*Matemática Contemporânea, Vol. 8, 1–19*
http://doi.org/10.21711/231766361995/rmc81
©1995, Sociedade Brasileira de Matemática

# ON A SYSTEM OF EVOLUTION EQUATIONS OF MAGNETOHYDRODYNAMIC TYPE

**José Luiz Boldrini** **Marko Rojas-Medar**

**Abstract**

By using the spectral Galerkin method, we prove the existence and uniqueness of strong local solutions for a system of equations of magnetohydrodynamic type. Several estimates for the solution and their approximations are given. These estimates could be used for derivation of error bounds.

## 1. Introduction

In several situations the motion of incompressible electrical conducting fluids can be modelled by the so called equations of magnetohydrodynamics, which correspond to the Navier-Stokes' equations coupled with the Maxwell's equations. In the case where there is free motion of heavy ions, not directly due to the electric field (see Schlüter [10] and Pikelner [9]), these equations can be reduced to the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\eta}{\rho_m}\Delta u + u.\nabla u - \dfrac{\mu}{\rho_m} h.\nabla h = f - \dfrac{1}{\rho_m}\nabla(p^* + \dfrac{\mu}{2}h^2), \\ \dfrac{\partial h}{\partial t} - \dfrac{1}{\mu\sigma}\Delta h + u.\nabla h - h.\nabla u = -\text{grad } w, \\ \text{div } u = 0, \\ \text{div } h = 0, \\ u|_{\partial\Omega} = 0 \ , \ h|_{\partial\Omega} = 0 \\ u|_{t=0} = u_0 \ , \ h|_{t=0} = h_0 \end{cases} \tag{1.1}$$

In these equations we have assumed homogeneous boundary conditions just for simplicity. In standard ways we could treat the nonhomogeneous case.

**KEY WORDS:** Magnetohydrodynamic, Galerkin method, strong solution

Here, $u$ and $h$ are respectively the unknown velocity and magnetic fields; $p^*$ is the unknown hydrostatic pressure; $w$ is an unknown function related to the motion of heavy ions (in such way that the density of electric current, $j_0$, generated by this motion satisfies the relation $rot\ j_0 = -\sigma\nabla w$); $\rho_m$ is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\eta > 0$ is the constant viscosity of the fluid; $f$ is an given external force field; $\Omega$ is the domain where the physical phenomena ocurr, $\partial\Omega$ its boundary; $u_0$ and $h_0$ are given initial conditions for $u$ and $h$, respectively.

The stationary problem corresponding to (1.1) was considered by Chizhonkov [2]; while the question of (local) existence of solution of the evolution problem (1.1) was analysed by Lassner [8] by making the use of semigroup techniques as the ones in Fujita and Kato [6] (he also studied the asymptotic behavior of the solution as $t \to 0^+$).

In this paper we will consider the problem of local existence of strong solutions of (1.1) as in Lassner [8], with homogeneous boundary condition for $u$ an $h$ for simplicity of exposition, but under weaker assumptions on the initial data. Also, differently from [8], we will use the more constructive spectral Galerkin method of approximation. Thus, the results in this paper form the theoretical basis for future numerical analysis of the problem: here we will obtain estimates for the approximate solutions that will be fundamental in a forthcoming paper in which we will obtain optimal error estimates for such Galerkin approximations. These estimates will also play a role in the proof of global existence of solutions of (1.1).

Finally, the paper is organized as follows: in Section 2 we state the basic assumptions and results that will be used later in the paper; we also rewrite (1.1) in a more suitable weak form; we describe the approximation method and state the results of the paper (Theorems 2.1, 2.2 and 2.3). Each one of the following sections will be devoted to their proofs.

## 2. Preliminaries and Results

Let $\Omega \subseteq \mathbb{R}^n$, $n = 2$ or 3, be a bounded domain with boundary $\partial\Omega$ of Class $C^{1,1}$. We denote by $H^s(\Omega)$ the Sobolev Spaces on $\Omega$ with norm $\|\cdot\|_s, (\cdot,\cdot)$ denotes the $L^2$-norm on $\Omega$. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_1$. Also, we denote by $L^p(\Omega)$ for $1 \leq p \leq \infty$ the usual Lebesgue Spaces and by $\|\cdot\|_{L^p}$ the $L^p$-norm on $\Omega$. With the same symbols we denote the spaces of $n$-dimensional vector functions.

We put

$$\begin{aligned} C_{0,\sigma}^\infty &= \{v \in C_0^\infty(\Omega)/\mathrm{div} v = 0\} \\ H &= \text{closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } (L^2(\Omega))^n \\ V &= \text{closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } (H^1(\Omega))^n . \end{aligned}$$

It is possible to show that

$$V = \{v \in H_0^1(\Omega) | \mathrm{div}\ v = 0\}.$$

We recall the Helmholtz decomposition of vector field: $L^2(\Omega) = H \oplus G$, being $G = \{\phi | \phi = \nabla p\ ,\ p \in H^1(\Omega)\}$.

Throughout the paper $P$ will denote the orthogonal projection from $L^2(\Omega)$ onto $H$. Then, the operator $A : H \to H$ given by $A = -P\Delta$ with domain $D(A) = H^2(\Omega) \cap V$ is called the Stokes' operator. It is well know that $A$ is a positive definite self-adjoint operator and it is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v) \quad \text{for all} \quad w \in D(A),\ v \in V.$$

The operator $A^{-1}$ is linear continuous from $H$ into $D(A)$, and, since the injection of $D(A)$ in $H$ is compact, $A^{-1}$ can be considered as a compact operator in $H$. As an operator in $H, A^{-1}$ it is also self-adjoint. By a well know theorem of Hilbert, there exists a sequence of positive numbers $\mu_j > 0, \mu_{j+1} \leq \mu_j$ and an orthonormal basis of $H, (w_j)$ such that $A^{-1}w_j = \mu_j w_j$. We denote $\lambda_j = \mu_j^{-1}$. Since $A^{-1}$ has range in $D(A)$ we obtain that $Aw_j = \lambda_j w_j$ , $w_j \in D(A)$ , $0 < \lambda_1 < \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots$ and $\lim_{j \to +\infty} \lambda_j = +\infty$. Also, $\{w_j\}_{j=1}^\infty$ is an orthonormal

basis for $H$, and $\{w_j/\sqrt{\lambda_j}\}_{j=1}^{\infty}$ and $\{w_j/\lambda_j\}_{j=1}^{\infty}$ form orthonormal basis in $V$ (with the inner product $(\nabla u, \nabla v), u, v \in V$) and $H^2(\Omega) \cap V$ (with the inner product $(Au, Av), u, v \in D(A)$), respectively.

