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SOLVING CERTAIN GROUP EQUATIONS IN $PGL(2,k)$ – A COMPUTATIONAL APPROACH

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1. Introduction

Let F_2 be the free group of rank 2, freely generated by x, y. Let $k \in \mathbb{N}$, $i_1, \ldots, i_k, j_1, \ldots, j_k$ integers different from zero,

$$\omega(x,y) = x^{i_1} y^{j_1} \ldots x^{i_k} y^{j_k}$$

element an of F_2, and

$$G = < x, y \mid \omega(w,y) >$$

the quotient group of F_2 by the normal closure of $\omega(x,y)$ in F_2. The questions we treat here fall under the following general problem.

Problem. Describe all representations (or projective representations)

$$\rho_k : G \to GL(2,K) \qquad (\text{or } PGL(2,K))$$

over a variable field K. In particular, determine those ρ_K for which $\rho_K(G)$ is a nonsolvable group.

The abelianized group $\bar{G} = G/G'$, is nontrivial and has faithful representations in $GL((2,K)$ for appropriate fields K depending upon the type of $\bar{G}$.

Let us fix some notation: (conjugate) $x^y = y^{-1}xy$, (commutator) $[x,y] = x^{-1}y^{-1}xy$, (iterated commutator) $[x,ky] = [[x,(k-1)y],y]$.

Two groups studied by G. Baumslag define two extremes of the problem. The first

$$< x, y \mid [x,y] = x >$$

is isomorphic to a subgroup of $GL(2, \mathbf{Q})$ through

$$x \to \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad y \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, the second group

$$< x, y \mid [x, x^y] = x >$$

has a derived group which is not finitely generated. Also, all its finite quotients are cyclic; (see [1]).

It follows from a theorem of Mal'cev [6] that all images by linear representations of this group are also cyclic.

We refer the reader to W. Magnus [5] in which presentations into $PGL(2, \mathbf{C})$ of diverse classes such as Knot and Fuchsian groups are reviewed, and a survey is given of Fricke characters.

In this lecture we describe complete solutions of group equations of the type

$$[x^y, kx] = x^{y^{-1}} \qquad (k = 1, 2, 3)$$

and of

$$[x^{y^{-1}}, x^y] = x,$$

mainly in $PGL(2, K)$. These solutions were obtained with the help of the algebraic software MAPLE [7] complemented with a package of subroutines which were prepared by Christoph Seidler during 1988-89 with a scholarship from CNPq.

2. The Group Equations

The second group of Baumslag belongs to

$$\mathcal{L}_y(n; i, j) =< x, y \mid x^n = e, \;\; [x^{y^i}, x^{y^j}] = x >$$

which was used by R. Lyndon as a test for the Kervaire Lauderbach problem [4].

The group $\mathcal{L}_y$ $(n; -1, 1)$ is an overgroup of the groups

$$\mathcal{VE}(m,n) =< x, y \mid y^m = x^n = e,\ [x^{y^{-1}}, x^y] = x,\ [x^y, x] = e > .$$

$\mathcal{VE}$ stands for "verbal embedding" or codification of the ring $\mathbf{Z}_n$ within the group; see [2]. The linear group $SL(3, \mathbf{Z}_n)$ satisfies the conditions of $\mathcal{VE}(6,n)$ due to the existence of an automorphism η of order 6 such that

$$E_{12}(\alpha)^\eta = E_{13}(\alpha),\ \ E_{13}(\alpha)^\eta = E_{23}(\alpha),$$

and to the well-know facts

$$[E_{12}(\alpha),\ E_{23}(\beta)] = E_{13}(\alpha\beta), \quad [E_{23}(\alpha), E_{13}(\beta)] = I.$$

Another class of groups generalize two groups proposed by H. Heineken; these have their origin in *variety of groups* problems. We define

$$\text{Hein}(n,k) =< x, y \mid y^n = 1,\ [x^y, kx] = x^{y^{-1}} > .$$

Clearly, Hein (n,k) has as a quotient group the cyclic group of order n, by setting $x = e$.

It is known that Hein $(3,1)$ is cyclic of order three, whereas the finiteness of Hein(3,2) is still open. Finite quotients of the latter group were investigated by J. Neubüser, and its $SL(2,-)$ representation by Schönert and Sidki (see [8]).

