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ALGEBRAIC ASPECTS OF TANGENT CONES

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Contents

1 Introduction

The present notes form an enlarged update of a series of lectures at the XII Escola de Álgebra, for a mixed audience of specialists whose expertise lied

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somewhere between Commutative Algebra and Algebraic Geometry.

The notes do no claim complete originality. As a matter of fact, almost entirety based on a forthcoming joint paper by B. Ulrich, W. Vasconcelos and myself [32]. Since this paper is of a somewhat difficult nature and requires a good deal of background from deeper themes of Commutative Algebra, I then thought it appropriate to offer a souped-up version of the (prerequisites for the) paper. In particular, all the results are stated here over a base field, while in [32] everything is developed for algebras flat over a general noetherian base ring (with an eye in theorems for flat families, cf. the forthcoming [19]). This on itself constitutes a distinction and, in a sense, these lectures took the easy way out.

Anyhow, when asked to include the notes in the present Proceedings, I pondered whether a set of (mostly unproved) statements of technical flavour would be of any help for potential readers. Thrust into this perplexity, nonetheless still willing to contribute to the Proceedings, I arrived at a certain formula by which I was to expand and give most proofs of the easiest parts and let the more difficult statements stand as a reminder of the aforementioned joint paper.

The net result is an account that may be of some avail to newcomers as well as experts, in the way of a semi-original surveylike source. Here, the reader may enjoy elementary versions of quite a few among difficult results published in the recent literature in the field, for which I am myself responsible together with many co-authors.

To the latter, specially B. Ulrich and W. Vasconcelos, I wish to thank for letting me freely use some of our published (and unpublished) work. Alas, I cannot ask them to be accessory to my mistakes!

I would also like to aknowledge some exchange with F. Gaeta, who helped me out in understanding a little better the classical geometric connections to the subject.

The rest of my thanks go to the organizers of the XII Escola: by succeeding in carrying it through, they showed the same vigor and stubborness of the early portuguese explorers when faced with the need for founding Diamantina in the

middle of nowhere!

1.1 Early geometric ground

We may start by considering some of the geometric ideas lying on the background.

Thus, let X be an algebraic variety over a field k. For $x \in X$ a point on the variety, one has the notion of the cone to X over x. Apparently, this notion was made crystalline only in the late fifties or early sixties thanks to the general effort at the period to give solid foundation to the shaky concepts of the preceding periods.

Most of this effort was concentrated in the abstract setup, that is to say, without imposing any restriction on the base field k. A seemingly isolated trial was carried out by H. Whitney to convey various alternatives for the notion of a cone over a point, assuming an analytic setup [45], [46].

Three notions of tangent cone seem to have gained more popularity than the others, ever since the main parts of modern Algebraic Geometry started being developed. The first two are defined, respectively, in terms of the linear forms (Zariski tangent space) and the initial forms (normal cone) of the equations belonging to the ideal of X at the point x. These definitions, as can be shown by the standard machinery of local algebra, depend only on X and x (not on a particular embedding of X).

The third species, less known to algebraists, is one of which specialists in Segre classes and intersection theory are fond of – although, as a slight irony, Whitney himself didn't seem to believe in any important applications for it! Like the Zariski tangent space, it has the advantage of admitting an immediate global version which the cone of initial forms seems to lack. The reason is that it is definable directly in terms of the diagonal embedding $X \to X \times X$, without reference to the ambient space of X (in case, say, X is affine).

First systematically employed by K. Jonhson [17] – who suggested the name *tangent star* – , this third kind of cone has recently been considered in a possibly different perspective by G. Kennedy [18] and L. van Gastel [7].

The subject has strong ties with the theory of varieties of secant lines and a suitable generalization of it is related to the varietiy of chords. The special fiber of what is here called the *star cone* coincides in some cases with the secant variety.

1.2 The role of the star cone in Segre classes

Recall the "classical" set-up: X is a d-dimensional non-singular projective variety and $\mathcal{F}$ is a locally free sheaf of rank r on X. Then, the ith *Segre class* of $\mathcal{F}$ is defined as

$$s_i(\mathcal{F}) := p_*(\xi^{d-1+i} \cap [\mathbf{P}(\mathcal{F})]) \in A^i X,$$

where

- $\mathbf{P}(\mathcal{F}) = \text{Proj } (S_{\mathcal{O}_X}\mathcal{F})$ is the total bundle space, with natural projection map $p : \mathbf{P}(\mathcal{F}) \to X$;
- $A^{\cdot}Y$ is the *Chow cohomology ring* of a projective variety Y;
- $\xi \in A^1\mathbf{P}(\mathcal{F})$ is the *Chern class* of the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$.

The ith *Segre class* of X itself is, by convention, the ith *Segre class* of the sheaf of Kähler differentials $\Omega(X/k)$.

The non-classical case assumes that X is any purely d-dimensional variety and replaces $\mathbf{P}(\Omega(X/k)) = \mathbf{P}(\mathbf{D}/\mathbf{D}^2)$ by $\mathbf{P}(\oplus_{j\geq 0}\mathbf{D}^j/\mathbf{D}^{j+1})$, where $\mathbf{D}$ is the ideal sheaf of the image of the diagonal map $X \to X \times X$.

The main point about the usefulness of this construction is perhaps the following result:

Proposition. *Let $X \subset \mathbf{C}^n$ be a complex affine variety, let $\Delta : X \to X \times X$ denote the diagonal map and $\mathbf{D}$, the ideal sheaf of the subvariety $\Delta(X) \subset X \times X$. Let further $\mathbf{G}$ denote the Grassmannian of lines in $\mathbf{P}^n$ and consider the map*

$$(X \times X) \setminus \Delta(X) \longrightarrow \mathbf{G},$$

which takes a point $(x, y), x \neq y$ to the line $\overline{xy} \in \mathbf{G}$. Then:

(i) $X \widetilde{\times} X \simeq \mathbf{P}(\oplus_{j\geq 0}\mathbf{D}^j)$, *where* $X \widetilde{\times} X$ *denotes the closure of the graph of* Δ *in* $(X \times X) \times \mathbf{G}$;

(ii) *For any closed point* $x \in X$, *the fiber* $\mathbf{P}(\oplus_{j\geq 0}\mathbf{D}^j/\mathbf{D}^{j+1}) \times_X \mathrm{k}(x)$ *can be identified set theoretically with the union of lines* $l \subset \mathbf{P}^n$ *for which there exist sequences* $\{y_i\}, \{y_i'\}$ *of points in* X *converging to* x *such that the sequence of lines* $\overline{y_i, y_i'}$ *converges to* l.

Thus, roughly, the fibers of the exceptional locus of the blow-up along the diagonal embedding are limits of secant lines to the variety. That was the original viewpoint of Whitney who was mainly interested in this sort of condition for results of stratification nature.

The main focus of these notes is on the coordinate ring of the projecting cone over $\mathbf{P}(\oplus_{j\geq 0}\mathbf{D}^j/\mathbf{D}^{j+1})$. The following sections will develop the underlying algebraic tools to deal appropriately with it.

2 A second view of dimension theory

Symmetric algebras of modules constitute the algebraist's version of (not necessarily locally free) bundles. Even from the viewpoint of Algebraic Geometry there is enough reason to deal with modules that are not necessarily locally free everywhere. Thus, an ordinary vector bundle over a projective variety $V \subset \mathbf{P}^n$ is given by a module over the homogeneous coordinate ring $k[X_1, \ldots, X_n]/\mathcal{I}(V)$ which is locally free outside the maximal irrelevant ideal.

Although symmetric algebras are quite ubiquitous in the present theme, in order to have a deeper feeling for its meaningfulness – not just for its formal role in the definitions – it is important to look at its dimension. The main point throughout is that the symmetric algebra has an *expected dimension* or *virtual dimension*, which works like an obstruction for certain behaviour pattern.

We collect a few results of technical nature which clarify the dimension-theoretic background. Some of these will be useful in the following sections.

2.1 The role of the Fitting invariants

Let R be a noetherian ring and let E be a finitely generated R-module with a given presentation

$$G \xrightarrow{\varphi} F \longrightarrow E \longrightarrow 0, \tag{1}$$

where F and G are free modules of finite rank. Say, $\operatorname{rk} F = n$. The *tth Fitting invariant* of E is the determinantal ideal $I_{n-t}(\varphi)$ of a matrix associated to φ – a standard exercise shows that the definition depends only on E and not on the selected presentation.

In practice, it is more convenient to work from a fixed presentation and with the corresponding determinantal ideals, so we will allow ourselves to generously draw on that.

Lemma 2.1 *Let E admit a presentation as (1). Given a number $t \geq 0$ and a prime ideal $P \subset R$, one has $I_t(\varphi) \not\subseteq P \Leftrightarrow \mu(E_P) \leq \operatorname{rk} F - t$ (resp. $I_t(\varphi) \not\subseteq P$ and $I_{t+1}(\varphi) \subseteq P \Leftrightarrow \mu(E_P) = \operatorname{rk} F - t$).*

Proof. Localizing at P and possibly changing the bases of the free modules in (1), one arrives to a presentation of E_P of the form

$$R_P^k \oplus R_P^r \stackrel{\mathrm{id}\oplus\ \varphi}{\longrightarrow} R_P^k \oplus R_P^\mu \longrightarrow E_P \longrightarrow 0,$$

where $k = \operatorname{rk} F - \mu$ and $\mu = \mu(E_P)$. Then the matrix of φ has entries in P_P and, from this, one clearly sees that $I_t(\varphi) \subseteq P$ if and only if $t \geq t + 1$, which shows the main assertion. We leave the supplementary assertion as an exercise. □

Recall that E is said to *have a rank*, $\operatorname{rk} E = r$, if E_P is R_P-free of rank r for every associated prime P of R.

Lemma 2.1 admits a uniform version for all relevant values of t taken at once.

Proposition 2.2 *Let E be a finitely generated R-module having a rank. The following conditions are equivalent for an integer $k \geq 0$:*

(i) *For every* $P \in \operatorname{Spec}(R)$, *the inequality*

$$\mu(E_P) \leq \operatorname{ht} P + \operatorname{rk} E + k$$

holds (resp. and for some $P \in \operatorname{Spec}(R)$ *the equality is attained).*

(ii) *For* any *presentation as (1) and any* $1 \leq t \leq \operatorname{rk}(\varphi)$, *the inequality*

$$\operatorname{ht} I_t(\varphi) \geq \operatorname{rk} \varphi - t + 1 - k$$

holds (resp. and for some $1 \leq t \leq \operatorname{rk} \varphi$ *the equality is attained).*

(iii) *For* some *presentation as (1) and any* $1 \leq t \leq \operatorname{rk}(\varphi)$, *the inequality*

$$\operatorname{ht} I_t(\varphi) \geq \operatorname{rk} \varphi - t + 1 - k$$

holds (resp. and for some $1 \leq t \leq \operatorname{rk} \varphi$ *the equality is attained).*

Proof. We first argue for the inequalities.
(i) $\Rightarrow$ (ii) Given t in the required interval, pick a prime $P \supset I_t(\varphi)$ such that $\operatorname{ht} I_t(\varphi) = \operatorname{ht} P$. Then

$$\operatorname{ht} I_t(\varphi) \geq \mu(E_P) - \operatorname{rk} E - k = \operatorname{rk} \varphi - s_P - k,$$

where $s_P := \operatorname{rk} F - \mu(E_P)$. By Lemma 2.1, one has $s_P \leq t - 1$, as required.
(ii) $\Rightarrow$ (iii) Trivial.
(iii) $\Rightarrow$ (i) Let $P \in \operatorname{Spec}(R)$ and s_P as above. Then $I_{s_P} \subseteq P$ by Lemma 2.1. On the other hand,

$$\begin{aligned} s_P &= \operatorname{rk} F - \mu(E_P) \\ &\leq \operatorname{rk} F - \operatorname{rk} E = \operatorname{rk} \varphi. \end{aligned}$$

Then, by the assumption, $\operatorname{ht} P \geq \operatorname{rk} \varphi - s_P - k = \mu(E_P) - \operatorname{rk} E - k$, as needed to be shown.

Finally, the supplementary assertions as to when the equalities are attained follow from the corresponding ones in Lemma 2.1. □

Proposition 2.2 motivates the following notion.

Definition 2.3 The least $k \geq 0$ such that E satisfies (any of) the conditions of Proposition 2.2 is called the *Fitting deffect* of E, denoted $\operatorname{fd} E$. E is said to satisfy $(\mathcal{F}_{-k})$ if $k \geq \operatorname{fd} E$.

The Fitting defect is actually a dimension defect for the symmetric algebra $S(E)$ as we will soon see.

On the other hand, a version of the property $(\mathcal{F}_{-k})$ is available for negative values of k, as was extensively treated in [11], [35], [12]. Namely, one says that E satisfies $(\mathcal{F}_{-k})$ provided the inequality

$$\mu E_P \leq \operatorname{ht} P + \operatorname{rk} E + k$$

holds for every $P \in \operatorname{Spec}(R)$ not lying in the free locus of E.

For negative values of k, the latter property has no impact on the dimension. Its main bearing is to the finer properties of E.

Exercise 2.4 What is the relation between an ideal $I \subset R$ and the module I/I^2 regarding the condition $(\mathcal{F}_{-k})$? What condition is to be imposed on I so that both I and I/I^2 have ranks?

2.2 Variations on the dimension of symmetric algebras

We first consider the case of an ideal $I \subset R$. As it will turn out, this case already implies the general case.

Let $R[It] \subset R[t]$ denote the Rees algebra of I. There is a canonical surjective R-algebra homomorphism

$$S(I) \xrightarrow{\alpha} R[It]. \tag{2}$$

The ideal I is said to be linear type if the map (2) is injective. The basic model of an ideal of linear type is given by an ideal generated by a regular sequence.

Exercise 2.5 If I is an ideal of linear type then $\mu(I_P) \leq \operatorname{ht} P$ for every $P \in \operatorname{Spec}(R/I)$.

The following proposition is an easy consequence of the general dimension formula of Huneke–Rossi [15]. However, we give an independent simple proof.

Proposition 2.6 *If* grade $I \geq 1$ *then* $\dim S(I) = \max\{\dim R+1, \dim S(I/I^2)\}$.

Proof. Consider the canonical map (2): for every $P \in \operatorname{Spec}(R)$ such that $I \not\subset P$, the induced localization $\alpha_P : S(I)_P \to R_P[I_P t]$ is clearly an isomorphism. This shows that $I^t \ker\alpha = (0)$ for $t >> 0$, hence every prime ideal of the ring $S(I)$ contains either $IS(I)$ or $\ker\alpha$. Therefore,

$$\begin{aligned}\dim S(I) &= \max\{\dim R[It], \dim S_{R/I}(I/I^2)\}\\ &= \max\{\dim R+1, \dim S_{R/I}(I/I^2)\},\end{aligned}$$

since $\dim R[It] = \dim R + 1$ for ideals containing regular elements. □

Corollary 2.7 *Let* $R, \mathbf{m}$ *be a local ring. If* $I \subset R$ *is an* $\mathbf{m}$*-primary ideal then* $\dim S(I) = \max\{\dim R + 1, \mu(I)\}$.

