boundary condition $\mathbf{u}\cdot\nu = 0$ on $\partial\Omega$, the sub-system (2.9)-(2.11) can be recast as

$$-\operatorname{div}(\lambda(c)\nabla p) = f \quad \text{on} \quad \Omega, \tag{3.1}$$

$$\lambda(c)\nabla p \cdot \nu = 0 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

$$\int_\Omega p dx = 0. \tag{3.3}$$

Multiplying both sides of (3.1) by $\varphi \in H^1(\Omega)$ and integrating by parts, we obtain the following variational problem:

$$\begin{aligned} (\lambda(c)\nabla p, \nabla\varphi) &= (f,\varphi), \quad \forall\, \varphi \in H^1(\Omega), \\ (p,1) &= 0. \end{aligned} \tag{3.4}$$

Let $A(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbf{R}$ be a bilinear form associated with problem (3.4), given by

$$A(p,\varphi) := \int_\Omega \lambda(c)\nabla p \nabla\varphi dx. \tag{3.5}$$

From (2.1)-(2.4) and (3.5), we see that

$$A(\varphi,\varphi) \geq \frac{k^*}{\mu_0}\|\nabla\varphi\|^2, \quad \forall\, \varphi \in H^1(\Omega), \tag{3.6}$$

$$A(\varphi,\psi) \leq \frac{k_* M}{\mu_0}\|\nabla\varphi\|\|\nabla\psi\|, \quad \forall\, \varphi,\psi \in H^1(\Omega), \tag{3.7}$$

which show coercivity and continuity of $A(\cdot,\cdot)$, and by the Lax-Milgram theorem [3] there exists $p \in H^1(\Omega)$ unique solution of problem (3.4). If $f \in L^2(\Omega)$ and

Ω is a square we know, from theory of elliptic boundary-value problems, that $p \in H^2(\Omega)$.

3.1. Galerkin Method

Let $\{\tau_h\}$ be a family of polygonalization $\tau_h = \{K\}$ of Ω, satisfying the minimum angle condition [3], and indexed by the parameter h representing the maximum diameter of the elements $K \in \tau_h$. For a given positive integer $l \geq 1$ we introduce the finite element space

$$\mathcal{N}_h = \left\{ q_h \in C^0(\Omega);\ q_{h|K} \in P_l(K),\ \forall\, K \in \tau_h \right\},$$

where $P_l(K)$ is the set of polynomials on K of degree less than or equal to l, i.e., $\mathcal{N}_h$ is the space of piecewise continuous polynomial functions of degree l. By standard interpolation theory we have [3]: Given a function $v \in H^{l+1}(\Omega)$ there exists an interpolant $\hat{v}_h \in \mathcal{N}_h$ such that

$$\|v - \hat{v}_h\| + h\|v - \hat{v}_h\|_1 \leq K_0 h^{l+1} |v|_{l+1}, \tag{3.8}$$

where $|\cdot|_m$ denotes the usual semi-norm in the Hilbert space $H^m(\Omega)$.

The Galerkin finite element approximation of problem (3.4) in $\mathcal{N}_h$ reads:

$$\begin{aligned} &\text{Find } \ p_h \in \mathcal{N}_h \ \text{ such that} \\ (\lambda(c_h)\nabla p_h, \nabla \varphi_h) &= (f, \varphi_h), \quad \forall\, \varphi_h \in \mathcal{N}_h, \\ (p_h, 1) &= 0, \end{aligned} \tag{3.9}$$

where c_h is a finite element approximation for concentration assumed given. Since $\mathcal{N}_h$ is a conforming finite element space, i.e., $\mathcal{N}_h \in C^0(\Omega)$, those results obtained in Equations (3.6) and (3.7) can be immediately transferred to the discrete case, and existence and uniqueness of the discrete problem (3.9) is assured.

Let

$$p - p_h = p - \tilde{p}_h + \tilde{p}_h - p_h$$

where $\tilde{p}_h \in \mathcal{N}_h$ is a projection of p into $\mathcal{N}_h$ to be defined. Before estimating $p - \tilde{p}_h$ and $\tilde{p}_h - p_h$ we call attention to:

a) Since we need, as before, $\lambda(\cdot)$ to be a uniformly limited, Lipschitz and strictly positive function, we assume [5] that

$$-\epsilon_0 \leq c_h \leq \epsilon_0 + 1,$$

with $\epsilon_0 > 0$ a small arbitrary constant.

b) From interpolation theory, it holds that

$$\nabla \tilde{p}_h \in L^\infty(\Omega).$$

Following the ideas found in [5,25], where an auxiliary elliptic problem is introduced in the error analysis of problem (3.9), we define $\tilde{p}_h \in \mathcal{N}_h$ as an elliptic projection of p in $\mathcal{N}_h$ given by

$$\begin{aligned} (\lambda(c)\nabla \tilde{p}_h, \nabla \varphi_h) &= (\lambda(c)\nabla p, \nabla \varphi_h), \quad \forall \varphi_h \in \mathcal{N}_h, \\ (\tilde{p}_h, 1) &= 0, \end{aligned} \tag{3.10}$$

where (p, c) is the solution of the continuous problem (1.1)-(1.10). The following *a priori* error estimate for elliptic problem (3.10) is found

Lemma 3.1: *There exists a positive constant K_1 such that*

$$\|p - \tilde{p}_h\| + h\|\nabla(p - \tilde{p}_h)\| \leq K_1 M^{\frac{1}{2}} h^{l+1} |p|_{l+1}, \tag{3.11}$$

where $K_1 = K_1(\Omega, (k_*/k^*)^{\frac{1}{2}} K_0)$ and $|\cdot|_{l+1}$ is the usual $H^{l+1}(\Omega)$ seminorm.

Proof: Using the definition of the elliptic projection (3.10), the coercivity and continuity constants exibited in Equations (3.6)-(3.7), respectively, and Cea's Lemma [3], we have

$$\|\nabla(p - \tilde{p}_h)\| \leq (k_*/k^*)^{\frac{1}{2}} M^{\frac{1}{2}} \inf_{q_h \in \mathcal{N}_h} \|\nabla(p - q_h)\|.$$

From the interpolation theory (see Equation (3.8)), we obtain

$$\|\nabla(p - \tilde{p}_h)\| \leq K_1 M^{\frac{1}{2}} h^l |p|_{l+1},$$

if we assume $p \in H^{l+1}(\Omega)$, $l \geq 1$. Applying the Nitsche trick [3] we have

$$\|p - \tilde{p}_h\| \leq K_1 M^{\frac{1}{2}} h^{l+1} |p|_{l+1},$$

which completes the proof.