The group theory software SPAS [9] provides a "deep view" of the group H_1 = Hein$(0,1)$. It has a unique normal subgroup N such that $H_1/N \cong SL(2,5)$. In addition, N/N' is isomorphic to

$$\mathbf{Z}^4 \times \mathbf{Z}_2^4 \times \mathbf{Z}_4 \times \mathbf{Z}_8^4 \times \mathbf{Z}_{13}^6.$$

3. The Computational Approach

We define the ring $R = \mathbf{Z}[s, s^{-1}, x_{ij}, y_{ij}]$,

$$\lambda : x \to (x_{ij})(:= X), \qquad y \to (y_{ij})(:= Y).$$

Then, λ extends to a $PGL(2,-)$ representation when the equation

$$\omega(X,Y) = (\omega_{ij}) = sI$$

is satisfied. That is, if and only if, s, x_{ij}, y_{ij} are solutions of the system of equations

$$s = \omega_{11}, \quad \omega_{11} = \omega_{22}, \qquad \omega_{12} = \omega_{21} = 0;$$

The trivial representation corresponds to the solution

$$s = 1, \quad x_{ii} = y_{ii} = 1 \quad (i = 1,2), \quad x_{ij} = y_{ij} = 0 \text{ for } i \neq j.$$

On considering $\mathcal{T}$, the ideal of R generated by

$$\{s - \omega_{11}, \quad \omega_{11} - \omega_{12}, \quad \omega_{12}, \quad \omega_{21}\}$$

the problem translates immediately to that of understanding the quotient ring $\bar{R} = R/\mathcal{T}$. On defining

$$\bar{s} = \mathcal{T} + s, \quad \bar{x}_{ij} = \mathcal{T} + x_{ij}, \quad \bar{y}_{ij} = \mathcal{T} + y_{ij}, \quad \bar{X} = (\bar{x}_{ij}), \bar{Y} = (\bar{y}_{ij})$$

the representation

$$\lambda : G \to PGL(2, \bar{R})$$

where $x \to \bar{x}, \; y \to \bar{y}$ becomes the "universal" PGL_2-representation of G.

In order to diminish the number of variables, we look for solutions in algebraicly closed fields K, without previously fixing such fields. This restriction eliminates the Jacobson radical of $\bar{R}$, and thus also the corresponding normal nilpotent subgroup

$$\lambda(G) \cap (I + \mathcal{M}_{2\times 2}(J(\bar{R}))) \quad \text{(mod. scalars)}.$$

The restriction allows us to find simple representatives of the conjugacy classes of the pairs (X, Y),

$$\{(c_1 X^{\mathcal{U}}, c_2 Y^{\mathcal{U}}) \mid \mathcal{U} \in GL(2,K), \quad c_1, c_2 \in K^{\#}\}.$$

Such representatives (X_0, Y_0) can be chosen to satisfy

$$X_0 \in \left\{ \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\},$$

$$Y_0 \in \{Y^{\mathcal{U}} \mid \det(Y) = 1,\ \mathcal{U} \in GL_2(K),\ \mathcal{U}X_0 = X_0\mathcal{U}\}.$$

In addition, since we are interested in non-solvable quotients of G, the condition $y_{12} \neq 0$ should hold. Therefore the representatives are simply

$$X_0 = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} y_{11} & 1 \\ y_{11}y_{22} - 1 & y_{22} \end{pmatrix}, \quad \text{with } y_{11}y_{22} \neq 1,$$

or

$$X_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} y_{11} & y_{12} \\ y_{12}^{-1} & 0 \end{pmatrix}.$$

4. Solutions of two matrix equations

With the purpose of dealing with the group equations discussed above we have found *all* solutions of the following system

$$[X, Z] = rI, \quad [W, X]^Y = sZ \qquad (r, s \in K^{\#}).$$

We consider equivalent the solutions

$$(r, s, X, Z, Y, W), \quad (r, ss_2^{-1}, s_1X, s_2Z\ \ Y^S S_1, S_2W^S)$$

where $s_1, s_2 \in K^{\#}$, $[S_1, Z] = I$ (that is, $S_1 \in C(Z)$),

$$S_2 \in C(X), \qquad S \in C < X, Z > .$$

With respect to this equivalence, the solutions fall into eight families as is seen in the next table.