Proof. Since $\operatorname{Supp}(S(I/I^2)) = \operatorname{Supp}(S(I/\mathbf{m}I))$, the result follows from Proposition 2.6. □

Ideals satisfying the formula of Corollary 2.7 were referred to by this author as being of *Valla type* (cf. [29]).

Example 2.8 ([29, (4.6)]) If I is an ideal of a polynomial ring generated by degree 2 squarefree monomials corresponding to the edges of a simple graph G, then

$$\dim S(I) = \begin{cases} \dim R + 1 & \text{if } \operatorname{rk} G \leq 2 \\ \dim R - 1 + \operatorname{rk} G & \text{if } \operatorname{rk} G > 2 \end{cases}.$$

This is slightly more precise than just knowing that an edge–ideal is of Valla type, the latter having originally being shown by Villarreal [43].

Exercise 2.9 Let $A = k[X,Y,Z]_{(X,Y,Z)}, \mathbf{n} = (X,Y,Z)_{(X,Y,Z)}$ and let $J \subset A$ be $\mathbf{n}$-primary with 5 generators. Let Tbe an indeterminate over A, let $R = A[T]_{(\mathbf{n},T)}, \mathbf{m} = (\mathbf{n},T)R$ and let $I = JR$. Then $\dim S_R(I) = \dim S_A(J) + 1$, thus showing that the ideal I is not of Valla type.

Determinantal ideals also fail to be of Valla type, except for special row and column sizes [15].

Ideals of linear type are also important because of the next seemingly innocent result.

Proposition 2.10 *Let A be a noetherian ring and let $I \subset A$ be an ideal of linear type. Then*

$$\dim A \geq \sup_{P \supseteq I} \{\dim A/P + \mu(I_P)\}.$$

Proof. It follows straightforwardly from Exercise 2.5. □

Of course, one has to impose further conditions in order to obtain equality above. We say that an ideal $I \subset A$ is a *junction ideal* if there is a $P \in \mathrm{Min}(A/I)$ such that a maximal chain of primes of A passes through P. The notion is perhaps only interesting if $\dim A < \infty$.

Theorem 2.11 *Let A be a noetherian ring of finite dimension admitting a junction ideal I of linear type. Then*

$$\dim A = \sup_{P \supseteq I} \{\dim A/P + \mu(I_P)\}.$$

Proof. By Proposition 2.10, it suffices to show that $\dim A = \dim A/P + \mu(I_P)$ for some $P \supseteq I$. Let $P \in \mathrm{Min}(A/I)$ be such that $\dim A = \dim A/P + \mathrm{ht}\, P$. Then $\mathrm{ht}\, P = \mathrm{ht}\, I_P \leq \mu(I_P)$ by general reasons. Since I is of linear type, one must have $\mathrm{ht}\, P = \mu(I_P)$. □

The theorem is pointless without knowing when an ideal is a junction ideal. One has

Lemma 2.12 *Let A be noetherian (resp. graded ring of finite type over its zero part) and $I \subset A$ an ideal contained in the Jacobson radical (resp. the graded Jacobson radical) of A. Then I is a junction ideal.*

Proof. (According to Herrmann, Moonen and Villamayor) Set $B = \mathrm{gr}_I(A)$. Since I is contained in the Jacobson radical, $\dim B = \dim A$. By a well-known consequence of the dimension inequality [22, Theorem 2.3], for any $Q \in \mathrm{Spec}(B)$ and $P := Q \cap A$:

$$\begin{aligned} \dim B/Q &\leq \dim A/P + \dim B \otimes_A A_P/PA_P \\ &= \dim A/P + \dim \mathrm{gr}_{I_P}(A_P)/P\mathrm{gr}_{I_P}(A_P) \\ &\leq \dim A/P + \dim A_P = \dim A/P + \mathrm{ht}\, P. \end{aligned}$$

Applying with Q such that $\dim B = \dim B/Q$, we get $\dim A \leq \dim A/P + \mathrm{ht}\, P$, as required.

We leave the graded version as an exercise. □

We are now ready to give a proof of the Huneke–Rossi formula.

Theorem 2.13 ([15]) *Let R be a noetherian ring and let E be a finitely generated R-module. Then*

$$\dim S(E) = \sup_{P \in \mathrm{Spec}(R)} \{\dim R/P + \mu(E_P)\}.$$

Proof. Since $S(E)$ is graded with R as its zero part, we won't affect its dimension by passing to the ring of fractions $A := S(E)_{1+S(E)_+}$. The exteded ideal $I := (S(E)_+)S(E)_{1+S(E)_+}$ is clearly contained in the (graded) Jacobson radical of A. By Lemma 2.12, I is a junction ideal.

On the other hand, $S(E)_+$ is an ideal of linear type [10, Example 2.3], hence so is I. Thus, we can apply Proposition 2.11 by further noticing that

1. Any $Q \in \mathrm{Spec}(S)(E)$ containing $S(E)_+$ is of the form $(P, S(E)_+)$ for some $P \in \mathrm{Spec}(R)$

2. $S(E)/S(E)_+ \simeq E$.

The rest is standard. □

Exercise 2.14 Show that, conversely, the Huneke–Rossi formula implies the formula stated in Proposition 2.11 for ideals contained in the Jacobson radical.

Next is a nontrivial application of the Huneke–Rossi formula that has often been quoted in the related literature under various forms and hypotheses. We believe the one here gives its state-of-the-art.

Recall that a noetherian ring of finite Krull dimension is said to be *equicodimensional* if all maximal ideals of R have the same codimension. We say that R has the *dimensional complementarity* property if $\dim R = \operatorname{ht} P + \dim R/P$ holds for every prime $P \in \operatorname{Spec}(R)$.

Theorem 2.15 *Let R be a noetherian ring of finite Krull dimension and let $I \subset R$ be an ideal. Consider the following two conditions:*

(i) $\dim S_{R/J}(J/J^2) = \dim \operatorname{gr}_J(R)$

(ii) $\mu(J_P) \leq \operatorname{ht} P$ *for every prime ideal* $P \supseteq J$.

Then (ii) $\Rightarrow$ (i). *If R has the complementarity property and is further equicodimensional (or J is contained in the Jacobson radical of R) then* (i) $\Rightarrow$ (ii).

Proof. (ii) $\Rightarrow$ (i). Since there is a surjective homomorphism

$$\dim S_{R/J}(J/J^2) \rightarrow \dim \operatorname{gr}_J(R),$$

it suffices to show that $\dim S_{R/J}(J/J^2) \leq \dim \operatorname{gr}_J(R)$. By the Huneke–Rossi formula (Proposition 2.13), one has

$$\begin{aligned} \dim S_{R/J}(J/J^2) &= \sup_{P \supseteq J} \{\mu(J_P/J_P^2) + \dim(R/J)/(P/J)\} \\ &= \sup_{P \supseteq J} \{\mu(J_P) + \dim R/P\} \\ &= \leq \sup_{P \supseteq J} \{\operatorname{ht} P + \dim R/P\}. \end{aligned} \tag{3}$$

Now, for each $P \supseteq J$, choose a maximal ideal M of R containing P and such that $\dim R/P = \dim R_M/P_M$. We get

$$\operatorname{ht} P + \dim R/P = \operatorname{ht} P_M + \dim R_M/P_M \leq \dim R_M. \tag{4}$$

Replacing the estimate (4) in (3) yields

$$\begin{aligned} \dim S_{R/J}(J/J^2) &\leq \sup_{M \supseteq J} \{\dim R_M\} \quad M \text{ maximal} \\ &= \dim \operatorname{gr}_J(R), \end{aligned}$$

by the well-known formula for the dimension of the associated graded ring.

(i) $\Rightarrow$ (ii). Since R is equicodimensional (or J is contained in the Jacobson radical of R), the assumption reads

$$\sup_{P \supseteq J} \{\mu(J_P) + \dim R/P\} = \dim R.$$

Thus, for every prime $P \supseteq J$, one has $\mu(J_P) \leq \dim R - \dim R/P = \operatorname{ht} P$, by the complementarity hypothesis. □

Remark 2.16 Thus, the equality $\dim S_{R/J}(J/J^2) = \dim \operatorname{gr}_J(R)$, without additional hypotheses, is not sufficient to trigger the local estimates $\mu(J_P) \leq \operatorname{ht} P$, a condition easily implied by the linear type property. On the other hand, many rings satisfy the complementarity property, among them the Cohen–Macaulay rings (more generally, rings that are both equidimensional and equicodimensional – for example, equidimensional finite type algebras over a field) or rings that possess a finitely generated faithful Cohen–Macaulay module.

3 Zariski and tangent star algebras

In this section we introduce the main object of our study in a more systematic way.

The following notation will prevail thoughout.

k	a base field
R	a ring of fractions of a polynomial ring $k[\underline{X}] = k[X_1, \ldots, X_n]$
A	a k-algebra of the form R/I, $I \subset R$ an ideal
$\mathbf{D}$	the kernel of the multiplication map $A \otimes_k A \to A$, generated by the residues of the elements $X_i \otimes 1 - 1 \otimes X_i$ (*diagonal ideal*)
$\Omega(A/k)$	the module of Kähler k-differentials of A
$\mathbf{Z}$	the symmetric algebra of the A-module $\Omega(A/k)$
$\mathbf{T}$	the associated graded ring $\operatorname{gr}_{\mathbf{D}}(A \otimes_k A)$

The scheme defined by the algebra $\mathbf{Z}$ is called the *Zariski cone* of (the variety defined by) A, while the one defined by $\mathbf{T}$, following the suggestion of Johnson's, will be named the *tangent star cone* of (the variety defined by) A.

The algebras $\mathbf{Z}$ and $\mathbf{T}$ will be called the *Zariski tangent algebra* and the *tangent star algebra*, respectively, according to the terminology that has been introduced in [32].

The well-known identification $\Omega(A/k) \simeq \mathbf{D}/\mathbf{D}^2$ yields a surjective homomorphism

$$\mathbf{Z} \longrightarrow \mathbf{T}.$$

Various ways of measuring the kernel of this map were introduced in [32], based on earlier development of the theory of algebras of linear type.

Definition 3.1 (i) A has the *expected star dimension* if

$$\dim \mathbf{Z} = \dim \mathbf{T}.$$

(ii) A is *set theoretically starlike linear* if

$$(\mathbf{Z})_{\text{red}} = (\mathbf{T})_{\text{red}}.$$

(iii) A is *starlike linear* if

$$\mathbf{Z} = \mathbf{T}.$$

Clearly, $(iii) \Rightarrow (ii) \Rightarrow (i)$. We will see that none of these implications is reversible.

Although these definitions look rather inconspicuous, they become natural as soon as one works out their meaning in concrete cases. Thus, for example, (i) above at least implies that A is generically reduced, hence reduced if it is also unmixed (in particular, a 0-dimensional A having the expected star dimension must be a product of fields). If, moreover, $A \simeq k[\underline{X}]/I$, with I a perfect ideal of codimension two, then (i) can be simply be restated by saying that A is reduced and, locally in codimension one, (isomorphic to) a hypersurface ring – in higher codimension, the condition required by (i) is automatically satisfied in this case.

Example 3.2 The following examples may be worth keeping in mind for their behaviour pattern. In all of them, k stands for a field.

1. $A = k[X, Y, Z]/(XY, XZ, YZ)$

This is a Cohen–Macaulay reduced ring of dimension one.

The presentation ideal J of $\mathbf{T}$ over the polynomial ring $A[T, U, V]$ contains the element TUV which is a non-zero-divisor on the subideal $J(1)$ generated in degree one (by the linear relations coming from the transposed Jacobian matrix of the generators XY, XZ, YZ). Therefore, $\dim \mathbf{Z} > \dim \mathbf{T}$, so A has not the expected star dimension.

However, by the same token as Exercise 2.9, it is easy to check that the "cylinder" $A[T] = k[X, Y, Z, T]/(XY, XZ, YZ)$ on $A = k[X, Y, Z]/(XY, XZ, YZ)$ does have the expected star dimension. Geometrically, this may look rather intriguing but, from the algebraic viewpoint, it is rather explicable.

2. $A = k[X, Y]/(X^2, XY)$

Again this is a ring of dimension one, but not reduced (clearly, not Cohen-Macaulay since it has even an embedded prime).

The presentation ideal J of $\mathbf{T}$ over the polynomial ring $A[T, U]$ is generated by $J(1)$ plus the extra relations T^3, T^2U – a result that can be obtained by means of a computation in the program *Macaulay*. Since T is a zero-divisor on $J(1)$, it is clear that J and $J(1)$ have the same codimension (which is 2). Therefore, A has the expected star dimension. However, A is not set theoretically starlike linear since $\mathbf{Z}$ and $\mathbf{T}$ do not share the same minimal primes, e.g., (X, Y) is a minimal prime of $J(1)$ which does not even contain J.

The reduced structure (i.e., the underlying set theoretic geometry) is easily found: in 4-space X, Y, T, U, $(\mathbf{Z})_{\mathrm{red}}$ is the union of two concurrent planes while $(\mathbf{T})_{\mathrm{red}}$ is one single plane.

3. $A = k[X, Y]/(F(X, Y))$, with $F(X, Y)$ a square-free polynomial.

Geometrically, the situation is as good possible since one has a reduced plane curve. At any rate, it is easy to verify that A has the expected star dimension. Also, geometers will undoubtedly find no hardship in realizing that $\mathbf{Z}$ and $\mathbf{T}$ have the same reduced structure (i.e., that A is set theoretically starlike linear). However, most everybody will be stymied if asked whether the two cones are the same. The fact that this is indeed the case was originally proved in [18] as a special case of a hypersurface in n-space. In this work, it will come out as a special case of a more general class of varieties, namely, locally complete intersections in n-space (see Section 4).

Problem 3.3 How does one find, without computer resources, the ideal-theoretic defining equations of $\mathbf{T}$ in case A is a one-dimensional ring of the form $k[X,Y]/I$? (If I is principal, the calculation falls off the method of [18]).

3.1 Kähler differentials and the second symbolic power

In this portion, we assume that A is actually of finite type over k. Thus, $R = k[X_1, \ldots, X_n]$ and $A = R/I = k[X_1, \ldots, X_n]/I$. Since the diagonal ideal $\mathbf{D}$ is closely related to the module of Kähler differentials $\Omega(A/k)$, it maybe useful to develop some aspects of the latter.