By using the properties of $P$, we can reformulate the problem (1.1), with homogeneous boundary conditions, as follows: find $u, h$, in suitable spaces to be exactly defined later on, satisfying

$$\begin{cases} \alpha(u_t, \phi) + \alpha(u.\nabla u, \phi) - (h.\nabla h, \phi) + \nu(Au, \phi) = (\alpha f, \phi) \\ (h_t, \psi) + (u.\nabla h, \psi) - (h.\nabla u, \psi) + \gamma(Ah, \psi) = 0 \\ \text{for} \quad 0 < t < T \quad , \quad \forall \phi, \psi \in V \\ u(0) = u_0 \quad , \quad h(0) = h_0. \end{cases} \tag{2.1}$$

Here we have denoted

$$\alpha = \frac{\rho_m}{\mu} \quad , \quad \nu = \frac{\eta}{\mu} \quad , \quad \gamma = \frac{1}{\mu\sigma} \, .$$

Now, we define strong solutions of the problem (2.1).

**Definition.** *Let $u_0, h_0 \in V$ and $f \in L^2(0,T;L^2(\Omega))$. By a strong solution of the problem (2.1), we mean a pair of vector-valued functions $(u,h)$ such that $u, h \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$ and that satisfies (2.1).*

**Remark.** In what follows, we will prove that if $(u,h)$ is a strong solution of (2.1) then $u_t, h_t \in L^2(0,T;H)$. This condition, together with $u, h \in L^2(0,T;D(A))$, implies by interpolation (see, Temam [11] p. 260), that $u, h$ are almost everywhere equal to a continuous function from $[0,T]$ into $V$, consequently the initial conditions $u(0) = u_0$ and $h(0) = h_0$ are meaningful.

To prove (local) existence of strong solutions we will use the spectral Galerkin method applied to (2.1). That is, we consider the finite dimensional subspaces $V_k = \text{span}[w^1, \ldots, w^k]$, $k \in N$, the corresponding orthogonal projections $P_k : L^2(\Omega) \to V_k$ and the approximate solutions

$$u^k(x,t) = \sum_{i=1}^{k} c_{ik}(t) w^i(x) \, , \quad h^k(x,t) = \sum_{i=1}^{k} d_{ik}(t) w^i(x) \, ,$$

developed in terms of eigenfunctions of Stokes' operator, satisfying the following equations:

$$\begin{cases} \alpha u_t^k + \nu A u^k + \alpha P_k(u^k.\nabla u^k) - P_k(h^k.\nabla h^k) - \alpha P_k f = 0, \\ h_t^k + \gamma A h^k + P_k(u^k.\nabla h^k) - P_k(h^k.\nabla u^k) = 0, \\ u^k(0) = P_k u_0 \quad , \quad h^k(0) = P_k h_0. \end{cases} \tag{2.2}$$

This is equivalent to the weak form

$$\begin{cases} \alpha(u_t^k, \phi) + \nu(\nabla u^k, \nabla\phi) + \alpha(u^k.\nabla u^k, \phi) - (h^k.\nabla h^k, \phi) = \alpha(f, \phi) \\ \quad \forall \phi \in V_k \\ (h_t^k, \psi) + \gamma(\nabla h^k, \nabla\psi) + (u^k.\nabla h^k, \psi) - (h^k.\nabla u^k, \psi) = 0 \\ \quad \forall \psi \in V_k \\ u^k(0) = P_k u_0 \quad , \quad h^k(0) = P_k h_0. \end{cases} \tag{2.3}$$

By using these approximations, we will prove the following results.

**Theorem 2.1.** *Let the initial values $u_0, h_0 \in V$ and the external force $f \in L^2(0,T;L^2(\Omega))$. Then, on a (possibly small) time interval $[0,T_1]$, $0 < T_1 \leq T$, problem (2.1) has a unique strong solution $(u,h)$. This solution belongs to $C([0,T_1],V)$.*

*Moreover, there exist continuous functions $F$ and $G$ such that for any $t \in [0,T_1]$, there hold*

$$||\nabla u(t,\cdot)||^2 + ||\nabla h(t,\cdot)||^2 + \int_0^t (||Au(\tau,\cdot)||^2 + ||Ah(\tau,\cdot)||^2)d\tau \leq F(t),$$

*and*

$$\int_0^t (||u_t(\tau,\cdot)||^2 + ||h_t(\tau,\cdot)||^2)d\tau \leq G(t).$$

*Also, the same kind of estimates hold uniformly in $k \in N$ for the Galerkin approximations $(u^k, h^k)$.*

**Theorem 2.2.** *In addition to the assumptions of Theorem 2.1, assume that $u_0, h_0 \in V \cap H^2(\Omega)$ and that $f_t \in L^2(0,T;L^2(\Omega))$. Then, the functions $u, h$*

*satisfy*

$$||u_t(t,\cdot)||^2 + ||h_t(t,\cdot)||^2 + \int_0^t (||\nabla u_t(\tau,\cdot)||^2 + ||\nabla h_t(\tau,\cdot)||^2)d\tau \leq H(t);$$

$$||Au(t,\cdot)||^2 + ||Ah(t,\cdot)||^2 \leq L(t);$$

$$\int_0^t (||u_{tt}(\tau,\cdot)||^2_{V'} + ||h_{tt}(\tau,\cdot)||^2_{V'})d\tau \leq M(t),$$

*for every* $t \in [0,T_1]$, *where* $H(\cdot), L(\cdot)$ *and* $M(\cdot)$ *are continuous function in* $t \in [0,T_1]$. *Also,* $u, h \in C^1([0,T_1]; H) \cap C([0,T_1], D(A))$.