For the global error $p - p_h$ we prove

Lemma 3.2: *There exist positive constants K_2 and $\tilde{K}_2$ such that*

$$\|p - p_h\| \leq \|p - \tilde{p}_h\| + \tilde{K}_2 \|c - c_h\|, \tag{3.12a}$$

$$\|\nabla(p - p_h)\| \leq \|\nabla(p - \tilde{p}_h)\| + K_2 \|c - c_h\|. \tag{3.12b}$$

Proof: Subtracting (3.9) from (3.10) and considering Equation (3.4), we have

$$(\lambda(c_h)\nabla p_h - \lambda(c)\nabla \tilde{p}_h, \nabla \varphi_h) = 0,$$

which is equivalent to

$$(\lambda(c_h)\nabla(p_h - \tilde{p}_h), \nabla \varphi_h) = ([\lambda(c) - \lambda(c_h)]\nabla \tilde{p}_h, \nabla \varphi_h), \ \forall \varphi_h \ \in \mathcal{N}_h.$$

Taking $\varphi_h = p_h - \tilde{p}_h$ in the previous equation and applying Hölder's inequality, we obtain

$$k^*/\mu_0 \|\nabla(p_h - \tilde{p}_h)\|^2 \leq L\|c - c_h\| \|\nabla \tilde{p}_h\|_\infty \|\nabla(p_h - \tilde{p}_h)\|,$$

from which yields

$$\|\nabla(p_h - \tilde{p}_h)\| \leq K_2 \|c - c_h\|, \tag{3.13a}$$

with $K_2 = \mu_0 / k^* L \|\nabla \tilde{p}_h\|_\infty$ a positive constant independ of h. Taking Equation (3.13a) and Poincaré's inequality we get

$$\|p_h - \tilde{p}_h\| \leq \tilde{K}_2 \|c - c_h\|, \tag{3.13b}$$

with $\tilde{K}_2 = \tilde{K}_2 (K_2, \Omega)$. Using the triangular inequality in the definition of the global error $p - p_h$ and considering Equations (3.13 a-b), we obtain

$$\|p - p_h\| \leq \|p - \tilde{p}_h\| + \tilde{K}_2 \|c - c_h\|,$$

$$\|\nabla(p - p_h)\| \leq \|\nabla(p - \tilde{p}_h)\| + K_2\|c - c_h\|,$$

and the proof is complete.

From Lemmas 3.1 and 3.2 we have the following error estimates to problem (3.9)

$$\|p - p_h\| \leq \tilde{K}_1 M^{\frac{1}{2}} h^{l+1} |p|_{l+1} + K_2\|c - c_h\|, \tag{3.14a}$$

$$\|\nabla(p - p_h)\| \leq \tilde{K}_1 M^{\frac{1}{2}} h^{l} |p|_{l+1} + \tilde{K}_2\|c - c_h\|, \tag{3.14b}$$

for p sufficiently regular, i.e., $p \in H^{l+1}(\Omega)$, $l \geq 1$.

An approximation of the velocity field $\mathbf{u}_h^G$,

$$\mathbf{u}_h^G := -\lambda(c_h)\nabla p_h, \tag{3.15}$$

can be estimate from

$$\mathbf{u} - \mathbf{u}_h^G = \lambda(c)\nabla(p_h - p) + [\lambda(c_h) - \lambda(c)]\nabla p_h. \tag{3.16}$$

Taking the L^2-norm in both sides of the above equation, using Equation (3.14b) and the hypotheses on $\lambda(\cdot)$, we have

$$\|\mathbf{u}_h^G - \mathbf{u}\| \leq K_3 M^{\frac{3}{2}} h^{l} |p|_{l+1} + K_4 M\|c - c_h\|, \tag{3.17}$$

with $K_3 = k_* \tilde{K}_1/\mu_0$ and $K_4 = (\|\nabla p_h\|_\infty L + \tilde{K}_2 k_*/\mu_0)$ positive constants.

Remark 3.1: Concerning the velocity approximation $\mathbf{u}_h^G$, given by (3.15), we note that

a) It is a descontinuous field and does not obey the boundary condition $\mathbf{u}_h^G \cdot \nu = 0$ at the nodes in a strong sense;

b) It converges with one order lower than the pressure approximation p_h;

c) Its error estimate (3.17) depends on the error in the approximation of the concentration, $\|c - c_h\|$, and on the mobility ratio M. The factor $M^{\frac{3}{2}}$ deteriorates $\mathbf{u}_h^G$ as M increases.

Therefore, $\mathbf{u}_h^G$ is a poor approximation for the velocity field, specially for larger values of M, with severe consequence on the concentration approximation as illustrated in numerical experimentes presented in section 6.

To obtain velocity approximation with improved accuracy and regularity mixed methods based on Raviart-Thomas formulations will be considered next.

3.2. Mixed Methods

The idea behind mixed methods is to approximate velocity and pressure simultaneously. To this end we first introduce some additional notations. Let

$$V := \{\mathbf{v} \in H(\text{ div });\mathbf{v}\cdot\eta = 0 \quad \text{on} \quad \partial\Omega\},$$
$$W := \{\varphi \in \; L^2(\Omega); (\varphi, 1) = 0\},$$

where $H(\text{ div }) := \{\mathbf{v} \in (L^2(\Omega))^2;\ \text{div } \mathbf{v} \in L^2(\Omega)\}$ with norm given by:

$$\|\mathbf{v}\|^2_{H(\text{ div })} = \|v_1\|^2 + \|v_2\|^2 + \|\text{ div } \mathbf{v}\|^2.$$

For $\alpha, \beta \in V$ and $\varphi \in W$ we define the bilinear forms:

$$\begin{aligned} a(\theta;\alpha,\beta) &= \left(\tfrac{1}{\lambda(\theta)}\alpha,\beta\right), \\ b(\alpha,\varphi) &= -(\text{ div } \alpha,\varphi), \end{aligned} \tag{3.18}$$

with $\lambda(\cdot)$ as defined before and θ a smooth function assumed to be known.

The mixed formulation consists in finding $\{\mathbf{u}, p\} \in V \times W$, such that

$$\begin{aligned} a(c;\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= 0, \quad \forall \mathbf{v} \in V, \\ b(\mathbf{u},\varphi) &= -(f,\varphi), \quad \forall \varphi \in W, \end{aligned} \tag{3.19}$$

which is equivalent to solving a family of saddle-point problems. These two equations express Darcy's law and conservation of mass in a variational sense. The no-flow boundary condition (1.3) is strongly incorporated into V.

Let $\tilde{V}_h$ and $\tilde{W}_h$ be the Raviart-Thomas spaces [20] associated to the family τ_h and $V_h \times W_h \subset V \times W$, such that

$$V_h := \{\mathbf{v}_h \in \tilde{V}_h,\ \mathbf{v}_h\cdot\eta = 0 \quad \text{on } \partial\Omega\},$$
$$W_h := \{\varphi_h \in \tilde{W}_h; (\varphi_h, 1) = 0\},$$

with div $V_h \subset W_h$ by construction. The finite element approximation of problem (3.19) reads

$$
\begin{aligned}
&\text{Given } c_h, \text{ find } \{\mathbf{u}_h^M, p_h^M\} \in V_h \times W_h \quad \text{such that}\\
&a(c_h; \mathbf{u}_h^M, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^M) = 0, \quad \forall \mathbf{v}_h \in V_h,\\
&b(\mathbf{u}_h^M, \varphi_h) = -(f, \varphi_h), \quad \forall\, \varphi_h \in W_h.
\end{aligned} \tag{3.20}
$$

Existence, uniqueness and convergence of the solution of (3.20) relies on the well-known theory for mixed method [2,4,9,20]. For smooth data, the numerical analysis for this problem leads to

Lemma 3.3: *Let $\{\mathbf{u}, p\} \in V \times W$ and $\{\mathbf{u}_h^M, p_h^M\}$ be the solutions of (3.19) and (3.20), respectively. Then*

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h^M\|_{H(\operatorname{div})} + \|p - p_h^M\| \leq K_5 M(&\|\mathbf{u} - \mathbf{v}_h\|_{H(\operatorname{div})} + \|p - q_h\| + \\
&+ \|c - c_h\|), \ \forall \{\mathbf{v}_h, q_h\} \in V_h \times W_h,
\end{aligned} \tag{3.21}
$$

with K_5 independent of h, M and c.