The following is a well-known presentation of $\Omega(A/k)$ in this case (cf. [22], where it is called the *second fundamental exact sequence*).

$$I/I^2 \xrightarrow{\partial} \Omega(R/k) \otimes_R A \longrightarrow \Omega(A/k) \to 0.$$

Here, the map on the right is naturally induced from the canonical surjection $R \longrightarrow A$, while ∂ is induced from the structural derivation $d : R \longrightarrow \Omega(R/k)$. The kernel of the mapping

$$\partial : I/I^2 \longrightarrow \Omega(R/k) \otimes_R A \simeq A^n$$

may be identified to the torsion submodule, $\tau(I)$, of I/I^2 (as an A–module).

If further assumptions are imposed on I, then one can specify $\tau(I)$ a bit more.

Proposition 3.4 (Zariski holomorphic functions) *Assume that k has characteristic zero and I is a radical ideal. Then ker ∂ can be described by either one of the following:*
(i) *The A-submodule $\mathcal{D}(I)/I^2$, where $\mathcal{D}(I) := \{f \in I \mid \partial f/\partial X_i \in I \text{ for all } i\}$*
(ii) *The A-submodule $I^{(2)}/I^2$, where $I^{(2)}$ is the second symbolic power of I.*

Proof. (i) This description follows at once from the definitions. In fact, the R-module $\Omega(R/k)$ is free on the generators $\{d(X_i)\}$. Moreover, from the universal property of the Kähler differentials, one has $d(f) = \sum_i^n (\partial f/\partial X_i)d(X_i)$ for an element $f \in R$. Now, to say that $f \in \ker \partial$ amounts therefore to have $\partial f/\partial X_i \in I$ for $1 \leq i \leq n$, as required.
(ii) It is well known that, under the assumptions, $\tau(I) = I^{(2)}/I^2$ (cf. [38]). □

Effective calculation of $I^{(2)}$

Althogh Zariski theorem identifies, more generally, higher symbolic powers of I in terms of the higher partial derivatives (cf., e.g., [5]), it is in the case of the second symbolic power that one can set a remarkably simple and efficient tool for computing its non-trivial generators in terms of first order differentials. It is based on the following:

Proposition 3.5 *Let $I = (f_1, \ldots, f_m) \subset R$ be as in Proposition 3.4. Let Ψ stand for a lifting of the relation matrix of the transposed jacobian matrix of $\mathbf{f} = f_1, \ldots, f_m$, modulo I. Then the entries of the product matrix $\mathbf{f} \cdot \Psi$ form a (not necessarily minimal) set of generators of the symbolic power $I^{(2)}$.*

Proof. By Proposition 3.4, there is an exact sequence

$$I/I^{(2)} \longrightarrow \Omega(R/k) \otimes_R A \longrightarrow \Omega(A/k) \to 0.$$

The transposed jacobian matrix will give a map

$$\sum_{j=1}^{m}(R/I)e_j \simeq (R/I)^m \xrightarrow{\Theta} \sum_{i=1}^{n}(R/I)dX_i$$

with image module $I/I^{(2)}$. Let $K \subset \sum_{j=1}^m (R/I)e_j$ stand for the kernel of Θ and let

$$(R/I)^p \longrightarrow \sum_{j=1}^{m}(R/I)e_j \tag{5}$$

map surjectively onto K. Say, $\Psi = (\psi_{jk})$ is a corresponding lifted matrix of (5) over R. It follows that $\sum_j \psi_{jk} f_j \in I^{(2)}$.

Conversely, any element $\sum_j \alpha_j f_j \in I^{(2)}$, with $\alpha_j \in R$, is such that $\sum_j \bar{\alpha}_j e_j \in K$, where "$^-$" denotes residue modulo I. This means that $\bar{\alpha}_j = \sum_k \bar{\psi}_{jk}\bar{\beta}_k$ for certain $\bar{\beta}_k \in R/I$ with $j = 1, \dots, m$, $k = 1, \dots, p$. Write $\alpha_j = \sum_k \psi_{jk}\beta_k + g_j$, with $g_j \in I$. It follows that

$$\sum_j \alpha_j f_j = \sum_k \beta_k \left(\sum_j \psi_{jk} f_j \right) + \sum_j g_j f_j,$$

hence $I^{(2)} \subset ((f_1, \dots, f_m)\Psi)R + I^2$. This shows the contention. □

Remark 3.6 It is clear that, as long as one is not concerned with finding minimal generators, then one can liberally assume that the first syzygy matrix of the ideal I itself is a submatrix of the lifted relation matrix Ψ.

The preceding remark leads us to formulate the curious

Proposition 3.7 *Assume that $I = (f_1, \dots, f_m) \subset R$ is a perfect radical ideal of codimension two. Then the defining $m \times (m-1)$ matrix φ of I can be augmented to an $m \times p$ matrix Ψ, for suitable $p \geq m$, such that $I^{(2)}$ is generated by the maximal minors of Ψ fixing the columns of Φ.*

Proof. We can assume, by the earlier remark, that the $m \times (m-1)$ syzygy matrix φ of I is a submatrix of the lifted relation matrix Ψ as described above. Since the $(m-1) \times (m-1)$ minors of φ are, up to sign, the generators $\mathbf{f}$ of I, we can so arrange that $m \times m$ minors of Ψ be exactly the entries of the product matrix $\mathbf{f} \cdot \Psi$. Since, by Proposition 3.5, the latter generate $I^{(2)}$, we are through. □

Often, after throwing away the zero minors, the generators obtained by this procedure are indeed minimal.

Example 3.8 Let I stand for the ideal of an affine (or an arithmetically Cohen-Macaulay projective) monomial curve. It has been shown (cf. [41], [27], [24]) that $I^{(2)}/I^2$ is cyclic and a generator is given by the determinant of a 3×3 matrix obtained by suitably enlarging the presentation matrix of I. In characteristic zero, this matrix is nothing but the first syzygy matrix of the transposed jacobian matrix of the generators of I (modulo I). It would be nice to prove directly that the second syzygy of Ω is minimally three-generated in this case.

As a simple illustration of this example, consider the curve in $\mathbf{P}^3$ defined parametrically by the equations:

$$\begin{cases} x = s^5 \\ y = s^4 t \\ z = s^3 t^2 \\ w = t^5 \end{cases}$$

By the nowadays sufficiently well known theory of curves defined by monomial parametric equations, one can prove that the curve is arithmetically Cohen–Macaulay – the point being, for instance, that one of the eliminated binomial cartesian equations exhibits a pure y-power of degree not smaller than the degree of the other term.

Once this is known, determining three equations that generate the homogeneous ideal of the curve is a matter of standard calculation. Namely, one gets: $y^2 - xz$, $z^3 - xyw$, $yz^2 - x^2w$.

By computing the relation matrix of the transposed Jacobian matrix of these polynomials modulo the ideal I they generate, and lifting to R, one finds the following 3×6 matrix

$$\begin{pmatrix} 0 & x & y & z^2 & y^2 - xz & 0 \\ 0 & -y & -z & -xw & 0 & y^2 - xz \\ y^2 - xz & z^2 & xw & yzw & 0 & 0 \end{pmatrix},$$

which yields, by throwing away the null minors, a minimal set of generators of $I^{(2)}$, namely:

$$(y^2 - xz)^2,\ (y^2 - xz)(z^3 - xyw),\ (y^2 - xz)(yz^2 - x^2w)$$

and

$$z^5 + y^3zw - 3xyz^2w + x^3w^2.$$

The last generator is a cyclic generator of the torsion module $I^{(2)}/I^2$, as meant above.

3.2 A presentation of the Zariski tangent algebra

The preceding presentation of $\Omega(A/k)$ as an A-module yields a corresponding presentation of the Zariski tangent algebra as an A-algebra, namely, if $\underline{T} = T_1, \ldots, T_n$ are presentation variables such that $T_i \mapsto dX_i \pmod{\partial(I/I^2)}$ then

$$\mathbf{Z} \simeq A[\underline{T}]/(\underline{T}\Theta).$$

It may be worthwhile mentioning yet another related algebra, to wit, the symmetric algebra $S_{A\otimes_k A}(\mathbf{D})$ of the ideal $\mathbf{D} \subset A \otimes_k A$

In order to deal with a presentation of the latter, we identify $A \otimes_k A$ with the ring $k[\underline{X}, \underline{U}]/(I(\underline{X}) + I(\underline{U}))$, where $I(\underline{X}) = I$ and $I(\underline{U})$ is obtained from I by the substitution $\underline{X} \mapsto \underline{U}$. Let $\underline{T} = T_1, \ldots, T_n$ denote new variables (" presentation variables") over $k[\underline{X}, \underline{U}]/(I(\underline{X}) + I(\underline{U}))$ and define a surjective homomorphism

$$\rho : k[\underline{X}, \underline{U}, \underline{T}]/(I(\underline{X}) + I(\underline{U})) \to S_{A\otimes_k A}(\mathbf{D})$$

by the assignment $T_i \mapsto X_i - U_i \pmod{I(\underline{X}) + I(\underline{U})}$, where $X_i - U_i$ is taken in degree 1.

It is well known that the presentation ideal $\ker \rho$ is generated by $\underline{T}$-linear polynomials with coefficients in $k[\underline{X}, \underline{U}]/(I(\underline{X}) + I(\underline{U}))$. Now, for each of these generators whose coefficients actually belong to $\mathbf{D}$, we consider an arbitrary lifting to $k[\underline{X}, \underline{U}, \underline{T}]$ and denote by $\mathcal{D}(\underline{X} - \underline{U}) \subset k[\underline{X}, \underline{U}, \underline{T}]$ the ideal generated by these liftings. We informally call the latter the *syzygetic generators* of the presentation ideal $\ker \rho$.

We now introduce fresh generators, to be called the *Taylor generators* of $\ker \rho$. Namely, for each generator $f_j(\underline{X})$ of $I(\underline{X})$, consider the Taylor expansion of $f_j(\underline{X})$ at the point $\underline{U}$ and collect the polynomial coefficients of the linear terms $X_i - U_i$ in an arbitrary fashion to get an expression of the form $\sum g_{ij}(X_i - U_i)$ - modulo $\mathcal{D}(\underline{X} - \underline{U})$, it will be immaterial the way one collects them, due to the Koszul relations. Let $\mathcal{T}(I)$ denote the ideal of $k[\underline{X}, \underline{U}, \underline{T}]$ generated by the corresponding $\underline{T}$-linear forms $\sum g_{ij} T_i$.

Our result claims that, together the two kinds not only generate but are also natural generators.

Proposition 3.9 *Let char $k = 0$. Then, in the above notation, one has:*

1. *There is a presentation*

$$S_{A\otimes_k A}(\mathbf{D}) \simeq k[\underline{X}, \underline{U}, \underline{T}]/(\mathcal{D}(\underline{X} - \underline{U}) + \mathcal{T}(I) + I(\underline{X}) + I(\underline{U})).$$

2. *The above presentation of $S_{A\otimes_k A}(\mathbf{D})$ modulo $\mathbf{D}$ yields the earlier presentation of the Zariski tangent algebra by means of the transposed Jacobian matrix.*

Proof. (1) Quite generally, for any ring B and any ideal $\mathfrak{b}$, from a presentation

$$0 \longrightarrow Z \longrightarrow B^n \longrightarrow \mathfrak{b} \longrightarrow 0,$$

one obtains a presentation of the conormal module

$$0 \longrightarrow Z/Z \cap \mathfrak{b}B^n \longrightarrow B^n/\mathfrak{b}B^n \longrightarrow \mathfrak{b}/\mathfrak{b}^2 \longrightarrow 0.$$

This says that $Z = Z \cap \mathfrak{b}B^n + L$, where L is generated by lifted generators. Applying to the present situation, with $B = A \otimes_k A$ and $\mathfrak{b} = \mathbf{D}$, it will suffice to show that the Taylor expansions at $\underline{U}$ of the generators $f_1, \ldots, f_m$ of I, read modulo the ideal $I(\underline{X}) + I(\underline{U})$, are liftings of the differentials

$$\sum_{i=1}^{n} \frac{\partial f_1}{\partial X_i} dX_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial X_i} dX_i$$

read modulo $I = I(\underline{X})$.

Now, for that, in the Taylor expansion

$$f_j(\underline{X}) - f_j(\underline{U}) = \sum_{i=1}^{n} \frac{\partial f_m}{\partial X_i}(\underline{U})(X_i - U_i) + \frac{1}{2} \sum_{1\leq i\leq k\leq n} \frac{\partial^2 f_j}{\partial X_i \partial X_j}(\underline{U})(X_i - U_i)(X_k - U_k) + \ldots$$

one collects terms as follows:

$$\begin{aligned} f_j(\underline{X}) - f_j(\underline{U}) &= \left(\frac{\partial f_j}{\partial X_1} + \sum_{k\geq 2} \frac{\partial f_j}{\partial X_1 \partial X_k}(X_k - U_k) + \ldots\right)(X_1 - U_1) \\ &+ \left(\frac{\partial f_j}{\partial X_2} + \sum_{k\geq 3} \frac{\partial f_j}{\partial X_2 \partial X_k}(X_k - U_k) + \ldots\right)(X_2 - U_2) \\ &+ \ldots . \end{aligned}$$

It is clear that, modulo $\mathbf{D}$, these yield the above differentials.

(2) The second part follows immediately from the preceding proof. $\square$

Although much less ubiquitous than the tangent algebras studied in these notes, the symmetric algebra $S_{A\otimes_k A}(\mathbf{D})$ is nonetheless important. We next mention connections to central theories.

- (Relation to deformation theory) Under suitable conditions, the presentation ideal of $S_{A\otimes_k A}(\mathbf{D})$ is actually generated by the Koszul relations on the generators of $\mathbf{D}$ and the Taylor relations $\mathcal{T}(I)$. This requires that the ideal $\mathbf{D}$ be *syzygetic* in the terminology of [33] or, in the language of deformation functors [21], that $T_2(A|k, A) = 0$.

- (Relation to embedding dimension of affine varieties) Let A stand for a finitely generated algebra over an infinite field k. In[30, Theorem 1] the following estimate is obtained

  $$\operatorname{edim} A \leq \dim S_{A\otimes_k A}(\mathbf{D}).$$

  (This is not stated as such in *loc. cit*, but it turns out to be the same statement at least if $\operatorname{grade} \mathbf{D} > 0$ due to Proposition 2.6). Here $\operatorname{edim} A$ stands for the affine embedding dimension of A, i.e., the least $n \geq 0$ such that there is a surjective k-homomorphism

  $$k[X_1, \ldots, X_n] \longrightarrow A.$$

  The proof given in [30] hinges on a rather involved geometric argument, so one would naturally wonder if there is a simpler algebraic proof.