*Moreover, the same kind of estimates hold uniformly in k for the Galerkin approximations* $(u^k, h^k)$.

As a consequence of the above, by using the results of Amrouche and Girault [1], we conclude:

**Theorem 2.3.** *Under the hypothesis of Theorem 2.1, there exist unique functions* $p, w \in L^2(0,T; H^1(\Omega)/\mathbb{R})$ *such that by taking* $p^* = p - \frac{\mu}{2}h^2$ , $(u, h, p^*, w)$ *is solution of (1.1). Under the hypothesis of Theorem 2.2,* $p, w \in L^\infty(0,T_1; H^1(\Omega)/\mathbb{R}) \cap C([0,T_1], L^2(\Omega)/\mathbb{R})$.

## 3. The Proof of Theorem 2.1

The following estimates show the global existence (in $t$) of the approximate solutions $u^k, h^k$.

Setting $\phi = u^k$ and $\psi = h^k$ in (2.3) (i) and (ii), respectively, we have

$$\begin{aligned} \frac{\alpha}{2}\frac{d}{dt}||u^k||^2 + \nu||\nabla u^k||^2 &= \alpha(f, u^k) + (h^k.\nabla h^k, u^k) \\ \frac{1}{2}\frac{d}{dt}||h^k||^2 + \gamma||\nabla h^k||^2 &= (h^k.\nabla u^k, h^k), \end{aligned}$$

since $(u^k.\nabla u^k, u^k) = (u^k.\nabla h^k, h^k) = 0$.

Adding the above inequalities and observing that $(h^k.\nabla h^k, u^k) + (h^k.\nabla u^k, h^k) = 0$, we obtain

$$\frac{d}{dt}(\alpha||u^k||^2 + ||h^k||^2) + 2(\nu||\nabla u^k||^2 + \gamma||\nabla h^k||^2) \tag{3.1}$$

$$= 2\alpha(f, u^k) \leq C_\varepsilon ||f||^2 + \varepsilon ||u^k||^2,$$

for any $\varepsilon > 0$ and a suitable positive constant $C_\varepsilon$.

By using Poincaré's inequality and by taking $\epsilon$ small enough, after an integration in time and an application of Gronwall's inequality, (3.1) implies the global existence of $u^k$ and $h^k$ and also that

$$u^k, h^k \quad \text{are uniformly bounded in} \quad L^\infty(0,T;H) \cap L^2(0,T;V). \tag{3.2}$$

By taking $\phi = Au^k$ and $\psi = Ah^k$ in (2.3) (i) and (ii), respectively, we have

$$\frac{\alpha}{2}\frac{d}{dt}||\nabla u^k||^2 + \nu||Au^k||^2 = \alpha(f, Au^k) + (h^k.\nabla h^k, Au^k) - \alpha(u^k.\nabla u^k, Au^k), \tag{3.3}$$

$$\frac{1}{2}\frac{d}{dt}||\nabla h^k||^2 + \gamma||Ah^k||^2 = (h^k.\nabla u^k, Ah^k) - (u^k.\nabla h^k, Ah^k). \tag{3.4}$$

As before, for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$ we have

$$|(f, Au^k)| \leq C_\varepsilon ||f||^2 + \varepsilon ||Au^k||^2 \ .$$

Now, we can estimate the third term in the right hand side of (33) as follows (this term is estimated in Duff [4, p.464], but for completeness we also prove it here): we use the last estimate in the proof of Lemma 3.8, p. 313-314, in Temam [11] this is a result for $n = 3$ with $u = u^k$ and $\nu = Au^k$ to obtain

$$|(u^k \cdot \nabla u^k, Au^k)| \leq C_6 ||u^k||_{H^1}^{\frac{3}{2}} ||u^k||_{H^2}^{\frac{1}{2}} ||Au^k||.$$

Now, we use Poincaré's inequality and the equivalence of the $||Au^k||$ and $||u^k||_{H^2}$- norms to get

$$|(u^k \cdot \nabla u^k, Au^k)| \leq C_7 ||\nabla u^k||^{\frac{3}{2}} ||Au^k||^{\frac{3}{2}},$$

and finally by Young's inequality

$$|(u^k \cdot \nabla u^k, Au^k)| \leq C_\varepsilon ||\nabla u^k||^6 + \epsilon ||Au^k||^2$$

for any $\varepsilon > 0$ and suitable positive constant $C_\varepsilon$

To estimate the second term in the right-hand side of (3.3), we use Gagliardo-Nirenberg's inequality (Theorem 10.1, p.27, in Friedman [4]):

$$||D^j\varphi||_{L^p} \leq C||\varphi||^a_{W^{m,r}}||\varphi||^{1-a}_{L^q}$$

with $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$ if $m - j - n/r$ is not a nonnegative integer. ($\Omega \subset I\!R^n$)

We use this inequality with $j = 0, p = 4, m = 1, r = q = 2, n = 3$. In this case $m - j - n/r = 1 - 0 - 3/2 = -1/2$ and we obtain $a = 3/4$; that is, $||\varphi||_{L^4} \leq C||\varphi||^{\frac{3}{4}}_{H^1}||\varphi||^{\frac{1}{4}}$.