Proof: By construction of Raviart-Thomas spaces we have

$$
a(c_h; \mathbf{v}_h, \mathbf{v}_h) \geq \alpha M \|\mathbf{v}_h\|^2_{H(\operatorname{div})}, \quad \forall \mathbf{v}_h \in K_h(0),
$$

$$
\sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{v}_h, \varphi_h)}{\|\mathbf{v}_h\|_{H(\operatorname{div})}} \geq \beta \|\varphi_h\|, \ \forall \varphi_h \in W_h,
$$

with $\alpha > 0$, $\beta > 0$ independent of h and M, and

$$
K_h(f) = \{\mathbf{v}_h \in V_h;\ b(\mathbf{v}_h, \varphi_h) = -(f, \varphi_h)\ \forall\, \varphi_h \in W_h\}.
$$

From Bezzi's Theorem [2] problem (3.20) has a unique solution $\{\mathbf{u}_h^M, p_h^M\} \in V_h \times W_h$. Following the convergence analysis of finite element methods we decompose our estimates in two parts. First, we define a projection $\{\tilde{\mathbf{u}}_h^M, \tilde{p}_h^M\} \in V_h \times W_h$ of the solution $\{\mathbf{u}, p\} \in V \times W$ by

$$
\begin{aligned}
&a(c; \tilde{\mathbf{u}}_h^M, \mathbf{v}_h) + b(\mathbf{v}_h, \tilde{p}_h^M) = a(c; \mathbf{u}, \mathbf{v}_h) + b(p, \mathbf{v}_h), \ \forall \mathbf{v}_h \in V_h,\\
&b(\tilde{\mathbf{u}}_h^M, \varphi_h) = b(\mathbf{u}, \varphi_h), \ \forall\, \varphi_h \in W_h,
\end{aligned}
$$

which is easily estimated by Brezzi's Theorem [2], leading to

$$\begin{aligned} \|\mathbf{u}-\tilde{\mathbf{u}}_h^M\|_{H(\operatorname{div})} &+\|p-\tilde{p}_h^M\| \leq K_6 M(\|\mathbf{u}-\mathbf{v}_h\|_{H(\operatorname{div})}+\|p-q_h\|), \\ &\forall \mathbf{v}_h \in V_h,\ q_h \in W_h, \end{aligned} \tag{3.22}$$

with K_6 depending on α and β. To estimate $\mathbf{u}_h^M - \tilde{\mathbf{u}}_h^M$ and $p_h^M - \tilde{p}_h^M$ we have the system

$$\begin{aligned} a(c_h;\mathbf{u}_h^M-\tilde{\mathbf{u}}_h^M,\mathbf{v}_h) &+ b(\mathbf{v}_h,p_h^M-\tilde{p}_h^M) = a(c;\tilde{\mathbf{u}}_h^M,\mathbf{v}_h)-a(c_h;\tilde{\mathbf{u}}_h^M,\mathbf{v}_h), \\ &\qquad \forall \mathbf{v}_h \in V_h, \\ b(\mathbf{u}_h^M-\tilde{\mathbf{u}}_h^M,\varphi_h) &= 0,\ \forall\, \varphi_h \in W_h, \end{aligned}$$

which leads to

$$\|\mathbf{u}_h^M-\tilde{\mathbf{u}}_h^M\|_{H(\operatorname{div})}+\|p_h^M-\tilde{p}_h^M\| \leq K_7 M\|c-c_h\|, \tag{3.23}$$

with K_7 independent of h, M and c. The proof is completed combining (3.22) with (3.23).

From the interpolation theory for Raviart-Thomas spaces applied to estimates (3.21) we obtain

$$\|\mathbf{u}-\mathbf{u}_h^M\|_{H(\operatorname{div})}+\|p-p_h^M\| \leq K_8 M(h^{s+1}|p|_{s+3}+\|c-c_h\|), \tag{3.24}$$

with K_8 depending on K_5.

Remark 3.2:

a) From (3.24) we observe that the mixed method leads to the same rates of convergence for velocity and pressure in $H(\operatorname{div})$ and L^2, respectively.

b) Estimate (3.24) with $s=0$, which corresponds to the lowest order Raviart-Thomas element, shows first order rate of convergence for $\mathbf{u}_h^M$ in $H(\operatorname{div})$ norm, while (3.17) with $l=1$ shows the same rate of convergence for $\mathbf{u}_h^G$ but in L^2-norm only. Thus $\mathbf{u}_h^M$ is more accurate and regular than $\mathbf{u}_h^G$.

c) We should also note that $\mathbf{u}_h^M$ is less affected by the mobility ratio M than $\mathbf{u}_h^G$: estimate (3.17) is multiplied by $M^{\frac{3}{2}}$ while (3.24) is multiplied by M.

As an alternative to the mixed method we consider a post-processing approach [24] with a standard finite element implementation and improved accuracy and regularity to approximate the velocity after computing an approximation to the pressure field.

3.3. The Post-Processing Technique

Let p_h be a finite element approximation of p given by the Galerkin Method, for example, and $U_h \subset H(\text{ div })$. For simplicity we adopt $U_h \subset (H^1(\Omega))^2 \subset H(\text{ div })$, such that

$$U_h = \{\mathbf{v}_h \in \mathcal{N}_h \times \mathcal{N}_h, \quad \mathbf{v}_h \cdot \nu = 0 \quad \text{on} \quad \partial\Omega\},$$

with $\mathcal{N}_h \subset C^0(\Omega)$, as defined before. The post-processing technique consists in

$$\text{Finding} \quad \mathbf{u}_h^P \in U_h \quad \text{such that}$$

$$(\lambda^{-1}(c_h)\mathbf{u}_h^P + \nabla p_h, \mathbf{v}_h) + M^{\frac{1}{2}}\delta(\text{ div } \mathbf{u}_h^P - f, \text{ div } \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in U_h, \quad (3.25)$$

where δ is a positive constant depending on the mesh size. Since we have assumed that $\lambda(\cdot)$ is strictly positive $\lambda^{-1}(\cdot) = 1/\lambda(\cdot)$ makes sense.

By means of a simple calculation and applying Green's formula to Equation (3.25) we have the following expression for the post-processing method:

$$\begin{aligned} &\text{Given} \quad p_h \in \mathcal{N}_h, \quad \text{find} \quad \mathbf{u}_h^P \in U_h \quad \text{such that} \\ &A_\delta(\mathbf{u}_h^P, \mathbf{v}_h) = (p_h + M^{\frac{1}{2}}\delta f, \text{ div } \mathbf{v}_h), \quad \forall \mathbf{v}_h \in U_h, \end{aligned} \quad (3.26)$$

where $A_\delta(\mathbf{u}_h^P, \mathbf{v}_h) = (\lambda^{-1}(c_h)\mathbf{u}_h^P, \mathbf{v}_h) + M^{\frac{1}{2}}\delta(\text{ div } \mathbf{u}_h^P, \text{ div } \mathbf{v}_h)$.