We note *en passant* that if A is a finite type equidimensional k-algebra having the expected star dimension then the above estimate reduces to $\operatorname{edim} A \leq 2\dim A+1$ (cf. Proposition 3.12 and, specially, Remark 3.13), which is a classical result in the case of a smooth A. Thus, it seems even more natural to ask for a direct algebraic proof of this inequality for algebras A having the expected star dimension. It would suffice to show that, for such algebras, the following upper bound always works:

$$\operatorname{edim} A \leq \sup_{m \in \operatorname{Max} A} \{\operatorname{edim} A_m\} + 1.$$

Actually, one is tempted to formulate the

Conjecture 3.10 Let k be an infinite (maybe perfect) field and let A be a finite type equidimensional k-algebra. Then $\operatorname{edim} A \leq 2 \dim A$.

3.3 Zariski algebras having the expected dimension

Let k be a field and let A be a k–algebra essentially of finite type; write $A = W^{-1}B$, B finitely generated k–algebra, $B \subset A$, W a multiplicative set in B.

As before, write

$$0 \to \mathbf{D} \longrightarrow A \otimes A = A \otimes_k A \longrightarrow A \to 0;$$

then $\mathbf{D}/\mathbf{D}^2 = \Omega(A/k)$.

A typical prime ideal of A will be denoted by $\wp$. Identifying $\operatorname{Spec}(A)$ with $V(\mathbf{D}) \subset \operatorname{Spec}(A \otimes_k A)$, will allow for the slight confusion of denoting the inverse image of $\wp$ in $A \otimes_k A$ also by $\wp$.

Lemma 3.11 [32, Lemma 2.2] *Let k stand for a field and let $B \subset A$ be k-algebras such that B is of finite type over k and A is a ring of fractions of B. Let $\wp \in \operatorname{Spec}(A) = V(\mathbf{D}) \subset \operatorname{Spec}(A \otimes_k A)$; then*

$$\dim(A \otimes_k A)_\wp = 2 \dim A_\wp + \operatorname{trdeg}_k A_\wp/\wp A_\wp.$$

If A is locally equidimensional, then $(A \otimes_k A)_\wp$ is equidimensional and quasi-unmixed.

We refer to [32, *loc. cit.*] for a proof of this lemma which uses but standard machinery.

The next proposition also appears in [32], but since it is based on earlier results of independent interest, we provide a proof.

Proposition 3.12 *Assume that k is a perfect field, A is a k-algebra essentially of finite type, locally equidimensional and equicodimensional. Then the following are equivalent:*

(a) $\dim \mathbf{Z} = \dim \mathbf{T}$ *(i.e., A has the expected star dimension).*

(b) $\mathrm{edim}(A_{\wp}) \leq 2\dim A_{\wp}$, for all $\wp \in \mathrm{Spec}(A)$.

(c) $\mu(\mathbf{D}_{\wp}) \leq \dim(A \otimes_k A)_{\wp}$ *for every $\wp \in V(\mathbf{D})$.*

(d) *Let $B = k[\underline{X}]/(f_1, \ldots, f_m) \subset A$ such that A is a ring of fractions of B. Then*

$$\mathrm{ht}\, I_t(\Theta) \geq \mathrm{ht}\,(f_1, \ldots, f_m) - t + 1,$$

for $1 \leq t \leq \mathrm{ht}\,(f_1, \ldots, f_m)$, where Θ denotes the transposed Jacobian matrix of $f_1, \ldots, f_m$ modulo the ideal $(f_1, \ldots, f_m)$.

Proof. (a) $\Leftrightarrow$ (c). This is immediate by Theorem 2.15
(b) $\Leftrightarrow$ (c) It is well known (cf., e.g., [8]) that

$$\mu(\mathbf{D}_{\wp}) = \mu(\Omega(A/k)_{\wp}) = \mathrm{edim}(A_{\wp}) + \dim B/\wp.$$

The equivalence follows now from Lemma 3.11.
(c) $\Leftrightarrow$ (d): (Asuming that A is generically a complete intersection) This follows from Proposition 2.2 with $E = \Omega(A/k)$ and using the fact that this module has a rank equal to $\dim A$. □

Remark 3.13 We note that the equivalence $(a) \Leftrightarrow (b)$ holds with no assumption on the field k. Moreover, if A is k-affine (or local) equidimensional then all hypotheses are fulfilled and we can freely use this equivalence. More particularly, if A is a graded affine k-domain (or a local domain) then condition (b) is equivalent to $\dim \mathbf{Z} = 2\dim A$.

The next result explains the behaviour of set theoretically starlike linear rings under deformations. It will be used in the following sections.

Proposition 3.14 *Assume k is a perfect field and let A and C be k-algebras essentially of finite type such that $A = C/(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ is a quasiregular sequence in C. Assume that:*
(i) *A has the expected star dimension.*
(ii) *C_Q is equidimensional and set theoretically starlike linear for every maximal ideal Q containing $(a_1, \ldots, a_n)$. Write*

$$0 \to \tilde{\mathbf{D}} \longrightarrow C \otimes_k C \longrightarrow C \to 0.$$

Then A is set theoretically starlike linear as well.

The proof goes beyond the scope of these notes and will not be given here. We refer instead to the forthcoming [32, Proposition 6.4].

4 Homological background

The earlier sections were practically concerned with the Zariski algebra $\mathbf{Z}$ and its dimension, while the star algebra $\mathbf{T}$ was only mentioned in connection with Proposition 3.12. In this section we will deal with the finer properties of $\mathbf{T}$ using the method of the approximation complexes, which has proven efficient to study many classes of ideals and modules (cf. [9], [11], [13], [34], [35], [10], [39]).

The main ingredients of the method are the homology of the Koszul complex attached to the generators of an ideal and the conditions $(\mathcal{F}_k)$ introduced earlier. Our interest hinges on the diagonal ideal $\mathbf{D}$, hence the method depends largely on the techniques related to the so-called *reduction to the diagonal*, a device perhaps dating back to Weil and even others before him. Since the main focus is on the homological side of the technique, we largely draw on Serre [28] and Cartan–Eilenberg [4].

It may be convenient to review both the basic features of the approximation complexes and technique of Serre, as they will appear slightly adapted to our present needs.

4.1 The complexes of Herzog–Simis–Vasconcelos

Let R be a commutative ring, let $\varphi : G \to F$ be a map of R-modules and let M be a third R-module. The basic construction of the theory is the so-called *Koszul complex of φ with coefficients in M*, denoted $\mathbf{K}(\varphi, M)$: the components of the complex are the modules $\wedge^r G \otimes_R S_t(F) \otimes_R M$ and the differential is

$$\begin{array}{rcl} \wedge^r G \otimes_R S_t(F) \otimes M & \to & \wedge^{r-1} G \otimes_R S_{t+1}(F) \otimes M \\ g_1 \wedge \ldots \wedge g_r \otimes \mathbf{f} \otimes m & \mapsto & \sum(-1)^i g_1 \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge g_r \otimes \varphi(g_i) \cdot \mathbf{f} \otimes m. \end{array}$$

Now, one reason for the ubiquity of this complex is the following.

Lemma 4.1 *Let F, G be free modules of finite rank and let $E := \operatorname{coker} \varphi$ (i.e., φ is a presentation map of E). Then:*

(i) $\mathbf{K}(\varphi, R)$ *has a natural structure of graded complex over $S(F)$ and, as such, it is a direct sum of R-complexes*

$$\mathbf{K}_t : \quad 0 \to \wedge^q G \otimes S_{t-q}(F) \to \ldots \to \wedge^1 G \otimes S_{t-1}(F) \to S_t(F),$$

with $q = \min\{t, \operatorname{rk} G\}$ and $H_0(\mathbf{K}_t) \simeq S_t(E)$.

(ii) *As a graded complex defined over the polynomial ring $S(F) = \sum_t S_t(F)$, $\mathbf{K}(\varphi, R)$ is isomorphic with the (ordinary) Koszul complex attached to a set of generators of the presentation ideal $J(\varphi) \subset S(F)$ of $S(E)$ coming from the presentation map φ. As such, $\mathbf{K}(\varphi, R)$ is a complex such that $H_0(\mathbf{K}(\varphi, R)) \simeq S(E)$.*

Proof. (i) As a graded complex over the polynomial ring $S(F)$, $\mathbf{K}(\varphi, R)$ is the direct sum of the complexes

$$\begin{array}{ccccccccc} & & & & & & 0 & \to & S_0(F) & \to & 0 \\ & & & & 0 & \to & G \otimes S_0(F) & \to & S_1(F) & \to & 0 \\ & & 0 & \to & \wedge^2 G \otimes S_0(F) & \to & G \otimes S_1(F) & \to & S_2(F) & \to & 0 \\ 0 & \to & \wedge^3 G \otimes S_0(F) & \to & \wedge^2 G \otimes S_1(F) & \to & G \otimes S_2(F) & \to & S_3(F) & \to & 0 \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

Therefore, $\mathbf{K}(\varphi, R)$ is of the form

$$0 \to \wedge^m G \otimes S(F)(-m) \to \ldots \to \wedge^2 G \otimes S(F)(-2) \to G \otimes S(F)(-1) \to S(F) \to 0.$$

We leave as an easy exercise the verification that $H_0(\mathbf{K}_t) \simeq S_t(E)$. Clearly, then $H_0(\mathbf{K}(\varphi, R)) \simeq \sum_{t\geq 0} S_t(E) = S(E)$.

(ii) Let $\{g_1, \ldots, g_m\}$ be a basis of G. Then $G \otimes_R S(F)$ is a free $S(F)$-module on the generators $\{g_1 \otimes 1, \ldots, g_m \otimes 1\}$. On the other hand, $J(\varphi) \subset S(F)$ is, by definition, generated by the polynomials

$$S_1(F) \cdot \varphi(g_1), \ldots, S_1(F) \cdot \varphi(g_m),$$

where $\cdot$ denotes matrix product. Therefore, we have a map of free $S(F)$-modules

$$G \otimes_R S(F) \xrightarrow{\Phi} S(F), \quad g_j \otimes 1 \mapsto S_1(F) \cdot \varphi(g_j), 1 \leq j \leq m.$$

If we now consider the ordinary Koszul complex $K(\Phi, S(F))$ associated to the map Φ, then there is an isomorphism of complexes over $S(F)$

$$\begin{array}{ccccccccccc} \mathbf{K}(\varphi, R): & 0 & \to & (\wedge^m G) \otimes S(F)(-m) & \to & \cdots & \to & G \otimes S(F)(-1) & \to & S(F) & \to 0 \\ \| & & & \| & & & & \| & & \| & \\ K(\Phi, S(F)): & 0 & \to & \wedge^m (G \otimes S(F))(-m) & \to & \cdots & \to & (G \otimes S(F))(-1) & \to & S(F) & \to 0 \end{array}$$

We leave the details as a rewarding exercise. □

Exercise 4.2 Let $G \xrightarrow{\varphi} F$ be a map of free R-modules of finite rank and let $E := \operatorname{coker} \varphi$. Let, as above, $J(\varphi) \subset S(F)$ denote the presentation ideal of $S(E)$ coming from the given presentation of E.

1. If $L = \ker \varphi \subset F$, show that $L \simeq J/JS(F)_+$.
2. If the first syzygies of the entries of the product matrix $(S(F)_+) \cdot \varphi$ have coordinates in $S(F)_+$ if and only if L is a free R-module.
3. From the previous item, one deduces that L is a free module if the entries of $(S(F)_+) \cdot \varphi$ form a regular sequence in $S(F)$. Write your smallest "sufficiently regular" counterexample for the converse (Hint: set $R = k[X]$ and try $\operatorname{rk} F = \operatorname{rk} G = 2$ out).
4. Can you guess sufficient conditions on the matrix of φ in order that the converse of the previous item holds? (If you think in terms of the Fitting ideals of φ, you are guessing right).

We now specialize a bit our settting, to wit, we assume that $F = G$ and that the given map $F \to F$ is the identity map. Clearly, in this case $\mathbf{K}(\mathrm{id}, R)$ is isomorphic to the ordinary Koszul complex on the variables of $S(F)$ (i.e., the generators of the irrelevant ideal $S(F)_+$), which is then obviously acyclic.

But now assume, moreover, one is given a second map $\varphi : F \to R$. Then $\wedge F \otimes S(G)$ is a double complex

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & \wedge^r F \otimes S_t(F) & \to & \wedge^{r-1} F \otimes S_{t+1}(F) & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & \wedge^{r-1} F \otimes S_t(F) & \to & \wedge^{r-2} F \otimes S_{t+1}(F) & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \end{array}$$

where the horizontal subcomplexes are acyclic and the vertical ones come from the ordinary Koszul complex $K(\varphi, R)$ by "symmetrization" (i.e., by extending coeffcients to $S(F)$).

Here, a typical square composed of horizontal and vertical differentials is skew-commutative. If we denote the cycles of $K(\varphi, R)$ by $Z(\varphi, R)$ then, from the (skew-) commutativity of the maps and the fact that extending coefficients by a flat map is left-exact, we see that the horizontal complexes induce subcomplexes

$$\mathcal{Z}_{r+t} : \ldots \to Z_r(\varphi, R) \otimes S_t(F) \to Z_{r-1}(\varphi, R) \otimes S_{t+1}(F) \to \ldots.$$

Therefore, we obtain a graded complex $\mathcal{Z}(\varphi; R)$ of the form

$$0 \to Z_m(\varphi, R) \otimes S(F)(-m) \to \ldots \to Z_1(\varphi, R) \otimes S(F)(-1) \to S(F) \to 0.$$

This complex has been dubbed the *$\mathcal{Z}$-complex* attached to φ (or to its image in R). There are similar versions using the boundaries and the homology of the complex $K(\varphi, R)$ (see Exercise 4.3).

Also, these complexes can be defined with coefficients in an R-module M. We leave the details to the reader.

Exercise 4.3 Let $B_r = B_r(\varphi, R)$ (resp. $H_r = H_r(\varphi, R)$) denote the boundaries (resp. the homology) in degree r of the complex $K(\varphi, R)$.
(i) Show that there are graded complexes $\mathcal{B}(\varphi; R)$ and $\mathcal{M}(\varphi; R)$ of the form, respectively

$$0 \to B_m(\varphi, R) \otimes S(F)(-m) \to \ldots \to B_1(\varphi, R) \otimes S(F)(-1) \to (\varphi(F))S(F) \to 0$$

and

$$0 \to H_m(\varphi, R) \otimes S(F)(-m) \to \ldots \to H_1(\varphi, R) \otimes S(F)(-1) \to S(F)/(\varphi(F)S(F) \to 0.$$

(ii) Show that the length of the $\mathcal{M}$-complex, as defined in (i), is usually much shorter.
(iii) Let $I = (\varphi(F)) \subset R$. Show that $H_0(\mathcal{Z}(\varphi, R)) \simeq S(I)$, $H_0(\mathcal{B}(\varphi, R)) \simeq IS(I)$ and $H_0(\mathcal{M}(\varphi, R)) \simeq S(I/I^2)$.