By using this with $\varphi = \nabla h^k$ and recalling that $||\nabla h^k||_{H^1} \leq C||h^k||_{H^2} \leq C||Ah^k||$, we can estimate as follows

$$\begin{aligned}
|(h^k.\nabla h^k, Au^k)| &\leq ||h^k||_{L^4}||\nabla h^k||_{L^4}||Au^k|| \\
&\leq C||\nabla h^k||(||\nabla h^k||^{1/4}||Ah^k||^{3/4})||Au^k|| \\
&\leq C||\nabla h^k||^{5/4}||Ah^k||^{3/4}||Au^k|| \\
&\leq C_\varepsilon||\nabla h^k||^{5/2}||Ah^k||^{3/2} + \varepsilon||Au^k||^2 \\
&\leq C_\varepsilon C_\delta||\nabla h^k||^{10} + \delta||Ah^k||^2 + \varepsilon||Au^k||^2,
\end{aligned}$$

where we have used Hölder's and Young's inequalities and $\delta > 0$ (with suitable $C_\delta > 0$).

Analogously, we can prove that the terms in the right hand side of (3.4) satisfy

$$|(u^k.\nabla h^k, Ah^k)| \leq C_{\delta\varepsilon}||\nabla u^k||^8||\nabla h^k||^2 + \delta||Ah^k||^2 + \varepsilon||Au^k||^2$$

$$|(h^k.\nabla u^k, Ah^k)| \leq C_{\varepsilon\delta}||\nabla h^k||^8||\nabla u^k||^2 + \varepsilon||Au^k||^2 + \delta||Ah^k||^2.$$

If we add equalities (3.3) and (3.4) and use the above estimates with suitable small enough $\varepsilon$ and $\delta$, we are left with the following differential inequality,

$$\frac{d}{dt}(\alpha||\nabla u^k||^2 + ||\nabla h^k||^2) + \nu||Au^k||^2 + \gamma||Ah^k||^2 \tag{3.5}$$

$$\leq C||f||^2 + C(||\nabla u^k||^6 + ||\nabla h^k||^{10} + ||\nabla u^k||^8||\nabla h^k||^2 + ||\nabla h^k||^8||\nabla u^k||^2).$$

That is,

$$\frac{d}{dt}\xi^k(t) \leq \tau(t) + C\xi^k(t)^5$$

where $\xi^k(t) = \alpha||\nabla u^k(t)||^2 + ||\nabla h^k(t)||^2$ and $\tau(t) = C||f(t)||^2$. By using Lemma 3 in of Heywood [6, p. 656], we conclude that there exists $T_1 \in [0, T]$ such that

$$\alpha||\nabla u^k(t)||^2 + ||\nabla h^k(t)||^2 \leq F_0(t, \xi(0))$$

where $\xi(0) = \alpha||\nabla u_0||^2 + ||\nabla h_0||^2$, and $F_0$ is the solution of the initial value problem

$$F_0' = CF_0^5 + \tau(t) \quad , \quad F_0(0) = \xi(0).$$

Returning to (3.5), we are left with

$$\alpha||\nabla u^k(t)||^2 + ||\nabla h^k(t)||^2 + \int_0^t (\nu||Au^k||^2 + \gamma||Ah^k||^2)ds \tag{3.6}$$

$$\begin{aligned} &\leq \alpha||\nabla u_0||^2 + ||\nabla h_0||^2 + \int_0^t \tau(s)ds + CF_0^5(t, \xi(0)) \\ &\equiv F(t). \end{aligned}$$

Thus,

$$u^k, h^k \quad \text{are uniformly bounded in} \quad L^\infty(0, T_1; V) \cap L^2(0, T; D(A)) \; . \tag{3.7}$$

Now, by taking $\phi = u_t^k$ and $\psi = h_t^k$ in (2.3) (i) and (ii), respectively, we get

$$\begin{aligned} \alpha||u_t^k||^2 &= \alpha(f, u_t^k) + (h^k.\nabla h^k, u_t^k) - \alpha(u^k.\nabla u^k, u_t^k) - \nu(Au^k, u_t^k), \\ ||h_t^k||^2 &= (h^k.\nabla u^k, h_t^k) - (u^k.\nabla h^k, h_t^k) - \gamma(Ah^k, h_t^k) \; . \end{aligned}$$

From this, we have

$$\int_0^t ||u_t^k(s)||^2 ds \leq C\int_0^t (||f(s)||^2 + ||h^k.\nabla h^k||^2 + ||u^k.\nabla u^k||^2 + ||Au^k||^2)ds, \tag{3.8}$$

$$\int_0^t ||h_t^k(s)||^2 ds \leq C\int_0^t (||h^k.\nabla u^k||^2 + ||u^k.\nabla h^k||^2 + ||Ah^k||^2)ds. \tag{3.9}$$

Now, bearing in mind (3.7) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$, we obtain the following estimates:

$$\begin{aligned} ||h^k.\nabla h^k||^2 &\leq ||h^k||^2_{L^\infty}||\nabla h^k||^2 \\ &\leq C||Ah^k||^2||\nabla h^k||^2 \\ &\leq C \sup_{0\leq t\leq T_1} F(t) \quad ||Ah^k||^2 . \end{aligned}$$

Similarly,

$$\begin{aligned} ||u^k.\nabla u^k||^2 &\leq C \sup_{0\leq t\leq T_1} F(t)||Au^k||^2, \\ ||h^k.\nabla u^k||^2 &\leq C \sup_{0\leq t\leq T_1} F(t)||Ah^k||^2, \\ ||u^k.\nabla h^k||^2 &\leq C \sup_{0\leq t\leq T_1} F(t)||Au^k||^2. \end{aligned}$$

Since, $\int_0^t (\nu||Au^k(s)||^2 + \gamma||Ah^k(s)||^2)ds \leq F(t)$ for all $t \in [0, T_1]$ (see (3.6)), we deduce from the above estimates, together with (3.8) and (3.9), that

$$u^k_t, h^k_t \quad \text{are uniformly bounded in} \quad L^2(0, T_1; H). \tag{3.10}$$

Now, by standard methods (see for instance [3], [7], [11]), these estimates enable us to take the limit as $k \to +\infty$ in (2.3) and conclude that a solution for (2.1) exists in the stated class.