Taking into account the bounds on $\lambda^{-1}(\cdot)$ (see Equation (2.4)) we obtain

$$A_\delta(\mathbf{v}_h, \mathbf{v}_h) \geq K_9 \|\mathbf{v}_h\|^2_{H(\text{ div })}, \quad \forall \mathbf{v}_h \in U_h, \quad (3.27a)$$

$$|A_\delta(\omega_h, \mathbf{v}_h)| \leq K_{10} \|\omega_h\|_{H(\text{ div })} \|\mathbf{v}_h\|_{H(\text{ div })}, \quad \forall \omega_h, \mathbf{v}_h \in U_h, \quad (3.27b)$$

with $K_9 = \min\left\{\mu_0/Mk_*, M^{\frac{1}{2}}\delta\right\}$ and $K_{10} = \max\left\{\mu_0/k^*, M^{\frac{1}{2}}\delta\right\}$.

Now we analyse problem (3.26) taking $\delta = h$. This error analysis will justify some numerical results presented at the end of this work. From Equation (3.26) we define the weighted norm

$$\|\mathbf{v}_h\|_* := \left(\lambda^{-1}(c_h)\mathbf{v}_h, \mathbf{v}_h\right)^{\frac{1}{2}}, \tag{3.28}$$

which is equivalent to L^2-norm, due to the limitations shown in Equation (2.4). By the definition of $A_\delta(\cdot,\cdot)$ we have

$$A_h(\mathbf{u}_h^P, \mathbf{v}_h) := (\lambda^{-1}(c_h)\mathbf{u}_h^P, \mathbf{v}_h) + M^{\frac{1}{2}}h(\text{ div } \mathbf{u}_h^P, \text{ div } \mathbf{v}_h)$$

and

$$A_h(\mathbf{v}_h, \mathbf{v}_h) = \|\mathbf{v}_h\|_*^2 + M^{\frac{1}{2}}h\|\text{ div } \mathbf{v}_h\|^2 =: |||\mathbf{v}_h|||_*^2.$$

Using (1.1), (1.2) and (3.26) we obtain the following "consistency equation" for the post-processing

$$A_h(\mathbf{u} - \mathbf{u}_h^P, \mathbf{v}_h) = -(\lambda(c)/\lambda(c_h)\nabla p, \mathbf{v}_h) - (p_h, \text{ div } \mathbf{v}_h), \quad \forall \mathbf{v}_h \in U_h, \tag{3.29}$$

from which follows

$$\begin{aligned} A_h(\mathbf{u}_h^P - \mathbf{v}_h, \mathbf{u}_h^P - \mathbf{v}_h) &= \left(\lambda^{-1}(c_h)[\lambda(c) - \lambda(c_h)]\nabla p, \mathbf{u}_h^P - \mathbf{v}_h\right) + \\ &+ (p_h - p, \text{ div } (\mathbf{u}_h^P - \mathbf{v}_h)) + (\lambda^{-1}(c_h)(\mathbf{u} - \mathbf{v}_h), \mathbf{u}_h^P - \mathbf{v}_h) - \\ &+ M^{\frac{1}{2}}h(\text{ div } (\mathbf{u} - \mathbf{v}_h), \text{ div } (\mathbf{u}_h^P - \mathbf{v}_h)). \end{aligned}$$

Applying Cauchy-Schwarz inequality and considering that $2ab \leq \sigma a^2 + \frac{1}{\sigma}b^2$, we have

$$\begin{aligned} \tfrac{1}{2}A_h(\mathbf{u}_h^P - \mathbf{v}_h, \mathbf{u}_h^P - \mathbf{v}_h) \leq\ & \frac{1}{M^{\frac{1}{2}}h}\|p_h - p\|^2 + \|\mathbf{u} - \mathbf{v}_h\|_*^2 + \\ & + M^{\frac{1}{2}}h\|\text{ div } (\mathbf{u} - \mathbf{v}_h)\|^2 + K_{11}\|c - c_h\|^2, \end{aligned} \tag{3.30}$$

with $K_{11} = (\|\nabla p\|_{L^\infty(\Omega)}L)^2\mu_0/k^*$. Since $A_h(\omega_h, \omega_h) = |||\omega_h|||_*^2$, taking into account Equation (3.30) and the triangular inequality we conclude that

$$\begin{aligned} \tfrac{1}{2}|||\mathbf{u} - \mathbf{u}_h^P|||_*^2 \leq\ & \frac{1}{M^{\frac{1}{2}}h}\|p - p_h\|^2 + \tfrac{3}{2}\|\mathbf{u} - \mathbf{v}_h\|_*^2 + \tfrac{3}{2}M^{\frac{1}{2}}h\|\text{ div } (\mathbf{u} - \mathbf{v}_h)\|^2 + \\ & + K_{11}\|c - c_h\|^2. \end{aligned} \tag{3.31}$$

On the other hand, using definition (3.28) and Equation (2.4) we have

$$\|\mathbf{u}-\mathbf{v}_h\|_*^2 \leq \mu_0/k^* \|\mathbf{u}-\mathbf{v}_h\|^2. \tag{3.32}$$

Then, from the interpolation theory (see Equation (3.8)), error estimates (3.14 a-b) and Equations (3.31)-(3.32), we obtain

$$\|\mathbf{u}-\mathbf{u}_h^P\| \leq K_{12} M^{\frac{3}{4}} \left\{ h^{l+1/2}(|p|_{l+1} + |\mathbf{u}|_{l+1}) + \|c - c_h\| \right\}, \tag{3.33}$$

with K_{12} a positive constant independent of h, M and c .

Remark 3.3: From estimate (3.33) we observe that:

a) depending on the concentration approximation the post-processing technique may present a gain of $O(h^{0.5})$ in the rates of convergence for velocity compared to the Galerkin and mixed methods defined in sections 3.1 and 3.2, respectively;

b) the velocity approximation $\mathbf{u}_h^P$ is also less affected by the mobility ratio M than $\mathbf{u}_h^G$ and $\mathbf{u}_h^M$.

Numerical experiments performed with a particular situation of known exact solution ($\lambda(\cdot) = \lambda_0$ =constant) have exhibited rates of convergence for velocity that are higher than those predicted by the numerical analysis. In this case we found both velocity and pressure with optimal rates of convergence since pressure is calculated by Galerkin's method.

4. The SUPG Method for the Concentration

Our objective is to determine an approximate solution c_h for the concentration c, assuming that $\mathbf{u}_h^n$ is given by the post-processing technique defined in the last section. Recalling that the concentration equation is predominantely convective, it is well known that standard numerical methods such as second order finite differences or classical Galerkin finite element formulations do not work well when applied to this problem [13-14]. Here we use the Streamline

Upwind Petrov Galerkin (**SUPG**) method to approximate the solution of the concentration equation. Such method was introduced in [13] and can be viewed as a variant of Galerkin's method.

A finite element approximation, based on **SUPG** method, for Equation (2.2) is defined by:

For $n = 0, 1, 2, \ldots,$ and $\sigma \geq 0$, given $\mathbf{u}_h^n$ and c_h^n we determine $c_h^{n+1} \in \mathcal{M}_h$ satisfying:

$$\left(\phi\frac{c_h^{n+1}-c_h^n}{\Delta t}, \eta_h\right) + a(c_h^{n+1}, \eta_h) - f(\eta_h) + (\sigma R(c_h^{n+1}), \mathbf{u}_h^n \cdot \nabla \eta_h)_h = 0, \quad \forall\, \eta_h \in \mathcal{M}_h, \tag{4.1}$$

where σ is a scalar, in general depending on the mesh; the product $(\cdot,\cdot)_h$ is calculated only inside each element K, i.e.,

$$(f_h, g_h)_h := \sum_{K=1}^{N_K} \int_\Omega f_K g_K dx, \tag{4.2}$$

with N_K the number of elements of the discretization, and f_K, g_K the restriction of f_h, g_h to element K. In addition,

$$f(\eta_h) := (f^{n+1}\hat{c}^{n+1}, \eta_h) \tag{4.3}$$

and

$$\begin{aligned} R(c_h^{n+1}) \;:=\; & \phi\frac{c_h^{n+1}-c_h^n}{\Delta t} + \mathbf{u}_h^n \cdot \nabla c_h^{n+1} - \operatorname{div}(D(\mathbf{u}_h^n)\nabla c_h^{n+1}) + \\ & +(c_h^{n+1} - \hat{c}_h^{n+1}) f^{n+1} \end{aligned} \tag{4.4}$$

is the residual equation for the test function c_h^{n+1}.