Exercise 4.4 Let $K(\varphi, M)$ denote the ordinary Koszul complex of a map $\varphi : F \to R$ with coefficients in an R-module M.
(i) Show that $H_0(\varphi, M) \simeq M \otimes_R H_0(\varphi, R)$.
(ii) Show, by mean of examples, that the analogue of (i) for the homology in degree 1 is false (Hint: consider $M = R/(\varphi(F))$).

The following condition on the depth of the Koszul homology has been known as sliding depth [11] (but cf. [35] and [10] for earlier appearances).

Thus, one is given a noetherian ring R and a map $\varphi : F \to R$, where F is a free R-module. Let $\mathbf{a}$ be a set of generators of the ideal $(\varphi(F))$.

Definition 4.5 The set $\mathbf{a}$ (or the map φ) is said to satisfy the *sliding depth condition* if

$$\operatorname{depth} H_i(K(\mathbf{a}, R)) \geq \dim R - \operatorname{rk} F + i, \tag{6}$$

for all $i \geq 0$.

Sliding depth *with coefficients* is defined in the same manner if the complex has coefficients in a module, with no change in the right hand side of (6).

Remark 4.6 (i) (Invariance [11]) Although the homology of the Koszul complex varies while the set of generators of $I = (\varphi(F)) \subset R$ changes, it does so in a stable fashion sufficient to imply that the sliding depth condition is independent of the chosen generators. This means that, if R is local then $\operatorname{rk} F$ may be replaced by $\mu(I)$ in the right hand side of (6) as soon as we replace $\mathbf{a}$ by a minimal generating set of I.
(ii) (Localization [11]) If R is local Cohen–Macaulay then the sliding depth condition implies a similar condition locally at every prime of R. Thus, taking in account (i), one would have similarly

$$\operatorname{depth} H_i(K(\mathbf{b}, E))_P \geq \dim R_P - \mu(I_P) + i,$$

for every $P \in \operatorname{Spec}(R)$, where $\mathbf{b}$ is a minimal generating set of the ideal I and E is a coefficient module.

Put together in a single statement, the facts in Remark 4.6 amount to saying that if sliding depth holds locally at every *maximal* ideal of a locally Cohen–Macaulay ring R then it holds locally at every *prime* ideal of R.

Next is the main application of the $\mathcal{M}$-complex. It serves as a template for a diversity of situations. For convenience, it is stated in a slightly more general form than in [10, Theorem 9.1].

Theorem 4.7 *Let C be a locally Cohen–Macaulay noetherian ring, let $\varphi : F \longrightarrow C$ be a map, let $J = (\varphi(F))$ and let E be a finitely generated C-module. Suppose the following conditions hold:*
(i) *J satisfies sliding depth with coefficients in E locally at every maximal ideal of C.*
(ii) *J satisfies $(\mathcal{F}_1)$ (i.e., $\mu(J_\wp) \leq \dim C_\wp$ for every $\wp \in \operatorname{Spec}(C)$).*
Then:
(a) *The $\mathcal{M}$-complex $\mathcal{M}_\bullet(\varphi; E)$ is acyclic.*

(b) *If, moreover,* $H_1(\varphi, E) \simeq E \otimes_C H_1(\varphi, C)$, *one has*

(b.1) $H_0(\mathcal{M}_\bullet(\varphi; E)) \simeq E \otimes_C S_{C/J}(J/J^2) \simeq E \otimes_R \operatorname{gr}_J(C) \simeq \operatorname{gr}_J(E)$.

(b.2) *If* C/J *is a Cohen–Macaulay ring then* $E \otimes_R S_{C/J}(J/J^2) \simeq E \otimes_R \operatorname{gr}_J(C)$ *is a Cohen–Macaulay* $S_{C/J}(J/J^2)$*-module;*

(b.3) *If* $\operatorname{Supp}(E) = \operatorname{Spec}(C)$ *then* $(S_{C/J}(J/J^2))_{\text{red}} \simeq (\operatorname{gr}_J(C))_{\text{red}}$.

Proof. (a) Suppose to the contrary and choose $P \in \operatorname{Spec}(S(F))$ such that P is minimal in the set

$$\bigcup_{i>0} \operatorname{Supp}(H_i(\mathcal{M}_\bullet(\varphi; E))).$$

Look at the localized complex $\mathcal{M}_\bullet(\varphi; E)_P$: its homology $H_i(\mathcal{M}_\bullet(\varphi; E))_P)$ (for $i > 0$) is either the null module or has depth zero. Now set $\wp = P \cap C$. By the preceding remarks on the localizability of the sliding depth conditions, one has

$$\begin{aligned} \operatorname{depth} H_i(K(\mathbf{b}, E))_\wp &\geq \dim R_\wp - \mu(J_\wp) + i \\ &\geq \mu(J_\wp) - \mu(J_\wp) + i, \text{ by (ii)} \\ &= i, \end{aligned}$$

for every $i \geq 0$, where $\mathbf{b}$ denotes a minimal generating set of $(\varphi(F))$. Therefore, we can apply the acyclicity lemma of Peskine–Szpiro to deduce that the complex $\mathcal{M}_\bullet(\mathbf{b}; E)_P$ is acyclic. It follows that the localized complex $\mathcal{M}_\bullet(\varphi; E)_P$ is acyclic as well since the homology of the $\mathcal{M}$-complex is independent of generating sets [10, p. 102].

(b.1) By the assumption and Exercise 4.4, it follows that the complexes $\mathcal{M}_\bullet(\varphi; E)$ and $E \otimes_C \mathcal{M}_\bullet(\varphi; C)$ coincide in degrees 0 and 1. By Exercise 4.3, $H_0(\mathcal{M}_\bullet(\varphi; C)) \simeq S_{C/J}(J/J^2)$. Therefore, we obtain a commutative diagram of canonical surjective homomorphisms

$$\begin{array}{ccc} H_0(\mathcal{M}_\bullet(\varphi; E)) & \simeq & E \otimes_C S_{C/J}(J/J^2) \\ \downarrow \beta & & \downarrow \\ \operatorname{gr}_J(E) & \leftarrow & E \otimes_C \operatorname{gr}_J(C). \end{array}$$

By [10, Theorem 4.6], β is an isomorphism, which implies that all the maps in the diagram are isomorphisms.

(b.2) We show that locally at every prime of $\mathrm{Supp}(E)$, $E \otimes_C S_{C/J}(J/J^2)$ is Cohen–Macaulay. Thus, let $\wp \in \mathrm{Supp}(E)$. We have, for every $i \geq 0$,

$$\begin{aligned} \operatorname{depth}(H_i(K(\varphi, E)) \otimes_C S(F))_{\wp} &\geq \operatorname{depth} \mathcal{M}_i(\mathbf{b}, E_{\wp}) \\ &\geq i + \dim C_{\wp} - \operatorname{ht} J_{\wp} + \mu(J_{\wp}), \text{ by part (a)} \\ &\geq i + \dim C_{\wp}, \qquad (7) \end{aligned}$$

where $\mathbf{b}$ denotes a minimal generating set of $J_{\wp}$.

Depth-chasing through the complex $\mathcal{M}_{\bullet}(\mathbf{b}; E)$, using (7) and part (b.1), yields $\operatorname{depth} E_{\wp} \otimes_{C_{\wp}} S(J_{\wp}/J_{\wp}^2) = \operatorname{depth} H_0(\mathcal{M}_{\bullet}(\mathbf{b}; E_{\wp})) \geq \dim C_{\wp}$.

On the other hand, again by part (b.1),

$$\begin{aligned} \dim E_{\wp} \otimes_{C_{\wp}} S(J_{\wp}/J_{\wp}^2) &= \dim E_{\wp} \otimes_{C_{\wp}} \operatorname{gr}_{J_{\wp}}(C_{\wp}) \\ &\leq \dim \operatorname{gr}_{J_{\wp}}(C_{\wp}) \leq \dim C_{\wp}. \end{aligned}$$

This proves the contention.

(b.3) The assumption on the support implies $\mathrm{Supp}(E \otimes_C S(F)) = \mathrm{Spec}(S(F))$, hence

$$\begin{aligned} \mathrm{Spec}(S(J/J^2)) &= \mathrm{Supp}(S(J/J^2)) = \mathrm{Supp}(S(J/J^2) \cap \mathrm{Spec}(S(F))) \\ &= \mathrm{Supp}(S(J/J^2)) \cap \mathrm{Supp}(E \otimes_C S(F)) \\ &= \mathrm{Supp}((E \otimes_C S(F)) \otimes_{S(F)} S(J/J^2)) = \mathrm{Supp}(E \otimes_C S(J/J^2)). \end{aligned}$$

Likewise, $\mathrm{Spec}(\operatorname{gr}_J(C)) = \mathrm{Supp}(E \otimes_C \operatorname{gr}_J(C))$. The result now follows from part (b.1). □

4.2 The principle of Cartan–Eilenberg–Serre

We now consider the technique of reduction to the diagonal. We note that it will be mainly used in the opposite direction to that used in the classical sense!

In considering ordinary Koszul complexes, we will henceforth focus on a set of generators of $\varphi(F) \subset R$ rather than on φ itself. Accordingly, given a sequence $\mathbf{b}$ of elements of a ring B and a B-module E, the Koszul complex (resp. the Koszul homology) of $\mathbf{b}$ with coefficients in E will be indicated by $K_{\bullet}(\mathbf{b}; E)$ (resp. $H_{\bullet}(\mathbf{b}; E)$).

Theorem 4.8 [28, Ch. V, B] *Let k be a field, let R stand for a ring of fractions of the polynomial ring $k[X_1,\ldots,X_n]$ and let M and N be R-modules. Then there is an isomorphism of graded R-modules*

$$H_\bullet(\Delta; M\otimes_k N)\simeq \operatorname{Tor}_\bullet^R(M,N),$$

where Δ denotes the sequence $\{X_i\otimes 1-1\otimes X_i \mid 1\leq i\leq n\}\subset R\otimes_k R$.

Proof. One may clearly assume that $R=k[X_1,\ldots,X_n]$. Quite generally, since Δ is a regular $R\otimes_k R$-sequence such that $R\otimes_k R/(\Delta)\simeq R$, one has

$$H_\bullet(\Delta; M\otimes_k N)\simeq \operatorname{Tor}_\bullet^{R\otimes_k R}(M\otimes_k N, R).$$

The rest follows from a well known result in [4, IX, 2.8] to the effect that

$$\operatorname{Tor}_\bullet^{R\otimes_k R}(M\otimes_k N, R)\simeq \operatorname{Tor}_\bullet^R(M,N).$$

□

Remark 4.9 We will be mainly interested in gathering information on the diagonal ideal $\mathbf{D}$ from the homology modules $\operatorname{Tor}_\bullet^R(M,N)$. The main cases will have $M=N=A$ or $N=A$, $M=\omega_A$. In such situations one has a reasonable grasp of the modules $\operatorname{Tor}_i^R(M,N)$.

We now go back to the notation that has been set up at the beginning of Section 3.

By definition, the diagonal ideal $\mathbf{D}=\ker A\otimes_k A\to A$ is generated by the residues of the elements of Δ. The next result could be rightly called *main theorem* as it draws practically on all the material developed so far.

Theorem 4.10 *Let k be a field, let A be a locally Cohen–Macaulay k-algebra, essentially of finite type over k and consider any fixed presentation $A\simeq R/I$, with R a ring of fractions of the polynomial ring $k[\underline{X}]=k[X_1,\ldots,X_n]$. Let $\Delta=\{X_i\otimes 1-1\otimes X_i \mid 1\leq i\leq n\}\subset k[\underline{X}]\otimes_k k[\underline{X}]$ and let further E be a finitely generated A-module. Assume that the following conditions hold:*

(i) depth $\operatorname{Tor}_i^R(A,E)_{\mathrm{m}} \geq \dim A_{\mathrm{m}} - \operatorname{ht} I_M + i$, *for every integer* $i \geq 0$ *and every maximal ideal* $M \supseteq I$, *where* $\mathrm{m} = M/I$.
(ii) $\mu(\Omega(A/k)_{\wp}) \leq \dim A_{\wp} + n - \operatorname{ht} I_P$, *for every prime ideal* $P \supseteq I$, *where* $\wp = P/I$.

Then:

(a) *The approximation complex* $\mathcal{M}_\bullet(\mathbf{\Delta}; E \otimes_k A)$ *is acyclic and*

$$\begin{aligned} H_0(\mathcal{M}_\bullet(\mathbf{\Delta}; E \otimes_k A)) &\simeq \mathbf{Z} \otimes_A E \\ &\simeq \mathbf{T} \otimes_A E \simeq \operatorname{gr}_{\mathbf{D}}(E \otimes_k A). \end{aligned}$$

(b) *If* A *is Cohen-Macaulay then* $\mathbf{Z} \otimes_A E \simeq \mathbf{T} \otimes_A E$ *is a Cohen–Macaulay* $\mathbf{Z}$*-module.*

(c) *If* $\operatorname{Supp}(E) = \operatorname{Spec}(A)$ *then* $(\mathbf{Z})_{\mathrm{red}} \simeq (\mathbf{T})_{\mathrm{red}}$ *(in other words,* A *is set theoretically starlike linear).*

Proof. It is an application of Theorem 4.7, with $C := A \otimes_k A$, $J := (\mathbf{Delta})/(I \otimes 1 + 1 \otimes I) = \mathbf{D}$ and $E := E$ (considered as an $A \otimes_k A$-module via the map $A \otimes_k A \to A$). The remaining identifications come from Theorem 4.8, from $\mathbf{D}/\mathbf{D}^2 = \Omega(A/k)$ and from $\dim A_{\wp} + n - \operatorname{ht} I_P = 2\dim A_{\wp} + \operatorname{trdeg}_k A_{\wp}/\wp A_{\wp} = \dim(A \otimes_k A)_{\wp}$ by Lemma 3.11.

The hypothesis in Theorem 4.7(b) is satisfied in the present contex since

$$\begin{aligned} H_1(\mathbf{\Delta}, E \otimes_k A) &\simeq \operatorname{Tor}_1^R(A,E) \simeq I \otimes_R E \simeq (I \otimes_R A) \otimes_A E \\ &\simeq \operatorname{Tor}_1^R(A,A) \otimes_A E \simeq H_1(\mathbf{\Delta}; A \otimes_k A) \otimes_A E \end{aligned}$$

□

Remark 4.11 Theorem 4.10 admits a formulation with k replaced by a noetherian ring and A being flat over it [32]. This has some interest because of deformation theory [19].