	r	s	X	Z	Y	W	S_1	S_2	S
(1)	1	1	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}$ $\zeta = \epsilon^2(1-\omega^2)$	$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & \alpha_1 \end{pmatrix}$	$\begin{pmatrix} \alpha_2 & 0 \\ \beta_2 & \alpha_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$
(2)	1	−1	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}$ $\zeta = -4/\epsilon_{12}^2$	$\begin{pmatrix} 0 & \epsilon_{12} \\ -\epsilon_{12}^{-1} & \epsilon_{22} \end{pmatrix}$	$\begin{pmatrix} \omega_{11} & 2 \\ 1/2 & 0 \end{pmatrix}$ $\omega_{11} = -(1+2\epsilon_{22}/\epsilon_{12})$	"	"	"
(3)	−1		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \zeta \\ 1 & 0 \end{pmatrix}$ $\zeta = -1/s^2$	$\begin{pmatrix} 1 & 0 \\ 0 & \omega/s \end{pmatrix}$	$\begin{pmatrix} \omega & 1 \\ -\frac{1}{2} & \frac{1}{2\omega} \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1\zeta^{-1} & \alpha_1 \end{pmatrix}$	$\begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2' \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(4)	1		$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ $\xi = 1/s$	$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}$ $\zeta = 1/s^2$	$\begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} -(s+1)\epsilon & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1' \end{pmatrix}$	$\begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2' \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$
(5)	1		$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ $\xi = 1/s$	"	$\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & \frac{s+1}{s}\epsilon \end{pmatrix}$	"	"	"
(6)	1		$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ $\xi = s$	"	$\begin{pmatrix} 0 & 1 \\ 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} -\frac{(s+1)}{s}\epsilon & 1 \\ -1 & 0 \end{pmatrix}$	"	"	"
(7)	1		$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ $\xi = s$	"	$\begin{pmatrix} \epsilon & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & \epsilon(s+1) \end{pmatrix}$	"	"	"
(8)	1		$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ $\xi \neq s, s^{-1}$	"	$\begin{pmatrix} \epsilon & 1 \\ 1 & \frac{s-\xi}{\epsilon(s\xi-1)} \end{pmatrix}$	$\begin{matrix} \frac{s-\xi}{\epsilon\xi(1-s)} & 1 \\ \frac{-\xi(s-1)^2}{s(\xi-1)^2} & \frac{\epsilon\xi(s-1)(s\xi-1)}{s(\xi-1)^2} \end{matrix}$	"	"	"

5. Representations of Hein (n,k)

Recall that Hein $(n,k) =< x,y \mid y^n = e, \quad [x^y, kx] = x^{y^{-1}} >$. Define $H_k =$ Hein $(0,k)$.

5.1. The group H_1. The polynomial necessary and sufficient condition is

$$f(s) = \frac{s^5+1}{s+1} = 0.$$

This leads to

$$o(X) = 10, \qquad Y^5 \text{ is scalar},$$

$$\lambda(H_1) \cong PSL(2,5).$$

5.2. The group H_2. The polynomial condition is

$$f(s) = f_1(s)f_2(s) = 0,$$

where

$$f_1(s) = \frac{s^5+1}{s+1}, \quad f_2(s) = s^6 - 2s^5 + 2s^4 - 3s^3 + 2s^2 - 2s + 1.$$

(2.1) $f_1 = 0 \Rightarrow$

$$o(X) = 10, \qquad Y^3 \text{ is scalar}$$

$$\lambda(H_2) \cong PSL(2,5).$$

(2.2) $f_2 = 0 \Rightarrow$

$$\lambda(H_2) \text{ is an infinite group}.$$

On imposing finiteness on the order of $\bar{Y}$ $(= Y$ modulo scalars), the first values which give nonsolvable groups are

(2.2.1) $o(\bar{Y}) = 6 \Rightarrow \text{char}(K) = 7, \quad s^2 - 3s + 1 = 0, \quad \lambda(H_2) \cong PSL(2,7^2)$,