In the following we prove the uniqueness of such a solution. Consider that $(u_1, h_1)$ and $(u_2, h_2)$ are two solutions of the problem (2.1) with the same $f$ and $u_0, h_0$ and define the diferences $w = u_1 - u_2$ and $v = h_1 - h_2$. They satisfy

$$\begin{aligned} \alpha(w_t, \phi) + \nu(Aw, \phi) &= -\alpha(w.\nabla u_1, \phi) - \alpha(u_2.\nabla w, \phi) \\ &\quad +(v.\nabla h_1, \phi) + (h_2.\nabla v, \phi) \\ (v_t, \psi) + \gamma(Av, \psi) &= -(u_1.\nabla v, \psi) - (w.\nabla h_2, \psi) + (v.\nabla u_1, \psi) + (h_2.\nabla w, \psi) \end{aligned}$$

for any $\phi, \psi \in V$; also $w(0) = v(0) = 0$.

By setting $\phi = w$ and $\psi = v$ in the above inequalities, we obtain

$$\frac{\alpha}{2}\frac{d}{dt}||w||^2 + \nu||\nabla w||^2 = -\alpha(w.\nabla u_1, w) + (v.\nabla h_1, w) + (h_2.\nabla v, w), \tag{3.11}$$

$$\frac{1}{2}\frac{d}{dt}||v||^2 + \gamma||\nabla v||^2 = -(w.\nabla h_2, v) + (v.\nabla u_1, v) + (h_2.\nabla w, v). \qquad (3.12)$$

Now, we observe that for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$

$$\begin{aligned} |\alpha(w.\nabla u_1, w)| &\leq ||w||||\nabla u_1||_{L^4}||w||_{L^4} \\ &\leq C||w||||Au_1||||\nabla w|| \\ &\leq C_\varepsilon||Au_1||^2||w||^2 + \varepsilon||\nabla w||^2, \end{aligned}$$

thanks to the Sobolev embedding $H^2 \hookrightarrow L^4$ and Hölder and Young inequalities.

Similarly,

$$\begin{aligned} |(v.\nabla h_1, w)| &\leq C_\varepsilon||Ah_1||^2||v||^2 + \varepsilon||\nabla w||^2, \\ |(h_2.\nabla v, w)| &\leq C_\delta||Ah_2||^2||w||^2 + \delta||\nabla v||^2, \\ |(w.\nabla h_2, v)| &\leq C_\delta||Ah_2||^2||w||^2 + \delta||\nabla v||^2, \\ |(v.\nabla u_1, v)| &\leq C_\delta||Au_1||^2||v||^2 + \delta||\nabla v||^2, \\ |(h_2.\nabla w, v)| &\leq C_\varepsilon||Ah_2||^2||v||^2 + \varepsilon||\nabla w||^2, \end{aligned}$$

for any positive $\varepsilon$ and $\delta$ and suitable positive constants $C_\varepsilon, C_\delta$.

Consequently, by taking $\varepsilon = \nu/6$ and $\delta = \gamma/6$, in the above inequalities, (3.11) and (3.12) imply the differential inequality

$$\begin{aligned} &\frac{d}{dt}(\alpha||w||^2 + ||v||^2) + \nu||\nabla w||^2 + \gamma||\nabla v||^2 \\ \leq\ & C(\alpha||w||^2 + ||v||^2)(||Ah_1||^2 + ||Ah_2|| + ||Au_1||^2 + ||Au_2||^2). \end{aligned}$$

By integration in time, the use of Gronwall's inequality and (3.6), we obtain

$$\begin{aligned} \alpha||w(t)|| \ +\ & ||v(t)||^2 + \nu\int_0^t ||\nabla w(s)||^2 ds + \gamma\int_0^t ||\nabla v(s)||^2 ds \\ &\leq (\alpha||w(0)||^2 + ||v(0)||^2)e^{CF(t)}, \end{aligned}$$

Since $w(0) = v(0) = 0$, we finally obtain $w(t) = v(t) = 0$ for all $t \in [0, T_1]$. Hence $u_1 = u_2$ and $h_1 = h_2$ and the uniqueness is proved. For the proof of the continuity of $u(t)$ and $h(t)$ in the $H^1(\Omega)$ -norm, we proceed as in the remark after the definition of strong solution. This completes the proof of the Theorem.

## 4. The Proof of the Theorem 2.2

To prove Theorem 2.2 we will need further estimates for the approximations $u^k, h^k$. To this end, we differentiate (2.3) (i) and (ii) with respect to $t$ and set $\phi = u_t^k$ and $\psi = h_t^k$. We are left with

$$\frac{\alpha}{2}\frac{d}{dt}||u_t^k||^2 + \nu||\nabla u_t^k||^2 = \alpha(f_t, u_t^k) - \alpha(u_t^k.\nabla u^k, u_t^k) \tag{4.1}$$

$$-\alpha(u^k.\nabla u_t^k, u_t^k) + (h_t^k.\nabla h^k, u_t^k) + (h^k.\nabla h_t^k, u_t^k),$$

$$\frac{1}{2}\frac{d}{dt}||h_t^k||^2 + \gamma||\nabla h_t^k||^2 = -(u_t^k.\nabla h^k, h_t^k) - (u^k.\nabla h_t^k, h_t^k) \tag{4.2}$$

$$+(h_t^k.\nabla u^k, h_t^k) + (h^k.\nabla u_t^k, h_t^k)\ .$$

By using the Cauchy-Schwarz and Young's inequalities, we obtain

$$\alpha|(f_t, u_t^k)| \leq \frac{1}{2}||f_t||^2 + \frac{\alpha}{2}||u_t^k||^2\ .$$

Also, we observe that

$$(u^k.\nabla u_t^k, u_t^k) = (u^k.\nabla h_t^k, h_t^k) = 0\ .$$

The second term in (4.2) can be estimated in the following form

$$\begin{aligned} |\alpha(u_t^k.\nabla u^k, u_t^k)| &\leq \alpha||u_t^k||||\nabla u^k||_{L^4}||u_t^k||_{L^4} \\ &\leq C_\varepsilon||u_t^k||^2||Au^k||^2 + \varepsilon||\nabla u_t^k||^2\ , \end{aligned}$$

for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$. Here we have used the Hölder's and Young's inequalities, together with the Sobolev embedding $H^1 \hookrightarrow L^4$.