On the other end of the spectrum, if k is a perfect field and A is equicodimensional – a property Nagata used to include in the definition of a Cohen–Macaulay noetherian ring – then by Proposition 3.12, condition (ii) of Theorem 4.10 is but the condition that A has the expected star dimension.

Corollary 4.12 *Let k be a field and let A be a k-algebra essentially of finite type over k satisfying the following conditions:*
(i) $\mathrm{Tor}_i^R(A, A)$ *is Cohen–Macaulay for every* $i \geq 0$.
(ii) $\mu(\Omega(A/k)_{\wp}) \leq \dim A_{\wp} + n - \mathrm{ht}\, I_P$, *for every prime ideal* $P \supseteq I$, *where* $\wp = P/I$.
(iii) A *is (locally) Gorenstein.*

Then A is starlike linear and $\mathbf{Z} = \mathbf{T}$ *is a Gorenstein ring.*

Proof. The starlike linear part follows from Theorem 4.10 with $E = A$ and the Gorenstein part is proved in [10, Theorem 9.1]. □

There are other consequences and variants of Theorem 4.10 which are of interest. However, most of them actually fall off Theorem 4.7, so we will not expand on them referring rather to [10].

5 Selecta

This section will briefly survey reasonably broad classes of ideals that can be approached from the point of view of the preceding section. At the end we will discuss a question posed by van Gastel and give counterexamples to it in the spirit of some of the chosen classes herein.

5.1 Ideals in the linkage class of a complete intersection

Following current terminology, we say that a noetherian local algebra A is *licci* over a field k if it admits a presentation $A \simeq R/I$ with R a localization at a prime ideal of a polynomial ring over k and I, an ideal belonging to the linkage class of a complete intersection.

Theorem 5.1 *Let k be a perfect field and let A be a k-algebra essentially of finite type over k satisfying the following conditions:*

(i) *A is licci locally at every one of its maximal ideals;*
(ii) edim $A_{\wp} \leq 2 \dim A_{\wp}$ *for every* $\wp \in \operatorname{Spec}(A)$.
Then:

(a) $\mathbf{Z} \otimes_A \omega_{A_m} \simeq \mathbf{T} \otimes_A \omega_{A_m}$ *for every maximal ideal* m *of A and this module is a Cohen–Macaulay* $(\mathbf{Z})_m$*-module.*

(b) *A is set theoretically starlike linear.*

(c) *Moreover, if A is Gorenstein then* $\mathbf{D} \subset A \otimes_k A$ *is normally torsion free if and only if* edim $A_{\wp} \leq 2 \dim A_{\wp} - 1$ *for every non-minimal* $\wp \in \operatorname{Spec}(A)$.

(d) *If A is Gorenstein then it is starlike linear and* $\mathbf{Z} \simeq \mathbf{T}$ *is a Gorenstein ring.*

Proof. By [2, 6.2.11], $\operatorname{Tor}_i^R(A_m, \omega_{A_m})$ is a maximal Cohen–Macaulay A_m-module for every $i \geq 0$ and every maximal ideal m of A. The assertions then follow from Theorem 4.10, Remark 4.11, Proposition 4.12 and [10, Theorem 9.1] □

Remark 5.2 (Locally complete intersections) A special case of the preceding theorem is that of a locally complete intersection A with the expected star dimension. Here, of course, the modules $\operatorname{Tor}_i^R(A, A)$ are locally free (i.e., projective) over A.

5.2 Cohen–Macaulay rings of codimension two

Perfect ideals of codimension two form the first class of licci ideals for which the assumption on having the expected star dimension actually implies starlike linearity (not just set theoretically). Here is a more precise version of this result.

Proposition 5.3 *Let k be a perfect field and let A be a k-algebra essentially of finite satisfying the following conditions:*
(i) *For every maximal ideal* m *of A,* $A_m \simeq R/I$ *where R is a polynomial ring over k localized at a prime ideal and I is a perfect ideal of codimension at most*

2.

(ii) *For every prime ideal* $\wp \in \mathrm{Spec}(A)$, $\mathrm{edim}\, A_\wp \leq 2 \dim A_\wp$.
Then

(a) $\mathbf{Z} = \mathbf{T}$ *is a Cohen–Macaulay ring;*

(b) $\mathbf{D}$ *is normally torsionfree if and only if* $\mathrm{edim}\, A_\wp \leq 2 \dim A_\wp - 1$ *for every non-minimal prime* $\wp \in \mathrm{Spec}(A)$.

Proof. By [10, Theorem 9.1] and Remark 4.11, it suffices to show the depth condition on the modules $\mathrm{Tor}_i^R(A_m, A_m)$, with m a maximal ideal of A. We may then assume that A is local and that $\mathrm{ht}\, I = 2$. Therefore, we have only to deal with $\mathrm{Tor}_i^R(A, A)$ $(i = 1, 2)$. By the result of [1], $H_1(I)$ is a Cohen–Macaulay module. Thus, on one hand, from the exact sequence

$$0 \longrightarrow H_1(I) \longrightarrow A^m \longrightarrow I/I^2 \longrightarrow 0$$

one gets $\mathrm{depth}\, \mathrm{Tor}_1^R(A, A) = \mathrm{depth}\, I/I^2 \geq \dim A - 1$.

On the other hand, the symmetric power $S_2(\omega_A)$ is also a Cohen–Macaulay module [37, 2.1(b)]. This implies that

$$\mathrm{Tor}_2^R(A, A) \simeq \mathrm{Hom}\,(\omega_A, A) \simeq \mathrm{Hom}\,(S_2(\omega_A), \omega_A)$$

is Cohen–Macaulay as well. □

Remark 5.4 Condition (ii) in the preceding Proposition can in the present circumstances be restated to the effect that A is a reduced ring and a hypersurface locally in codimension one. Likewise, the above condition in order that $\mathbf{T}$ be torsionfree over A is equivalent to requiring that A be normal and a hypersurface locally in codimension two.

Example 5.5 Let $\underline{X}$ be an $m \times m - 1$ matrix of indeterminants over a perfect field k, let $R := k[\underline{X}]$ and let $I \subset R$ stand for the ideal generated by the maximal minors of the matrix $\underline{X}$. Setting $A = R/I$, the assumptions of Proposition 5.3 are fulfilled, so A is starlike linear, $\mathbf{T}$ is a Cohen–Macaulay domain.

5.3 Cohen–Macaulay rings of codimension three

The exact conditions we ought to tailor in order to have starlike linearity for codimension three ideals are far from clear. In the next section we will see examples of such ideals that are of the expected star dimension and nevertheless not even set theoretically starlike linear.

The next proposition gives one result in the positive direction. The main condition, homological in nature, may not look so natural but it takes place in many situations.

Proposition 5.6 *Let k be a perfect field of characteristic $\neq 2$ and let A be a k-algebra essentially of finite type satisfying the following conditions:*
(i) *For every maximal ideal m of A, $A_{\mathrm{m}} \simeq R/I$ where R is a polynomial ring over k localized at a prime ideal and I is a perfetct ideal of codimension at most 3 such that the first Koszul homology module $H_1(I; R)$ is Cohen–Macaulay.*
(ii) *For every prime ideal $\wp \in \mathrm{Spec}(A)$, $\mathrm{edim}\, A_\wp \leq 2 \dim A_\wp$.*
Then

(a) $\mathbf{Z} = \mathbf{T}$ *is a Cohen–Macaulay ring.*

(b) $\mathbf{D}$ *is normally torsionfree if and only if* $\mathrm{edim}\, A_\wp \leq 2\dim A_\wp - 1$ *for every non-minimal* $\wp \in \mathrm{Spec}(A)$.

Proof. Again it suffices to show the following depth inequalities

$$\mathrm{depth}\,\mathrm{Tor}_i^R(A, A) \geq \dim A - (3 - i), \;\; 1 \leq i \leq 3$$

where A is assumed to be local and $A \simeq R/I$ with $\mathrm{ht}\, I = 3$.

The cases $i = 1, 3$ are taken care of exactly as in the proof of Proposition 5.3. As for $i = 2$, by [33] one has $\mathrm{Tor}_2^R(A, A) \simeq \wedge^2 I$ since I is syzygetic and $1/2 \in R$. On the other hand, Weyman's resolution [44] implies that $\mathrm{pd}\, \wedge^2 I \leq 4$. Therefore

$$\mathrm{depth}\, \wedge^2 I \geq \dim R - 4 = \dim A - 1.\square$$

Example 5.7 The primeval example of a perfect ideal $I \subset R$ such that $H_1(I; R)$ is Cohen–Macaulay is one that has deviation $d(I) := \mu(I) - \mathrm{ht}\, I$ at most two.

Thus, codimension three perfect ideals generated by at most 5 elements satisfy condition (i) of Proposition 5.6. For them, $A = R/I$ is starlike linear provided it has the expected dimension. By Proposition 3.14, any specialization thereoff having the expected star dimension will be set theoretically starlike linear.

The assumption on $H_1(I; R)$ being Cohen–Macaulay is not necessary as the following instance puts forward.

Example 5.8 Consider the ideal $I \subset k[X_1, \ldots, X_6]$ generated by the following polynomials:

$$f_1 = X_2X_4 + X_3X_6, \quad f_2 = X_3X_5 + X_1X_6, \quad f_3 = X_1X_2 - X_2X_5 + X_3X_5 - X_5X_6,$$

$$f_4 = X_2X_3 + X_2X_4 + X_2X_6 + X_6^2 \quad f_5 = X_3^2 + X_3X_4 + X_3X_6 - X_4X_6,$$

$$f_6 = X_1X_3 + X_1X_4 + X_4X_5 + X_1X_6.$$

This ideal I is perfect of codimension three (and deviation three) and has analytic spread $\ell_{(\underline{X})}(I) = 5$ (in particular, I is not of linear type).

Actually, $H_1(I; R)$ is *not* Cohen-Macaulay as I is not syzygetic: a quadratic relation lying outside $(\underline{X})R[\underline{T}]$ being

$$T_1T_2 + T_2T_4 - T_2T_5 + T_3T_5 + T_1T_6 - T_4T_6,$$

with $T_i \mapsto f_i$.

The corresponding projective variety cut by quadrics in $\mathbf{P}^5$ has degree 4 and is reduced and irreducible as is shown by means of Noether normalization using the method given in [40]. Alternatively, one can show under a calculation with *Macaulay* that the ring R/I is (R_2), hence R/I is a normal domain (since it is Cohen–Macaulay) and the corresponding projective variety is non-singular.

A second calculation in *Macaulay* grants that the Zariski tangent algebra and the tangent star algebra are isomorphic. Sufficient evidence from a wasteful computation in the same program indicates that the algebra is indeed Cohen–Macaulay (possibly, a normal domain too).

This example is curious in a number of other ways as well. Let us look at the ideal I more closely.

First, its transposed jacobian matrix has rank 5 over $R = k[\underline{X}]$ with cokernel L such that

$$0 \longrightarrow R \xrightarrow{\Phi} R^6 \xrightarrow{\Theta} R^6 \longrightarrow L \longrightarrow 0,$$

where $I = (\Phi^t)$. Another computation in *Macaulay* yields that $\operatorname{ht} I_5(\Theta) = 2$ when char $k \neq 2$, thus showing that L is isomorphic to an ideal of $k[\underline{X}]$ in this case. This ideal can be taken as the ideal generated by the cubic polynomials obtained by cancelling the common factor among the six 5×5 minors of a 5×6 submatrix of rank 5 of Θ (in particular, it has codimension two).

Of course, we have $\Omega(R/I) \simeq L/IL$, a locally free module in the punctured spectrum of R/I.

Further, L is *self-dual* in the sense of jacobian duality, i.e., it and its jacobian dual [31] are isomorphic.

Let us remark that if char $k = 2$ then the cokernel L has torsion and, moreover, $(\Phi^t) = (\underline{X})$.

5.4 Determinantal rings

Determinantal rings are representative of a more general pattern which is explained, at least in one direction, by Proposition 5.20. The coordinate rings of the Veronese and Segre varieties, to be considered later, are examples of special interest.

Typically, the results for this class are more or less definitive, at least in the generic case. We will content ourselves in quoting the following theorem, whose parts are collected from [32].

Theorem 5.9 *Let R be a regular domain and let $(\underline{X})$ be an $r \times s (r \leq s)$ matrix of indeterminants over R. Let t be an integer such that $1 \leq t \leq r$. Then:*

(a) $R[\underline{X}]/I_t(\underline{X})$ has the expected star dimension if and only if

$$\begin{cases} t = 1 & \text{or} \\ t = r & \text{or} \\ t = r-1, s \leq r+1 \end{cases}$$

(b) *If R is a localization of a polynomial ring over a field then $R[\underline{X}]/I_t(\underline{X})$ is set theoretically starlike linear if and only if*

$$\begin{cases} t = 1 & \text{or} \\ t = r \end{cases}$$

The proof of (a) is by induction, using the well-known *inversion-localization trick* for matrices of indeterminants. The argument for (b) is a consequence of Proposition 5.20 for one direction, but a lot more involved for the other direction.

5.5 Rings associated to graphs

To a simple graph $G = (V, E)$ (V = vertices, E = edges) one can attach a polynomial ring $k[\underline{X}]$ and an ideal $I = \mathcal{I}(G) \subset k[\underline{X}]$, where $\underline{X}$ is in bijection with V and I is generated by the squarefree monomials X_iX_j corresponding to the elements of E. The ideal I (resp. the ring $k[\underline{X}]/I$) is called the *edge-ideal* (resp. the *edge-ring*) of the graph G.

For further terminology and basic results in this section, we refer to [43] and [36].

Let then $I = \mathcal{I}(G) \subset k[\underline{X}]$ be as just explained. Set $A = k[\underline{X}]/I$. Further, let $\underline{T}$ and J denote, respectively, a set of presentation variables and the corresponding presentation ideal of $\mathbf{T}$ over A.

The *suspension* over G is the graph defined by doubling the set of vertices of G and then adding to the set of edges of G one extra edge for each pair of vertices (V_i, U_i), where V_i is an old vertex of G and U_i its duplicate. In terms of ideals, the suspension is defined by the ideal $(\mathcal{I}(G), X_1Y_1, \ldots X_nY_n) \subset k[\underline{X}, \underline{Y}]$, where $\underline{Y}$ is a doubling of $\underline{X}$.

Here is a result that explains the definite pattern for edge-ideals regarding the problem posed by van Gastel.