The last term in the left-hand side of Equation (4.1) is activated only when the hyperbolic features of this equation prevail over the parabolic ones. As such, the **SUPG** method improves stability compared to Galerkin's method applied to predominantly convective problems [13,14]. When this term is "off" **SUPG** reduces to the usual Galerkin's method.

In [14] the following *a priori* error estimates for **SUPG** method applied to a transport equation with $\mathbf{u}_h^n$ replaced by a given function $\mathbf{u}^n$ in (4.1) is proved

$$|||c^{n+1} - c_h^{n+1}||| \leq K_{13} h^{k+\frac{1}{2}} |c^{n+1}|_{k+1}, \tag{4.5}$$

where

$$|||\varphi_h||| := \bar{\sigma}^{\frac{1}{2}}||\varphi_h|| + \epsilon^{\frac{1}{2}}||\nabla\varphi_h|| + \sigma^{\frac{1}{2}}||\mathbf{u}^n \cdot \nabla\varphi_h||.$$

Here we have a different problem since $\mathbf{u}_h^n$ is approximated by a finite element method. In [17] the following error estimates was found to this case:

$$|||c^{n+1} - c_h^{n+1}|||_{**} \leq K_{14}h^{k+\frac{1}{2}}|c^{n+1}|_{k+1} + K_{15}||\mathbf{u}^n - \mathbf{u}_h^n||, \tag{4.6}$$

where

$$|||\phi_h|||_{**} := ||\phi_h||_{**} + ||\nabla\phi_h|| + \sigma^{\frac{1}{2}}||\mathbf{u}_h^n \cdot \nabla\phi_h||$$

with $||\varphi_h||_{**} = (\phi\varphi_h, \varphi_h)^{\frac{1}{2}}$ a weighted norm.

In Equations (4.5) and (4.6) we are assuming that $c^{n+1} \in H^{k+1}(\Omega)$, $k \geq 1$, with k the degree of the interpolation polynomial of the finite dimensional subspace $\mathcal{M}_h$. In the numerical analysis made in the previous sections we present different methods to approximate the velocity field. It has been experimentally observed that when M increases the velocity approximation becomes an important factor in the calculation of the concentration. Numerical experiments for different mobility ratios will be shown in section 6 to illustrate this influence.

5. Point Sources and Sinks

In realistic reservoir simulation, the external flow is concentrated at wells. In this case the right-hand side of Equation (3.1) is given by

$$f = \sum_{i=1}^{N_w} Q_i\delta(x_i, y_i), \tag{5.1}$$

where N_w is the number of wells, $\delta(x_i, y_i)$ denotes the Dirac measure at (x_i, y_i) and Q_i are the specified flow rates of the wells with $Q_i > 0$ in the injection wells and $Q_i < 0$ in the production wells. In this section we shall modify our formulation to recognize the existence of these point sources and sinks. Such modification is frequently called *removal or subtraction of singularities* [4,6,10]. Taking into account Equation (3.1) with f given by Equation (5.1), we can write:

$$p(x) := p_r(x) + p_s(x), \tag{5.2}$$

with p_s denoting the singular part of the pressure, which is given by

$$p_s(x) := \sum_{i=1}^{N_w} \frac{Q_i}{2\pi} \frac{1}{\lambda_i(c)} \ln|x - x_i| = \sum_{i=1}^{N_w} p_s^i, \tag{5.3}$$

with $\lambda_i(c) = k(x_i)/\mu(c(x_i))$. The regular part p_r will be approximated by a finite element method. Similarly we decompose the Darcy velocity as

$$\mathbf{u}(x) := \sum_{i=1}^{N_w} \frac{Q_i}{2\pi} \frac{\lambda(c)}{\lambda_i(c)} \nabla \ln|x - x_i| + \mathbf{u}_r(x), \tag{5.4}$$

where the first term on the right-hand side denotes the singular part $\mathbf{u}_s$. The regular part of the velocity field $\mathbf{u}_r$ satisfies

$$\operatorname{div} \mathbf{u}_r = 0 \quad \text{on } \Omega, \tag{5.5}$$

$$\mathbf{u}_r = -\lambda(c)\nabla p_r + \sum_{i=1}^{N_w} (\lambda(c)/\lambda_i(c) - 1)\, \mathbf{u}_s^i, \tag{5.6}$$

$$\mathbf{u}_r \cdot \nu = -\mathbf{u}_s \cdot \nu, \quad \text{on } \partial\Omega. \tag{5.7}$$

To define a finite element method to approximate p_r and $\mathbf{u}_r$ we recall some properties of the solutions $\mathbf{u}$ and p of the subsystem (1.1)-(1.2), with f given by (5.1). According to [12,22] there exists positive constants K_{16} and K_{17} such that

$$\begin{aligned} \|p\|_{1-\epsilon} &\leq K_{16}, \\ \|\mathbf{u}\|_{(L^{2-\epsilon}(\Omega))^2} &\leq K_{17}, \end{aligned} \tag{5.8}$$

with $\epsilon > 0$ an arbitrary small constant, and the above norms defined in the fractional-index Sobolev spaces $H^r(\Omega)$, $r \in \mathbf{R}$ [1].

Using Equations (5.5), (5.6) and (5.3) we obtain

$$-\operatorname{div}\,(\lambda(c)\nabla p_r) = \sum_{i=1}^{N_w} \operatorname{div}\,\left([\lambda(c) - \lambda_i(c)]\, \nabla p_s^i\right), \tag{5.9}$$

with

$$-\lambda(c)\nabla p_r \cdot \nu = \lambda(c)\nabla p_s \cdot \nu \quad \text{on} \quad \partial\Omega, \tag{5.10}$$

whose variational form is given by

$$
\begin{aligned}
(\lambda(c)\nabla p_r, \nabla\varphi) = & -\sum_{i=1}^{N_w}\left([\lambda(c)-\lambda_i(c)]\,\nabla p_s^i, \nabla\varphi\right) \\
& -\sum_{i=1}^{N_w}\int_{\partial\Omega}\lambda_i(c)\nabla p_s^i\cdot\nu\varphi dx.\ \forall\,\varphi\in H^1(\Omega).
\end{aligned}
\tag{5.11}
$$

The Galerkin method associated to problem (5.11) reads

Find $p_{rh} \in \mathcal{N}_h$ such that

$$
\begin{aligned}
(\lambda(c_h)\nabla p_{rh}, \nabla\varphi_h) = & -\sum_{i=1}^{N_w}\left([\lambda(c_h)-\lambda_i(c_h)]\,\nabla p_s^i, \nabla\varphi_h\right) \\
& -\sum_{i=1}^{N_w}\int_{\partial\Omega}\lambda_i(c_h)\nabla p_s^i\cdot\nu\varphi_h dx, \quad \forall\,\varphi_h \in \mathcal{N}_h,
\end{aligned}
\tag{5.12}
$$

with $\mathcal{N}_h$ as defined in section 3.