Similarly, we have

$$\begin{aligned} |(h_t^k.\nabla h^k, u_t^k)| &\leq C_\varepsilon||h_t^k||^2||Ah^k||^2 + \varepsilon||\nabla u_t^k||^2, \\ |(h_t^k.\nabla u^k, h_t^k)| &\leq C_\delta||h_t^k||^2||Au^k||^2 + \delta||\nabla h_t^k||^2, \\ |(u_t^k.\nabla h^k, h_t^k)| &\leq C_\delta||u_t^k||^2||Ah^k||^2 + \delta||\nabla h_t^k||^2, \end{aligned}$$

for any $\varepsilon, \delta > 0$ and suitable $C_\varepsilon, C_\delta > 0$.

To estimate the fifth term in (4.1), we use the Hölder's and Young's inequalities together with the Sobolev embedding $H^2 \hookrightarrow L^\infty$. We obtain for any $\delta > 0$ and suitable $C_\delta > 0$,

$$\begin{aligned} |(h^k.\nabla h_t^k, u_t^k)| &\leq ||h^k||_{L^\infty}||\nabla h_t^k||||u_t^k|| \\ &\leq C_\delta||Ah^k||^2||u_t^k||^2 + \delta||\nabla h_t^k||^2 . \end{aligned}$$

Similary, for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$, we have

$$|(h^k.\nabla u_t^k, h_t^k)| \leq C_\varepsilon||Ah^k||^2||h_t^k||^2 + \varepsilon||\nabla u_t^k||^2 .$$

By taking small enough $\varepsilon > 0$ and $\delta > 0$, by adding (4.1) and (4.2) and by using the above estimates, we are left with the following differential inequality

$$\begin{aligned} &\frac{d}{dt}(\alpha||u_t^k||^2 + ||h_t^k||^2) + (\nu||\nabla u_t^k||^2 + \gamma||\nabla h_t^k||^2) \\ &\leq ||f_t||^2 + C(\alpha||u_t^k||^2 + ||h_t^k||^2)(||Au^k||^2 + ||Ah^k||^2 + 1), \end{aligned}$$

where $C > 0$ is a generic constant. Consequently, for $0 \leq t \leq T_1$, we obtain

$$\begin{aligned} \alpha||u_t^k||^2 &+ ||h_t^k||^2 + C\int_0^t (\nu||\nabla u_t^k||^2 + \gamma||\nabla h_t^k||^2)ds \\ &\leq \alpha||u_t^k(0)||^2 + ||h_t^k(0)||^2 + \int_0^t C||f_t||^2 ds \\ &+ C\int_0^t (\alpha||u_t^k||^2 + ||h_t^k||^2)(||Au^k||^2 + ||Ah^k||^2 + 1)ds . \end{aligned}$$

Now, analogously as in Heywood [7, p. 665], we can prove that

$$\alpha||u_t^k(0)||^2 + ||h_t^k(0)||^2 \leq L ,$$

where $L > 0$ is a constant independent of $k$.

Thus, by applying Gronwall's inequality to the above integral inequality, we obtain

$$\alpha||u_t^k||^2 + ||h_t^k||^2 + C\int_0^t (\nu||\nabla u_t^k||^2 + \gamma||\nabla h_t^k||^2)ds \tag{4.3}$$

$$\begin{aligned} &\leq C(\int_0^t (||f_t||^2 ds + L)e^{\int_0^t \theta(s)ds} \\ &\equiv G(t) \end{aligned}$$

for all $t \in [0, T_1]$, with $\theta(t) = C(||Au^k||^2 + ||Ah^k||^2 + 1)$. By the estimates in (3.6), we have

$$\int_0^t \theta(s)ds \leq Ct + CF(t)$$

for all $t \in [0, T_1]$, where $F(t)$ is a function independent of $k$. Consequently, we conclude that

$$u_t^k, h_t^k \quad \text{are uniformly bounded in} \quad L^\infty(0, T_1; H) \cap L^2(0, T_1; V). \tag{4.4}$$

Now, by taking $\phi = Au^k$ and $\psi = Ah^k$ in (2.3) (i) and (ii), respectively, we obtain

$$\nu||Au^k||^2 \leq C(||f||^2 + ||\dot{u}_t^k||^2 + ||u^k.\nabla u^k||^2 + ||h^k.\nabla h^k||^2), \tag{4.5}$$

$$\gamma||Ah^k||^2 \leq C(||h_t^k||^2 + ||u^k.\nabla h^k||^2 + ||h^k.\nabla u^k||^2)\ . \tag{4.6}$$

We observe that

$$\begin{aligned} ||u^k.\nabla u^k||^2 &\leq ||u^k||^2_{L^4}||\nabla u^k||^2_{L^4} \\ &\leq C||\nabla u^k||^2(||\nabla u^k||^{1/2}||Au^k||^{3/2}) \\ &\leq C||\nabla u^k||^{5/2}||Au^k||^{3/2} \\ &\leq C_\varepsilon||\nabla u^k||^{10} + \varepsilon||Au^k||^2 \end{aligned}$$

for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$.