Proposition 5.10 *(k a perfect field) Let $I \subset R := k[\underline{X}, \underline{Y}]$ stand for the edge-ideal of a suspension over a graph G and let $A = R/I$. Then*

(1) *A has the expected star dimension.*

(2) *If G is not bipartite then A is not set theoretically stalike linear.*

Proof. (1) Let G denote a graph and let $S(G)$ denote its suspension. Then $I = \mathcal{I}(S(G)) = (\mathcal{I}(G), X_1Y_1, \dots X_nY_n) \subset R$.

Proceed by induction on $n = \#\underline{X}$. For $n = 1$, G is an isolated vertex, so $I = (XY)$. Then $\mathbf{Z} \simeq k[X,Y,T,W]/(XY, XT + YW)$ which is certainly of dimension two.

Since A is Cohen–Macaulay [43], it suffices by Proposition 3.12 to argue that $\operatorname{edim} R_P/I_P \geq 2\operatorname{ht} I_P$ for every prime $P \supset I$.

If $P = (\underline{X}, \underline{Y})$ the result follows from the nature of the suspension since one has here $\operatorname{ht} I = n$. Let $P \neq (\underline{X}, \underline{Y})$. Say, first $X_1 \notin P$. Clearly then $R_P/I_P = (R_{X_1}/I_{X_1})_{PR_{X_1}}$. But $I_{X_1} = (Y_1, X_2, \dots, X_d, X_iX_j, X_{d+1}Y_{d+1}, \dots, X_nY_n)$, where d is the degree of the vertex corresponding to X_1 in $S(G)$ and $i, j \neq 1, 2, \dots, d$.

Cancelling the free variables $Y_1, X_2, \dots, X_d$, the ring R_{X_1}/I_{X_1} is isomorphic to a localization of the ring of the suspension over a subgraph of G with $n - d$ vertices.

Suppose now that, say, $Y_1 \notin P$. A similar argument shows that R_{Y_1}/I_{Y_1} is isomorphic to a localization of the ring of the suspension over a subgraph of G with $n - 1$ vertices.

In either case, one is done by the induction hypothesis.

(2) If G is not bipartite, it contains an odd cycle with vertices $X_1, \dots, X_{2m+1}$ (say) and $m \geq 2$. We claim that the corresponding monomial $T_1 \cdots T_{2m+1}$ belongs to J.

For this, let $\underline{U}$ be the other set of variables in the enveloping algebra of $k[\underline{X}]$. The ideal of the diagonal is generated by the differences $X_i - U_i$ plus similar generators for the $\underline{Y}$.

Since everything is homogeneous, it suffices to show that upon substitution $T_i \mapsto X_i - U_i$, the monomial $T_1 \cdots T_{2m+1}$ lands inside $\mathcal{I}(G) + \mathcal{I}(G') \subset k[\underline{X}, \underline{U}]$, where G' stands for a copy of G in the "vertices" $\underline{U}$.

But, note that any variable product in

$$\prod_{i=1}^{2m+1} (X_i - U_i)$$

is divisible by at least one monomial of the form X_iX_j (or U_iU_j) with $i \leq j$. If $j = i+1$, we are done; otherwise, it must be the case that the variable product is

$$X_1U_2 \cdots X_{2m-1}U_{2m}X_{2m+1} \quad \text{or} \quad U_1X_2 \cdots U_{2m-1}X_{2m}U_{2m+1},$$

in which case we are again done.

To reach the desired conclusion one now invokes the fact that set theoretic starlike linearity implies that there are no equations of analytic dependence for the generators of $\mathbf{D}$ (cf. next subsection). □

Remark 5.11 One can actually prove that (for char $k \neq 2$) if G is an odd cycle then $J(1) = J \cap (\underline{X}, \underline{Y})R[\underline{T}]$ where, as above, J denotes a presentation ideal of $\mathbf{T}$, with A the edge–ring of the suspension of G, and $J(1)$ denotes the ideal generated by the linear $(\underline{T})$-forms of J. This euality is not true for an arbitrary suspension, an easy example being given by the suspension over a graph consisting of a pentagon with a chord.

To finish these short considerations, we would like to file the following.

Conjecture 5.12 (k perfect, char $k \neq 2$) The following conditions are equivalent:

(1) A is starlike linear.

(2) $\operatorname{ht}(\underline{X})\mathbf{Z} \geq 1$.

(3) G is bipartite.

Note *en passant* that condition (3) implies, by [36], that $\operatorname{ht}(\underline{X})R[It] \geq 2$ which is at least interesting to compare with condition (2).

Actually, the above conjecture falls within the expectation that, if char $k \neq 2$, the generators of J not contained in $J(1)$ are monomials corresponding to the odd cycles present in G. A large computation of examples indicate this might be the case.

Moreover, some examples also point out in the direction that the equality $J(1) = J \cap (\underline{X})R[\underline{T}]$ might hold for large classes of graphs (see below).

5.6 Hall of starlike counterexamples

The rings considered in this section will be again finite type algebras over a field, i.e., of the form $A := R/I$, where $R = k[\underline{X}]$ (k perfect) and $I \subset (\underline{X})$ an arbitrary ideal.

Dimension theoretic versus set theoretic The following question was raised by L. van Gastel [6].

Question 5.13 Let A be a *reduced* affine ring having the expected star dimension. Is A set theoretically starlike linear?

The preceding sections dealt with this and similar questins from their positive side (Proposition 5.10 being an exception). Here we we show that the answer to this question is negative in general.

The question can be so rephrased as to ask whether the two tangent algebras have the same minimal primes. We observe initially that the hypothesis on the expected star dimension already implies generic reducedness of A.

Let now $J(1)$ and J stand, respectively, for the presentation ideals of $\mathbf{Z}$ and $\mathbf{T}$ on the polynomial ring $k[\underline{X}, \underline{T}]$. We know (see, e.g., [14]) that a necessary condition for the set theoretic equality of $\mathbf{Z}$ and $\mathbf{T}$ is the analytic independence of the generators of $\mathbf{D}$ at the maximal ideal

$$(\underline{X}, \underline{U})/I(\underline{X}) + I(\underline{U})) \subset k[\underline{X}, \underline{U}]/I(\underline{X}) + I(\underline{U})) \simeq A \otimes_k A.$$

This latter condition translates to $J \subset (\underline{X})$.

All the counterexamples to follow will trespass this condition by exhibiting directly a (global) relation of analytic independence. In particular, the following weaker version of van Gastel's is momentarily open:

Question 5.14 Assume $A = k[\underline{X}]/I$ is reduced and has the expected star dimension. If the generators of **D** are analytically independent at $(\underline{X}, \underline{U})/I(\underline{X}) + I(\underline{U}))$, is it true that **Z** and **T** have the same reduced structure?

If one is willing to sacrifice unmixedness, then a "smallest" example that answers the question negatively is very likely to be the following ideal of codimension two.

Example 5.15 Let $\underline{X} = X_1, X_2, X_3, X_4$ and let I be generated by

$$X_1X_2, \quad X_2X_3, \quad X_3X_1, \quad X_1X_4.$$

It is easily seen directly that $\dim \mathbf{Z} = \dim \mathbf{T}(= 4)$. Let $\underline{T} = T_1, T_2, T_3, T_4$ be the presentation variables. Then $T_1T_2T_3 \in J$.

In order to construct a negative instance which is moreover *prime*, one can upgrade the example to a Cohen-Macaulay one. In particular, codimension three must be the case.

Example 5.16 (char $k \neq 2$) Let $\underline{X} = X_1, X_2, X_3, X_4, X_5, X_6$ and let I be generated by

$$X_1X_2, \quad X_2X_3, \quad X_3X_1, \quad X_1X_4, \quad X_2X_5, \quad X_3X_6.$$

I is Cohen-Macaulay of codimension three and deviation $(= \mu(I) - \operatorname{ht} I)$ three (moreover, it is of linear type).

Again A has the expected star dimension, as is readily checked from the heights of the various Fitting ideals, using Proposition 3.12.

The presentation ideal J of the tangent star algebra **T** has an extra generator as before, namely, $T_1T_2T_3$ corresponding to the (odd) cycle structure in the graph defined by the generators of I.

Actually, a computation in *Macaulay* shows that (provided char $k \neq 2$) then

$$J = (J(1), T_1T_2T_3) \text{ and } J(1) = M \cap J,$$

where $M = (\underline{X})k[\underline{X}, \underline{T}]$.

Remark 5.17 We note *en passant* that the tangent star algebra $\mathbf{T}$ is still Cohen-Macaulay. This holds even if k has characteristic 2, although in the latter case one has $J = (J(1), X_3T_1T_2, T_1T_2\dot{T}_3)$ hence the finer equality $J(1) = M \cap J$ is no longer valid.

Out of Example 5.16, using the deformation-linkage argument in the last section of [31], one can produce many prime ideals preserving similar data.

Example 5.18 Here is an example linking the preceding one with the regular sequence

$$X_1X_2 + X_1X_4, \quad X_2X_3 + X_2X_5, \quad X_3X_6.$$

After introducing generic variables Z_1, Z_2, Z_3 and linking again, one finds the following (non-homogeneous) prime ideal in a polynomial ring in 9 variables.

$$X_3X_4Z_1 + X_4X_5Z_1 + X_3X_6Z_1 + X_4X_6Z_1 + X_5X_6Z_1 - X_1X_4Z_2 - X_1X_6Z_2 + X_1X_4,$$
$$X_2X_3Z_1 + X_2X_5Z_1 + X_2X_6Z_1 - X_3X_6Z_1 - X_1X_2Z_2 + X_1X_6Z_2 + X_1X_2Z_3 + X_1X_4Z_3 + X_1X_2,$$
$$X_3^2Z_1 + X_3X_5Z_1 + X_3X_6Z_1 - X_1X_3Z_2 + X_1X_3Z_3 + X_1X_5Z_3 + X_1X_3,$$
$$X_4X_5Z_2 + X_2X_6Z_2 + X_4X_6Z_2 + X_5X_6Z_2 - X_2^2Z_3 - X_2X_4Z_3 + X_2X_5,$$
$$X_3X_4Z_2 + X_3X_6Z_2 + X_2^2Z_3 + X_2X_4Z_3 + X_2X_3,$$
$$X_3X_4Z_3 + X_4X_5Z_3 + X_2X_6Z_3 + X_3X_6Z_3 + X_4X_6Z_3 + X_5X_6Z_3 + X_3X_6.$$

The symmetric algebra $\mathbf{Z}$ still has the expected dimension (which is now 12). On the other hand, A is not set theoretically starlike linear by Proposition 3.14 as the present link is also a deformation of the original ideal.

Remark 5.19 For the purpose of getting smaller counterexamples, one may specialize down the Z-variables. Again, it would not be too difficult to apply same primality test as in [40] to show that a suitable specialization is still prime.

The preceding examples seem somewhat fragmentary. Next we present two counterexamples to van Gastel's question which are important from the viewpoint of deformation theory and free resolutions.

1. The projecting cone of the Veronese surface

This surface is certainly ubiquitous in classical Algebraic Geometry. The approach taken here is rather simple.

As is well known, the projecting cone of the surface is ideal-theoretically defined by the 2×2 minors of the symmetric matrix

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_6 \end{pmatrix}.$$

Let $R := k[\underline{X}] = k[X_1, \ldots, X_6]$ and $A := R/I$, where I is the ideal generated by these minors. Also well known is that A is a normal Cohen–Macaulay isolated singularity of of dimension 3. Let $\underline{T} = T_1, \ldots, T_6$ be presentation variables for $\mathbf{Z}$ (or $\mathbf{T}$) over A. Finally, let $D(\underline{T})$ stand for the determinant of the above matrix evaluated on the variables $\underline{T}$.

A classical argument will show that $D(\underline{T})$ belongs to the presentation ideal of $\mathbf{T}$ over A. This yields the required relation of analytic dependence.

More generally, one has the following intriguing phenomenon:

Proposition 5.20 *Let k be a field, let $R = k[\underline{X}] = [X_1, \ldots, X_n]$ and let $I \subset (\underline{X})$ be any ideal of R. If there exists an $r \times r$ matrix M, whose entries are linear forms in R and whose determinant is non-zero, such that moreover $I_{[(r+1)/2]}(\mathrm{M}) \subset I$, then the ring $(R/I)_{(\underline{X})}$ is not set theoretically starlike linear.*

Proof. The canonical generators $\{x_i \otimes 1 - 1 \otimes x_i |\ 1 \leq i \leq n\}$ of $\mathbf{D}$ (here x_i denotes the residue of X_i in $B = (R/I)_{(\underline{X})}$) are minimal generators. Therefore, as remarked above, they would have to be analytically independent at the maximal ideal $(x_i \otimes 1, 1 \otimes x_i |\ 1 \leq i \leq n)$ were B to be set theoretically starlike linear. Now, expanding the determinant of the $r \times r$ matrix $M \otimes 1 - 1 \otimes M$ by Laplace, using multilinearity and the assumption that the $[(r+1)/2] \times [(r+1)/2]$ minors of M belong to the given ideal I, one finds

$$\det(M \otimes 1 - 1 \otimes M) \subset I_{[(r+1)/2]} \otimes R + R \otimes I_{[(r+1)/2]} \subset I \otimes R + R \otimes I,$$

which yields a relation of analytic dependence. □

Going back to the example, it remains to verify that A has the expected star dimension. But this is clear by Proposition 3.12, (b), since A is an isolated singularity (alternatively, by a computation in *Macaulay* using (d) of the same proposition).

Remark 5.21 (1) The ideal I above is interesting from the purely algebraic viewpoint. First, the preceding argument shows *a posteriori* that its first Koszul homology module $H_1(I; R)$ is *not* Cohen–Macaulay (cf. Proposition 5.6). Nonetheless, it is known [20] that it is an ideal of linear type, i.e., its blowing-up algebra coincides with the residual scheme (symmetric algebra of I). This seemingly makes the counterexample very tight.
(2) The star algebra $\mathbf{T}$ is a Cohen–Macaulay domain (computation in *Macaulay*).
(3) As in Example 5.16, here too the finer equality $J(1) = M \cap J$ hold true provided char $k \neq 2$.

2. The projecting cone of the Segre $\Sigma_{2,2}$

It is well known that this cone is ideal-theoretically defined by the 2×2 minors of a generic 3×3 matrix. The argument is then *ipsis-literis* the same as the one for the Veronese, a relation of analytic dependence being given once more by the determinant evaluated at $\underline{T}$. Thus, this yields an additional counterexample to van Gastel's question. .

We close this section with a remark about the characteristic of the ground field k.

The difference between characteristic 2 and other characteristics turns out to be deeper than it appears on the surface, at least for ideals generated by quadratic polynomials.