The post-processing technique to evaluated $\mathbf{u}_{rh}$ is now defined by

Given $p_{rh} \in \mathcal{N}_h$, find $\mathbf{u}_{rh} \in \tilde{U}_h$, such that,

$$
\begin{aligned}
(\mathbf{u}_{rh}, \mathbf{v}_h) + M^{\frac{1}{2}}\delta(\operatorname{div}\mathbf{u}_{rh}, \operatorname{div}\mathbf{v}_h) = & -(\lambda(c_h)\nabla p_{rh}, \mathbf{v}_h) \\
& +\sum_{i=1}^{N_w}\left([\lambda(c_h)/\lambda_i(c_h)-1]\,\mathbf{u}_s^i, \mathbf{v}_h\right), \quad \forall\mathbf{v}_h \in \tilde{U}_h,
\end{aligned}
\tag{5.13}
$$

where $\tilde{U}_h := \{\mathbf{v}_h \in \mathcal{N}_h \times \mathcal{N}_h; \mathbf{v}_h\cdot\nu = -\mathbf{u}_{sh}\cdot\nu \text{ on } \partial\Omega\}$, with $\mathbf{u}_{sh}$ denoting the interpolant the $\mathbf{u}_s$.

Using the previously defined notations we can re-write Equation (5.13) as

$$
\begin{aligned}
A_h(\mathbf{u}_{rh}, \mathbf{v}_h) = & -(\nabla p_{rh}, \mathbf{v}_h) + \sum_{i=1}^{N_w}\left([\lambda_i^{-1}(c_h)-\lambda^{-1}(c_h)]\mathbf{u}_s^i, \mathbf{v}_h\right), \\
& \forall\mathbf{v}_h \in \tilde{U}_h,
\end{aligned}
\tag{5.14}
$$

with $A_h(\mathbf{u}_{rh}, \mathbf{v}_h) := (\lambda^{-1}(c_h)\mathbf{u}_{rh}, \mathbf{v}_h) + M^{\frac{1}{2}}h(\operatorname{div}\mathbf{u}_{rh}, \operatorname{div}\mathbf{v}_h)$.

The pressure and velocity approximations are then given by

$$p_h = p_s + p_{rh}$$

$$\mathbf{u}_h = \mathbf{u}_s + \mathbf{u}_{rh}.$$

In particular when $\lambda(c) = \lambda_0$, λ_0 a positive constant, the velocity field does not depend on the concentration since system (1.1)-(1.10) is naturally uncoupled. In this case, the elliptic sub-system has solution $p \in H^{3-\epsilon}(\Omega)$. The analysis presented in section 3 applied to this case leads to following error estimates:

a) For the pressure

$$\|p - p_h\| = \|p_r - p_{rh}\| \leq K_{18} h^{m+1} |p_r|_{m+1}$$

$$\|\nabla(p - p_h)\| = \|\nabla(p_r - p_{rh})\| \leq K_{19} h^{m} |p_r|_{m+1}.$$

b) For the post-processing velocities

$$\|\mathbf{u} - \mathbf{u}_h\| = \|\mathbf{u}_r - \mathbf{u}_{rh}\| \leq K_{20} h^{m+0.5} |p_r|_{m+2}. \tag{5.15}$$

with $m = \min\{l, 2 - \epsilon\}$.

6. Numerical Results

To confirm the rates of convergence predicted in the analysis presented in section 3, corresponding to the finite element approximations of pressure and velocity fields based on Galerkin's method and on the post-processing technique, we solve the Poisson problem (Equations (3.1)-(3.3) with $\lambda = 1$) in a square domain $\Omega = \{0, L\} \times \{0, L\}$ ($L = 1.854075$) with a point source at $x = y = 0.0$ and a point sink at $x = y = L$, whose analytical solution is given by [18]

$$p(x, y) = \frac{1}{4\pi} ln \left(\frac{1 - cn^2 x cn^2 y}{cn^2 x cn^2 y} \right)$$

with cn denoting the elliptic cossine with modulus $1/\sqrt{2}$ [23]. The finite element approximations were computed with a sequence of 2×2, 4×4, 8×8 and 16×16 uniform meshes of bilinear quadrilateral elements ($l = 1$).

First we solved the problem using the methods defined in section 3, without removing the singularities. In this case $p \in H^{1-\epsilon}(\Omega)$, and the maximum rate of convergence expected is $1-\epsilon$ for the pressure approximation, and no convergence for the gradient in L^2−norm. To check these conclusions and the influence of the singularities on the approximate solution we computed the error in $\Omega^* = \Omega/B$, with

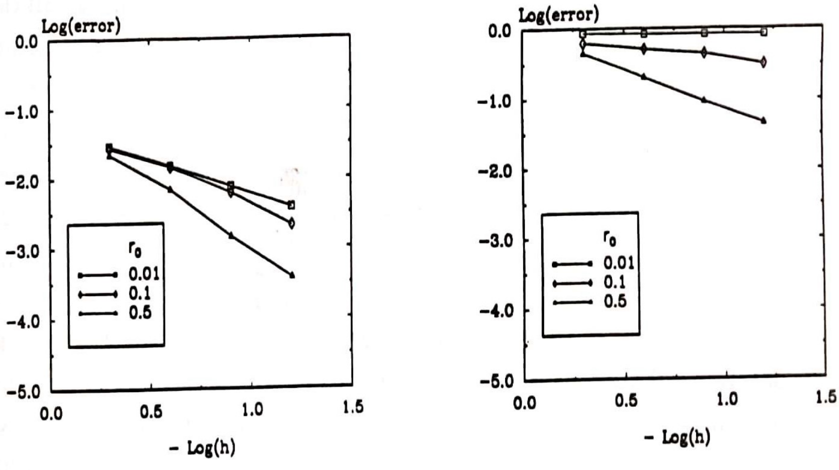
$$B := \left\{\{x,y\} \in \Omega;\ x^2 + y^2 \le r_0 \text{ or } (x-L)^2 + (y-L)^2 \le r_0\right\}$$

for different values of r_0. Figures 6.1 (a)-(b) exhibit the convergence rates for pressure and its gradient, respectively. Note the influence of r_0 on these rates. For $r_0 \ge 0.5$ the solution becomes more accurate and optimal convergence rates are recovered.

Then we combine the Galerkin method with the technique of removal of singularities defined in section 5. In Figure 6.2 a) we compare the convergence rates for the pressure obtained using this approach with those given by the mixed method presented in reference [8] in which the removal of singularities technique is only applied to the velocity field. Futhermore, Figure 6.2 b) exhibits the rates of convergence for the velocity field evaluated with the mixed method and and the post-processing technique combined with removal of singularities. We clearly observe the improved accuracy of the post-processing technique. As predicted in the analysis the mixed method presents first order rates of convergence for both velocity and pressure approximations while the post- processing technique shows second order rate of convergence for the velocity field which is even higher than that derived in section 5, that is $h^{1.5}$ according to equation (5.15).