Analogously as above, we obtain

$$\begin{aligned} ||h^k.\nabla h^k||^2 &\leq C_\delta||\nabla h^k||^{10} + \delta||Ah^k||^2, \\ ||u^k.\nabla h^k||^2 &\leq C_\delta||\nabla u^k||^8||\nabla h^k||^2 + \delta||Ah^k||^2, \\ ||h^k.\nabla u^k||^2 &\leq C_\varepsilon||\nabla h^k||^8||\nabla u^k||^2 + \varepsilon||Au^k||^2. \end{aligned}$$

Now, by adding the inequalities (4.5) and (4.6) and by using the above estimates, together with a suitable choice of small enough $\varepsilon$ and $\delta$, we are left with.

$$\nu||Au^k||^2 + \gamma||Ah^k||^2 \leq C(||f||^2 + ||u_t^k||^2 + ||h_t^k||^2$$

$$
\begin{aligned}
&+ \ ||\nabla u^k||^{10} + ||\nabla h^k||^{10} + ||\nabla u^k||^8 ||\nabla h^k||^2 \\
&+ \ ||\nabla h^k||^8 ||\nabla u^k||^2) \\
&\leq \ C(||f||^2 + F(t)^5 + G(t)) \\
&\equiv \ H(t)
\end{aligned}
$$

for all $t \in [0, T_1]$.

Here we have used the estimates obtained in the proof Theorem 1 and (4.3). Hence, we have

$$
u^k, h^k \quad \text{are uniformly bounded in} \quad L^\infty(0, T_1; D(A)) \ . \tag{4.7}
$$

Now, by standard methods [3], [7], [11], these estimates enable us to assert that the solution $(u, h)$ of the problem (2.1) satisfies the stated estimates.

To prove the continuity of $u_t(t)$ in the $L^2$-norm, we only need to show that $u_{tt}$ is in $L^2(0, T_1; V^*)$. In fact, if $u_{tt} \in L^2(0, T_1; V^*)$ then the fact that $u_t$ is in $L^2(0, T_1; V)$, implies that $u \in C^1([0, T]; H)$ (Lemma 1.2, p. 260 in Temam [11]).

To prove that $u_{tt} \in L^2(0, T_1, V^*)$, we observe that it is enough to show the existence of $C > 0$ (independent of $k$) such that

$$
\int_0^{T_1} ||u_{tt}^k(s)||_{V^*}^2 ds \leq C \ .
$$

To this end, we differentiate equation (2.2) with respect to $t$; we obtain

$$
\begin{aligned}
u_{tt}^k &= P_k(\alpha f_t + h_t^k . \nabla h^k + h^k . \nabla h_t^k - \alpha u_t^k . \nabla u^k - \alpha u^k . \nabla u_t^k) - \nu A u_t^k \\
&\equiv g^k \ .
\end{aligned}
$$

The above estimates for $u^k$ and $h^k$ imply that $g^k$ is uniformly bounded in $L^2(0, T_1; V^*)$. In fact, we have

$$
\begin{aligned}
||P_k(h_t^k . \nabla h^k)||_{V^*} &= \sup_{||v||_V \leq 1} |(P_k h_t^k . \nabla h^k, v)| \\
&\leq \sup_{||v||_V \leq 1} |(h_t^k . \nabla h^k, P_k v)| \\
&\leq C \sup_{||v||_V \leq 1} ||h_t^k||_{L^4} ||\nabla h^k|| ||v||_{L^4} \\
&\leq C ||\nabla h_t^k|| \ .
\end{aligned}
$$

Here we have used the Sobolev embedding $H^1 \hookrightarrow L^4$, the estimate (3.7) and the continuity of $P_k$ in $L^4$ (von Wahl [12, p. XXIII]); $C$ denotes a general constant depending only the previous estimates.

Consequently, due to estimates (4.4) we obtain

$$\int_0^{T_1} ||P_k(h_t^k.\nabla h^k)||_{V^*}^2 ds \leq C \int_0^{T_1} ||\nabla h_t^k||^2 ds \leq C,$$

where $C > 0$ is independent of $k \in N$.

Also, we have

$$\begin{aligned} ||P_k u^k.\nabla u_t^k||_{V^*} &= \sup_{||v||_V \leq 1} |(u^k \nabla u_t^k, P_k v)| \leq ||u^k||_{L^\infty} ||\nabla u_t^k|| \\ &\leq C||\nabla u_t^k||. \end{aligned}$$

Consequently, $\int_0^{T_1} ||P_k u^k.\nabla u_t^k||_{V^*}^2 ds \leq C$ thanks to the above estimates.

Also, from

$$||Au_t^k||_{V^*} = \sup_{||v||_V \leq 1} |(Au_t^k, v)| = \sup_{||v||_V \leq 1} |(\nabla u_t^k, \nabla v)| \leq ||\nabla u_t^k||,$$

we conclude that $\int_0^{T_1} ||Au_t^k||_{V^*}^2 ds \leq C$.

The other terms in the $g^k$ are analogously estimated.

To prove the continuity of $h_t(t)$ in the $L^2$-norm, we work exactly as before.

To finish the proof we have to show the continuity of $u(t)$ and $h(t)$ in the $H^2(\Omega)$-norm. We will only prove the continuity of $h(t)$; the proof for $u(t)$ is quite similar. Also, we will prove this continuity only at $t_0 = 0$; for other $t_0 > 0$ the argument is analogous.