Practically all the explicit examples discussed so far have in common the following feature. Let $I \subset R = k[\underline{X}]$ be a homogeneous ideal generated by quadrics and let E stand for the R-module defined as the cokernel of the transposed jacobian matrix of a set of generators of I. Assume that E has rank zero as R-module. Then the maximal minors of the jacobian matrix Θ, viewed in the $\underline{T}$-variables, belong to the presentation ideal J of $\mathbf{T}$ (and, clearly, not to $J(1)$).

In the case of the veronesean and the Segre threefold, the module E indeed has rank zero if char $2 \neq 0$ (but positive rank otherwise!) and, moreover, the maximal minors of Θ are indeed powers of generators of J not lying in $J(1)$.

This yields an efficient, as yet not completely understood, method of verifying the deficiency of the corresponding secant variety.

Set theoretic versus ideal theoretic We will briefly deal with the implication "set theoretically starlike linear $\Rightarrow$ starlike linear". To the best of our feeling, this implication should fail even for $A \simeq k[\underline{X}]/I$ with I a perfect codimension three ideal. The examples may require a good deal of ingenuity for their construction.

We will next give a flavour for the question by showing a counterexample in codimension two which, in spite of not being unmixed, keeps many features common to determinantal ideals.

Example 5.22 Let $R = k[X_1, \ldots, X_5]$ and let $I \subset R$ denote the ideal of maximal minors fixing the first column of the catalecticant

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_5 \end{pmatrix}.$$

Then $I = (X_1, X_2) \cap D$, where D is the ideal of all maximal minors. Hence, I is prime (but not C–M, not even unmixed).

We have found directly the minimal primes of both algebras, as follows. Letting J stands for either presentation ideal, then:

1. Every minimal prime of $\mathbf{Z}$ (resp. $\mathbf{T}$) contains a minimal prime of either $(X_1, X_2)\mathbf{Z}$ (resp. $(X_1, X_2)\mathbf{T}$) or $D\mathbf{Z}$ (resp. $D\mathbf{T}$);

2. A trivial computation shows that the minimal primes of $(X_1, X_2)\mathbf{Z}$ (resp. $(X_1, X_2)\mathbf{T}$) are $(\underline{X})$ and (X_1, X_2, T_1, T_2) (modulo J of course);

3. Since ht $J = \text{ht}\,(X_1, X_2, T_1, T_2)(= 4)$, it follows that the latter is a minimal prime of J.

4. Now, a computation following the script of Vasconcelos [42] yielded that the radical of the extension of D/I is $(\underline{X}) \cap (X_1, X_2, X_3, X_4, T_1)$ (It is immediate that these are minimal primes of the extension of D/I but not trivial that they are the only ones). These are primes of codimension 5.

5. To see that the primes in (2) are actually minimal primes of J is more involved: still following the same script, we checked that they are minimal primes of a larger ideal $J_2 \cap J_3'$ sharing same radical as $J_2 \cap J_3$ – the latter standing for the intersection of the minimal primes of codimension 5 or 6.

6. Therefore, the minimal primes of J (for both $\mathbf{Z}$ and $\mathbf{T}$) are

$$(X_1, X_2, T_1, T_2),\ \ (\underline{X}),\ \ (X_1, X_2, X_3, X_4, T_1);$$

7. It follows that $\mathbf{Z}$ and $\mathbf{T}$ have the same set structure, as required.

To conclude, we give below the explicit equations of the respective presentation ideals, thus showing they are different.

Remark 5.23 The homological dimension of J (for both Zariski and star) is 6. We have checked that $(\underline{X}, q(\underline{T}))$ (cf. below) is an *embedded* prime of both presentation ideals. Is it the only prime in cod. 6 ?

We next give explicit sets of generators of both presentation ideals.

Here is the presentation ideal of the star algebra (the one of the Zariski algebra just leaves out the last polynomial)

$$X_2X_4 - X_1X_5,\ X_2X_3 - X_1X_4,\ X_2^2 - X_1X_3$$
$$X_5T_1 - X_4T_2 - X_2T_4 + X_1T_5,\ X_4T_1 - X_3T_2 - X_2T_3 + X_1T_4,$$
$$X_3T_1 - 2X_2T_2 + X_1T_3,$$
$$X_2T_1T_3^3 - X_1T_2T_3^3 - 2X_2T_1T_2T_3T_4 + 2X_1T_2^2T_3T_4 + X_2T_1^2T_4^2 - X_1T_1T_2T_4^2$$
$$+X_2T_1T_2^2T_5 - X_1T_2^3T_5 - X_2T_1^2T_3T_5 + X_1T_1T_2T_3T_5.$$

In particular one sees that all coefficients lie in $(\underline{X})$ which implies analytic independence of $\mathbf{D} \subset k[\underline{X},\underline{U}]/(I(\underline{X})+(\underline{U}))$ globally on $(\underline{X},\underline{U})/(I(\underline{X})+(\underline{U}))$ (recall that this fact on itself is insufficient to have set theoretic starlike linearity).

Next are the generators of the primary part $J_2 \cap J_3'$ needed for the higher codimension primes.

$$X_2X_4 - X_1X_5,\ X_2X_3 - X_1X_4,\ X_2^2 - X_1X_3,\ X_3^3 - X_1X_3X_5$$
$$X_3X_4^3 - X_3^2X_4X_5,\ X_3^2X_4^2 - X_1X_3X_5^2,\ X_4^4 - 2X_3X_4^2X_5 + X_3^2X_5^2$$
$$X_5T_1 - X_4T_2 - X_2T_4 + X_1T_5,\ X_4T_1 - X_3T_2 - X_2T_3 + X_1T_4,$$
$$X_3T_1 - 2X_2T_2 + X_1T_3,$$
$$X_3^2X_4T_3 - 2X_1X_4X_5T_3 + 2X_1X_3X_5T_4 - X_1X_2X_5T_5,$$
$$X_4^3T_3 - 2X_2X_5^2T_3 - X_3X_4^2T_4 - X_3^2X_5T_4 + 4X_1X_5^2T_4 + X_3^2X_4T_5 - 2X_1X_4X_5T_5,$$
$$X_3X_4^2T_3 - 1/2X_3^2X_5T_3 - X_1X_5^2T_3 - X_3^2X_4T_4 + 2X_1X_4X_5T_4 - 1/2X_1X_3X_5T_5,$$
$$X_2T_1T_3^3 - X_1T_2T_3^3 - 2X_2T_1T_2T_3T_4 + 2X_1T_2^2T_3T_4 + X_2T_1^2T_4^2 - X_1T_1T_2T_4^2$$
$$+X_2T_1T_2^2T_5 - X_1T_2^3T_5 - X_2T_1^2T_3T_5 + X_1T_1T_2T_3T_5.$$

In order to calculate in turn the radical of the extension of the minimal prime coming from the Fitting of 2×2 minors, we have used the following. regular sequence of length 5:

$$X_2^2 - X_1X_3,\ X_3^2 - X_1X_5,\ X_4^2 - X_3X_5,$$
$$X_5T_1 - X_4T_2 - X_2T_4 + X_1T_5,\ X_3T_1 - 2X_2T_2 + X_1T_3,$$

The result was the ideal generated by the polynomials below, which are indeed awful and do not present any explicit signs of the final result.

$$X_2^2 - X_1X_3,\ \ X_3^2 - X_1X_5,$$
$$X_3T_1 - 2X_2T_2 + X_1T_3,\ \ X_5T_1 - X_4T_2 - X_2T_4 + X_1T_5,$$
$$X_2X_4T_2 - 2X_1X_5T_2 + X_2X_3T_3 + X_1X_3T_4 - X_1X_2T_5,$$
$$X_4T_1T_2 - 4X_3T_2^2 + 4X_2T_2T_3 - X_1T_3^2 + X_2T_1T_4 - X_1T_1T_5,$$
$$X_3X_5T_2 - 1/3X_3X_4T_3 - 2/3X_2X_5T_3 - 1/3X_2X_4T_4 - 2/3X_1X_5T_4$$
$$+2/3X_2X_3T_5 + 1/3X_1X_4T_5,$$

$$X_2X_5T_2T_3 - 1/3X_2X_4T_3^2 - 2/3X_1X_5T_3^2 - 2/3X_2X_3T_3T_4$$
$$-1/3X_1X_4T_3T_4 - 1/3X_2X_4T_1T_5 + 2/3X_1X_4T_2T_5 + 2/3X_1X_3T_3T_5,$$
$$X_4X_5T_2^2 - 5/9X_3X_4T_3^2 - 4/9X_2X_5T_3^2 - 5/3X_2X_5T_2T_4$$
$$-5/9X_2X_4T_3T_4 + 2/9X_1X_5T_3T_4 + 2/3X_2X_3T_4^2 + 5/3X_1X_5T_2T_5$$
$$-2/9X_2X_3T_3T_5 + 5/9X_1X_4T_3T_5 - 4/3X_1X_3T_4T_5 + 2/3X_1X_2T_5^2,$$
$$X_4T_2^4T_3 - 11/3X_3T_2^3T_3^2 + 23/6X_2T_2^2T_3^3 - 1/6X_2T_1T_3^4$$
$$-X_1T_2T_3^4 - 5/3X_2T_2^3T_3T_4 + 4/3X_2T_1T_2T_3^2T_4$$
$$+4/3X_1T_2^2T_3^2T_4 - 1/3X_1T_1T_3^3T_4 + 1/6X_2T_1^2T_3T_4^2$$
$$+4/3X_2T_1T_2^2T_3T_5 - X_1T_2^3T_3T_5 - 1/3X_2T_1^2T_3^2T_5$$
$$-2/3X_1T_1T_2T_3^2T_5 - 1/3X_1T_1^2T_3T_4T_5 - 1/6X_2T_1^3T_5^2$$
$$+1/3X_1T_1^2T_2T_5^2,$$
$$X_4T_2^3T_3^2 - 8/3X_3T_2^2T_3^3 + 2X_2T_2T_3^4 - 1/3X_1T_3^5$$
$$-5/3X_2T_2^2T_3^2T_4 + 2/3X_2T_1T_3^3T_4 + 4/3X_1T_2T_3^3T_4$$
$$+1/3X_1T_1T_3^2T_4^2 + 4/3X_2T_1T_2T_3^2T_5 - X_1T_2^2T_3^2T_5$$
$$-2/3X_1T_1T_3^3T_5 + 2/3X_2T_1^2T_3T_4T_5$$
$$-4/3X_1T_1T_2T_3T_4T_5 - 2/3X_2T_1^2T_2T_5^2 + 4/3X_1T_1T_2^2T_5^2$$
$$-1/3X_1T_1^2T_3T_5^2,$$
$$X_4T_2^5 - 14/3X_3T_2^4T_3 + 37/6X_2T_2^3T_3^2 - 2/3X_2T_1T_2T_3^3$$
$$-23/12X_1T_2^2T_3^3 + 1/12X_1T_1T_3^4 - 5/3X_2T_2^4T_4$$
$$+2X_2T_1T_2^2T_3T_4 + 4/3X_1T_2^3T_3T_4 - 1/6X_2T_1^2T_3^2T_4$$
$$-2/3X_1T_1T_2T_3^2T_4 + 1/6X_2T_1^2T_2T_4^2 - 1/12X_1T_1^2T_3T_4^2$$
$$+4/3X_2T_1T_2^3T_5 - X_1T_2^4T_5 - 2/3X_2T_1^2T_2T_3T_5$$
$$-2/3X_1T_1T_2^2T_3T_5 + 1/6X_1T_1^2T_3^2T_5 - 1/6X_2T_1^3T_4T_5$$
$$+1/12X_1T_1^3T_5^2,$$
$$X_4T_2^2T_3^3 - 65/96X_4T_1T_3^4 - 95/96X_3T_2T_3^4 + 2/3X_2T_3^5$$
$$+65/16X_4T_2^3T_3T_4 - 123/53X_3T_2^2T_3^2T_4 - 68/67X_2T_2T_3^3T_4$$
$$-7334X_1T_3^4T_4 - 65/96X_4T_1^2T_3T_4^2 - 65/48X_2T_2^2T_3T_4^2$$
$$-3329X_2T_1T_3^2T_4^2 + 4/3X_1T_2T_3^2T_4^2 - 65/8X_4T_2^4T_5$$
$$+102/73X_3T_2^3T_3T_5 - 65/48X_4T_1^2T_3^2T_5 + 7292X_2T_2^2T_3^2T_5$$
$$+4/3X_2T_1T_3^3T_5 - 20/107X_1T_2T_3^3T_5 + 116/107X_2T_2^3T_4T_5$$
$$+13316X_2T_1T_2T_3T_4T_5 - 57/77X_1T_2^2T_3T_4T_5$$

$$-11/8X_1T_1T_3^2T_4T_5 - 65/48X_2T_1^2T_4^2T_5 - 65/96X_4T_1^3T_5^2$$
$$-8/3X_2T_1T_2^2T_5^2 + 21/8X_1T_2^3T_5^2 + 2/3X_2T_1^2T_3T_5^2$$
$$+49/6X_1T_1T_2T_3T_5^2 + 65/48X_1T_1^2T_4T_5^2$$
$$X_4T_1T_3^4 - X_3T_2T_3^4 - 6X_4T_2^3T_3T_4 + 22X_3T_2^2T_3^2T_4$$
$$-23X_2T_2T_3^3T_4 + 6X_1T_3^4T_4 + X_4T_1^2T_3T_4^2$$
$$+2X_2T_2^2T_3T_4^2 - 6X_2T_1T_3^2T_4^2 + 12X_4T_2^4T_5$$
$$-56X_3T_2^3T_3T_5 + 2X_4T_1^2T_3^2T_5 + 58X_2T_2^2T_3^2T_5$$
$$-15X_1T_2T_3^3T_5 - 20X_2T_2^3T_4T_5 + 24X_2T_1T_2T_3T_4T_5$$
$$+6X_1T_2^2T_3T_4T_5 + 4X_1T_1T_3^2T_4T_5 + 2X_2T_1^2T_4^2T_5$$
$$+X_4T_1^3T_5^2 + 4X_1T_2^3T_5^2 - 16X_1T_1T_2T_3T_5^2$$
$$-2X_1T_1^2T_4T_5^2.$$

The following is the determinant of the specialized symmetric 3×3 matrix whose 2×2 define the rational normal quartic (also generated by the 2×2 of the catalecticant):

$$q(\underline{T}) := T_3^3 - 2T_2T_3T_4 + T_1T_4^2 + T_2^2T_5 - T_1T_3T_5.$$

Its geometric role is not so clear, neverheless the ideal $(\underline{X}, q(\underline{T}))$ is an embedded prime (of codimension 6) of the presentation ideal of both the Zariski and the star algebras. Also, it is a generator of the presentation ideal of the star algebra of the homogeneous coordinate ring of the rational normal quartic (cf. [19]).

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