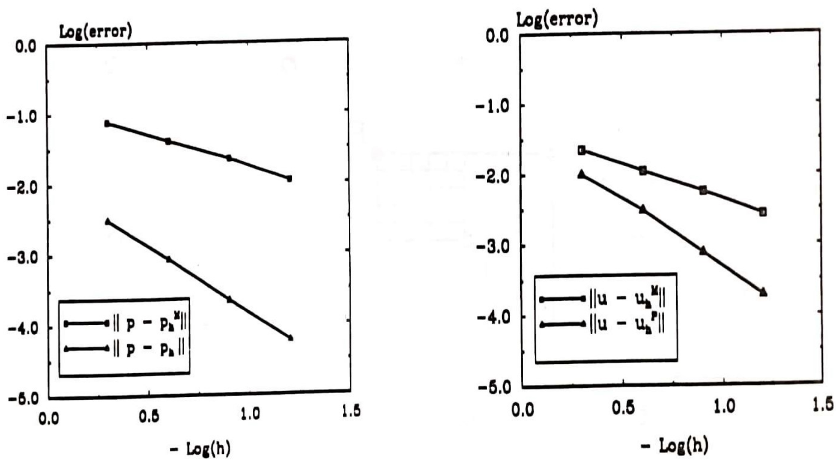
From the computational point of view the first order mixed method and the post-processing approach with bilinear lagrangian element, adopted in the present experiment demand about the same effort. For example, taking an $n \times n$ mesh of bilinear elements wich corresponds to a $2n \times n$ mesh of first order Raviart-Thomas triangles, if $n \ge 3$ the mixed methods will have more velocity degrees of freedom than the post-processing.

To illustrate the influence of the velocity on the concentration we solve a five-spot problem with different mobility ratios. The domain (see Figure 6.3) consists of a square (unit thickness) with side L, corresponding to one quarter



a) pressure b) gradient

Figure 6.1 - Rates of convergence for classical Galerkin method



a) pressure b) velocity

Figure 6.2 - Rates of convergence for mixed method and post-processing technique

of a five-spot arrangement. The injector well **I** is at the lower-left corner ($x = y = 0$) and the producer well **P** at the upper-right ($x = y = L$). In all the plots that follows we use 40×40 uniform grid of bilinear quadrilateral elements and a time step of 5 days, with: $L = 1000.0ft$, permeability $k = 100.0mD$, viscosity of oil $\mu(1) = 1.0cP$, molecular diffusion $d_m = 0.0ft$, $d_l = 10.0ft$ and $d_t = 1.0ft$. The porosity is set to a constant value of 0.1 and the well flow rates are all 200 square feet per day, thus a pore volume of solvent is injected in 2000 days.

The results are presented as isoconcentration curves of the injected fluid at 250, 500 750 and 1000 days for $M = 1.0$ and $M = 20.0$. For $M = 1.0$ the velocity field is independent of the concentration and the finite element approximations are more stable than for $M > 1$, as confirmed in the numerical results that follow.

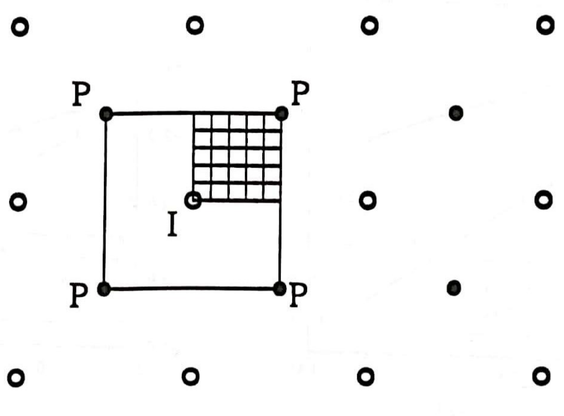


Figure 6.3 - Five-spot arrangement

Figure 6.4 and 6.5 show the concentration maps obtained with **SUPG** combined with velocity approximations given by Equation (3.15), corresponding to Darcy's law with pressure approximation calculated by (3.9). Figure 6.4, corresponding to $M = 1.0$, presents satisfatory results while Figure 6.5, corresponding to $M = 20.0$, shows the influence of this inaccurate velocity approximation on the concentration maps in this case. On the other hand, combining the **SUPG** with the velocity field given by the post-processing technique plus removal of singularity improved accuracy is obtained for the concentration as show in Figure 6.6 for $M = 20.0$.

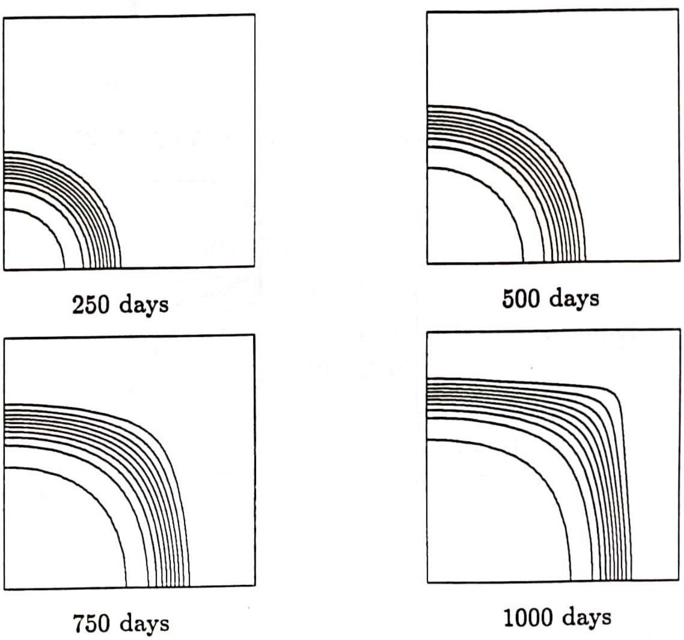
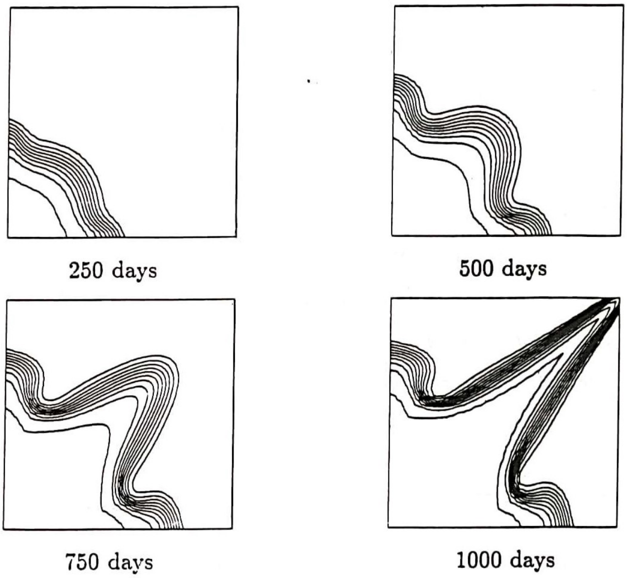


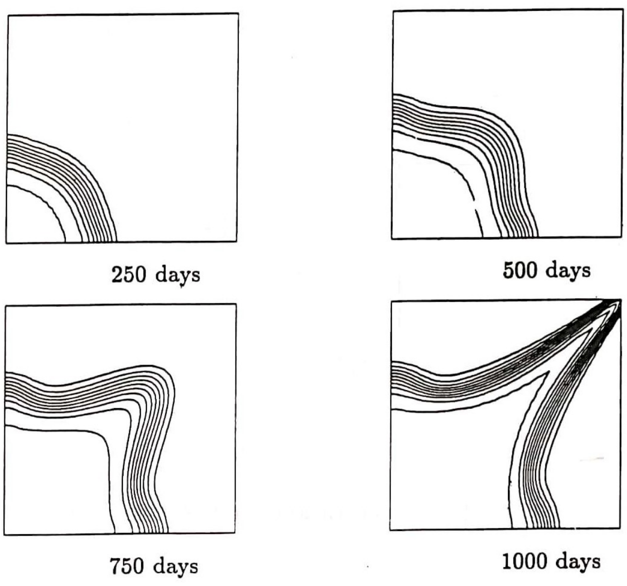
Figure 6.4 - Concentration maps with $\mathbf{u}_h^G$, M=1.0



250 days 500 days

750 days 1000 days

Figure 6.5 - Concentration maps with $\mathbf{u}_h^G$, M=20.