We observe that $h \in L^\infty(0, T_1, D(A))$ (see (4.7)). Thus, given any sequence $\{t_k\}_{k=0}^\infty \subset \mathbb{R}^+$, with $t_k \to 0^+$ we can extract a subsequence such that $h(t_{n_k}) \to \overline{h}$ weakly in $H^2$ for some $\overline{h} \in H^2$. Since we know (Theorem 2.1) that $h(t_{n_k}) \to h_0$ strongly, the above implies that $\overline{h} = h_0$. Moreover, since this holds for any sequence $\{t_k\}_{k=0}$, with $t_k \to 0^+$, we conclude that $h(t) \to h_0$ weakly in $H^2$ as $t \to 0^+$. Consequently, due to the lower semicontinuity with respect to the weak topology of the norm, we have $||Ah_0|| \leq \liminf_{t \to 0^+} ||Ah(t)||$. Now, if we are able to prove that

$$\limsup_{t \to 0+} ||Ah(t)|| \leq ||Ah_0||, \tag{4.8}$$

then we will have $\lim_{t\to 0^+} ||Ah(t)|| = ||Ah_0||$, which together with the fact that $Ah(t) \to Ah_0$ weakly in $L^2$ will imply that $Ah(t) \to Ah_0$ strongly in $L^2$.

In order to prove (4.8) we proceed as follows: put $\psi = P\Delta h_t^k$ in (2.3) to obtain

$$\begin{aligned} ||\nabla h_t^k||^2 \; + \; & \frac{1}{2}\gamma \frac{d}{dt}||Ah^k||^2 = (u^k.\nabla h^k - h^k.\nabla u^k,\; P\Delta h_t^k) \\ = \; & \frac{d}{dt}(u^k.\nabla h^k - h^k.\nabla u^k,\; P\Delta h^k) \\ - \; & (u_t^k.\nabla h^k + u^k.\nabla h_t^k - h_t^k.\nabla u^k - h^k.\nabla u_t^k, P\Delta h^k)\,. \end{aligned}$$

By integration with respect to time, and by using our previous estimates for $u^k$ and $h^k$, we obtain

$$\begin{aligned} \gamma||Ah^k(t)||^2 \; \leq \; & \gamma||Ah_0||^2 + 2\{(u^k.\nabla h^k - h^k.\nabla u^k, P\Delta h^k) \\ - \; & (u_0^k.\nabla h_0^k - h_0^k.\nabla u_0^k, P\Delta h_0^k)\} + Mt, \end{aligned}$$

where $M$ is a positive constant depending on the previous estimates in $k$. From this, we conclude

$$\begin{aligned} \gamma||Ah(t)||^2 \; \leq \; & \gamma||Ah_0||^2 + 2\{(u.\nabla h - h.\nabla u, P\Delta h) \\ - \; & (u_0.\nabla h_0 - h_0.\nabla u_0, P\Delta h_0)\} + Mt \end{aligned}$$

Now, since $u.\nabla h \to u_0.\nabla h_0$, $h.\nabla u \to h_0.\nabla u_0$ in $L^2$ and $P\Delta h \to P\Delta h_0$ weakly in $L^2$ as $t \to 0^+$. We obtain the desired result.

## 5. The Proof of the Theorem 2.3

The observe that (2.1) (i) and (ii) are equivalent to $Au = P(\overline{f})$ and $Ah = P(\overline{g})$, where $\overline{f} = \alpha f - u_t - \alpha u.\nabla u + h.\nabla h$ and $\overline{g} = h.\nabla u - u.\nabla h - h_t$, respectively.

Now, we observe that under the hypothesis of the Theorem 2.1 (respectively of Theorem 2.2), we have $\overline{f}, \overline{g} \in L^2(0, T_1; L^2(\Omega))$ (respectively $\overline{f}, \overline{g} \in L^\infty(0, T_1; L^2(\Omega))$).

Therefore, Amrouche and Girault's results [1] imply that there are unique $p, w \in L^2(0, T_1; H^1(\Omega)/\mathbb{R})$ (respectively $p, w \in L^\infty(0, T_1; H^1(\Omega)/\mathbb{R} \cap$

$C([0,T_1];L^2(\Omega)/I\!R))$ such that

$$-\nu\Delta u + .\nabla p = \overline{f}$$
$$\text{div } u = 0$$
$$u|_{\partial\Omega} = 0$$

and

$$-\gamma\Delta h + \nabla r = \overline{g}$$
$$\text{div } h = 0$$
$$h|_{\partial\Omega} = 0.$$

Now, it is enough to take $p^* = p - \frac{\mu}{2}h^2$ and Theorem 2.3 is proved.

## References

[1] Amrouche, G. and Girault, V., *On the existence and regularity of the solutions of Stokes problem in arbitrary dimension*, Proc. Japan Acad., 67, Ser. A (1991), 171-175.

[2] Chizhonkov, E. V., *On a system of equations of magnetohydrodinamic type*, Soviet Math. Dokl. 30 (1984), 542-545.

[3] Constantin, P. and Foias C., *Navier-Stokes equations*, The University of Chicago Press, Chicago and London, 1988.

[4] Duff, G. F. D., *Particle Path Length Estimates for the Navier-Stokes equations in three space dimensions*, Hokkaido Math. J., 12 (1983), 461-471.

[5] Friedman, A., *Partial Differential Equations*, Robert E. Krieger Publ. Co., Huntington, New York, 1976.

[6] Fujita, H. and Kato, T., *On the Navier-Stokes initial value problem*, I, Arch. Rational Mech. Anal., 16 (1964), 269-315.

[7] Heywood, J. G., *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Math. J., 29 (1980), 639-681.

[8] Lassner G., *Über ein rand-anfangswert-problem der magnetohydrodinamik*, Arch. Rational Mech. Anal., 25 (1967), 388-405.

[9] Pikelner, S. B., *Grundlagen der Kosmischen elektrodynamik*, Moscou, 1966.

[10] Schlüter, A., *Dynamik des plasmas, I and II*, Z. Naturforsch. 5a., (1950), 72-78; 6a., (1951), 73-79.

[11] Temam, R., *Navier-Stokes equations*, North-Holland, Amsterdam, Rev. Edit., 1979.

[12] Von Wahl, W., *The equations of Navier-Stokes and abstract parabolic equations*, Aspects of Math. 58, Vieweg, Braunschweig-Wiesbaden, 1985.

UNICAMP - IMECC
C.P. 6065
13081-970, Campinas, SP
Brazil