250 days 500 days

750 days 1000 days

Figure 6.6 - Concentration maps with $\mathbf{u}_h^P$, M=20.

7. Conclusions

Numerical analysis and error estimates for finite element approximations of the nonlinear system of partial differential equations governing miscible displacement in a porous medium are presented.

The influence of the mobility ratio M on the constants multiplying the error estimates of the velocity approximations is analized, showing that the post-processing technique presents a more accurate approximation which is also less influenced by M. A numerical experiment with a specific situation of known exact solution showed optimal rate of convergence for the post-processed velocity, which is even higher than the one derived in the present analysis.

The results presented in Figures 6.5 and 6.6 lead us to a better understanding of the longstanding problem for adverse mobility ratio computations. From Figure 6.5 we observe that the grid orientation effect, usually attribute only to numerical dispersion in oil reservoir literature, in this case is **mainly due to a poor approximation of the velocity field.**

Acknowledgments

We acknowledge PETROBRAS S.A., the Brazilian Oil Company, for the partial support through grant 650.401.792-1 . This work is part of a join research project on Finite Element for Reservoir Simulation, between LNCC/CNPq, COPPE/UFRJ and CENPES/PETROBRAS.

We also would like to acknowledge the valiable comments of anonymous referees.

References

[1] Adams, R.A., *Sobolev Spaces*, (Academic Press, N.Y.,1975).

[2] Brezzi, F., On the existence, uniqueness and approximation of saddle-point problems arising from Lagragian multipliers,*R.A.I.R.O., Anal. Numer.*, **2**(1974), 129-151.

[3] Ciarlet, P.G.,*The Finite Element Method for Elliptic Problems*, (North-Holland, Amsterdan, 1978).

[4] Douglas, J. Jr.; Ewing, R.E. and Wheeler, M.F., The Approximation of the Pressure by a Mixed-Method in the Simulation of Miscible Displacement, *R.A.I.R.O. Analyse Numerique*, **17** (1983), 17-33.

[5] Ewing, R.E. and Wheeler, M.F., Galerkin Methods for Miscible Displacement Problems in Porous Media, *SIAM J. Numer. Anal.*, **17** (1980), 351-365.

[6] Ewing, R.E. and Wheeler, M.F., Galerkin Methods for Miscible Displacement Problems with Point Sources and Sinks - unit mobility ratio case, (Proceeding of the Special Year in Numerical Analysis, no. 20, Univ. of Maryland, Baltimore, Md., 1981), pp. 151-181.

[7] Ewing, R.E. and Russel, T.F., Efficient Time-Stepping Methods for Miscible Displacement Problems in Porous Media, *SIAM J. Numer. Anal.*, **19**(1982), 1-45.

[8] Ewing, R.E.; Russel, T.F. and Wheeler, M.F., Simulation of Miscible Displacement Using Mixed Methods and a Modified Methods of Charactheristics, **SPE 12241**, Proc. Seventh Symp. on Reservoir Simulation, S. Francisco, L.A., (1983), 71-81.

[9] Ewing, R.E.; Russel, T.F. and Wheeler, M.F., Convergence Analysis of an Approximation of Miscible Displacement in Porous Media by a Mixed Finite Elements and a Modified Method of Characteristics, *Comput. Methds. Appl. Mech. and Engrg.*, **47**(1984), 73-92.

[10] Ewing, R.E., *The Mathematics of Reservoir Simulation*, Frontiers in Applied Mathematics, (SIAM, Philadelphia, 1983).

[11] Feng, X., *On Miscible Fluids in Porous Media and Absorbing Boundary Conditions for Eletromagnetic Wave Propagation and on Elastic and*

Nearly Waves in the Frequency Domain, Ph.D. Thesis, Purdue University, 1992.

[12] Grisvard, P., *Elliptic Problems in Non-Smooth Domains*, (Pitman Publishing Inc., London, 1985).

[13] Hughes, T.J. and Brooks, A., A theorical framework for Petrov- Galerkin methods with dicontinuous weighting function: Application to the Streamline Upwind Procedure, (Finite Elements in Fluids, vol IV, Wiley, 1982).

[14] Johnson, C.; Nävert U. and Pitkäranta, J., Finite element for linear hyperbolic problems, *Comupt. Methds. Appl. Mech. Engrg.*, **45** (1984), 285-312.

[15] Loula, A.F.D.; Garcia, E.L.M.; Murad, M.A.; Malta, S.M.C. and Silva, R.S., Métodos de Elementos Finitos Aplicados à Simulação de Reservatórios de Petróleo, (Projeto CENPES/LNCC, Relatório I, 1992).

[16] Loula, A.F.D.; Garcia, E.L.M. and Malta, S.M.C., Métodos de Elementos Finitos Aplicados à Simulação de Reservatórios de Petróleo, (Projeto CENPES/LNCC, Relatório II, 1992).

[17] Malta, S.M.C., Análise Numérica de Escoamentos Miscíveis, Ph. Thesis, to appear.

[18] Morel-Seytoux, H.J., Analytical-Numerical Method in Waterflooding Predictions, *SPE Journal*,**5** (1965), 247-285.

[19] Peaceman, D.W., *Fundamental of Numerical Reservoir Simulation*, (Elsevier, Amsterdan, 1977).

[20] Raviart, P.A. and Thomas, J.M., A Mixed Finite Element Method for 2nd Order Elliptic Problems,*Mathematical Aspect of the Finite Element Method*, (Lecture Notes in Mathematics 606, Springer-Verlag, 1977).

[21] Russel, T.F.; Wheeler, M.F. and Chiang, C., Large-Scale Simulation of Miscible Displacement by Mixed and Characteristic Finite Element Methods, (Proceeding of SEG/SIAM/SPE Conference, Philadelphia, 1985), pp. 85-107.

[22] Sammon, P.H., Numerical Approximations for a Miscible Displacement Process in Porous Media, *SIAM J. Numer. Anal.*, **23** (1986), 508-542.

[23] Spainer, J. and Oldham, K.B., *Atlas of Functions*, (Springer-Verlag, N.Y., 1987).

[24] Toledo, E.M., *Novos Métodos de Elementos Finitos com Pós- Processamento*, Ph.D. Thesis, Univ. Federal of Rio de Janeiro, 1990.

[25] Wheeler, M.F., A priori L^2 Error Estimates for Galerkin Approximations to Parabolic Partial Differential Equations, *SIAM J. Numer. Anal.*, **10** (1973), 723-759.

Laboratório Nacional de Computação Científica
Rua Lauro Müller, 455, Botafogo,
22290-160, Rio de Janeiro, RJ, Brazil.