(2.2.2) $o(\bar{Y}) = 11 \Rightarrow \text{char}(K) = 43, \quad s^2 - 9s + 1 = 0, \quad \lambda(H_2) \cong PSL(2,43^2)$, (2.2.3) $o(\bar{Y}) = 13 \Rightarrow \mathrm{char}(K) = 307, \quad s^2 + 39s + 1 = 0, \quad \lambda(H_2) \cong PSL(2, 307^2)$,

5.3. The group H_3. The polynomial condition is

$$f(s) = f_1(s)f_2(s) = 0$$

where

$$f_1(s) = \frac{s^5+1}{s+1},$$

$$f_2(s) = s^{18} - 7s^{17} + 24s^{16} - 58s^{15} + 113s^{14} - 187s^{13} + 272s^{12}$$

$$-352s^{11} + 409s^{10} - 431s^9 + 409s^8 - 352s^7 + 272s^6$$

$$-187s^5 + 113s^4 - 58s^3 + 24s^2 - 7s + 1.$$

Again, $f_2(s) = 0$ gives us an infinite group. On imposing finiteness on $o(\bar{Y})$, the first values have the following implications

(5.3.1) $o(\bar{Y}) = 3 \Rightarrow \mathrm{char}(K) = 2,$ or 239

(5.3.2) $o(\bar{Y}) = 4 \Rightarrow \mathrm{char}(K) = 41893.$

6. Representations of $\mathcal{VE}$

Define the *verbal embedding* group

$$\mathcal{VE} =< x, y \mid [x^{y^{-1}}, x^y] = x, \quad [x^y, x] = e > .$$

6.1. Let φ be a projective representation of $\mathcal{VE}$ into $PGL(2, K)$,

$$\varphi : x \mapsto \bar{X}, \quad y \mapsto \bar{Y},$$

such that the image is nonsolvable.

We invert our previous strategy by concentrating on the normal form of Y first and then simplify that of X.

As $< x^{y^{-1}}, x, x^y >$ is a nilpotent class 2 group, we conclude that $o(\bar{X}) = 2$, $x_{22} = -x_{11}$.

6.1.1. Let $Y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The solution is

$$\mathrm{char}(K) = 17, \qquad X = \begin{pmatrix} 0 & 4/3 \\ 2/3 & 0 \end{pmatrix},$$

$$< \bar{X}, \bar{Y} > \cong PSL(2, 17).$$

6.1.2. Let $Y = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}$. We have the solution

$$X = \begin{pmatrix} \frac{(1+\eta)(1+\eta^2)}{1-\eta^3} (= x_{11}) & 1 \\ -2\frac{(1+\eta)^2(1+\eta^2)\eta}{(1-\eta^3)^2} & -x_{11} \end{pmatrix}$$

where η is a root of the polynomial

$$p(t) = 2(t+1)^2(t^2+1) + t^2$$

(irreducible over $\mathbf{Q}$) and K is such that

$$2\eta(1+\eta)(1+\eta^2)(1-\eta^3) \neq 0.$$

We note that for η to be invertible, $\mathrm{char}(K) \neq 2$. Also, for $1-\eta$ to be invertible, $\mathrm{char}(K) \neq 17$.

Let $D = \mathbf{Z}[\frac{1}{2}, \frac{1}{17}]\ [t] / (p(t))$ Then, in D,

$$t,\ t+1,\ t-1,\ t^2+1,\ t^2+t+1.$$

are invertible. Thus, the above the representation is realizable over D. Also, since $p(t)$ is not a cyclotomic polynomial, t has infinite order in D. Therefore,

$$< \bar{X}, \bar{Y} > \leq\ PGL(2, D)$$

is an infinite group.

6.2. It turns out that the projective representation in 6.1.2 is a special case of the 3-dimensional representations φ into $SL(3,K)$

$$y \to \begin{pmatrix} \eta_1 & & \\ & \eta_2 & \\ & & 1 \end{pmatrix} (= Y),$$

$$x \to (x_{ij}) \ (= X),$$

such that $X^2 = I$. Here the solution is equivalent to one where

$$x_{11} = \frac{\eta_2 + \eta_1}{\eta_2 - \eta_1}, \quad x_{12} = -2\,\frac{\eta_2}{\eta_2 - \eta_1}\,\frac{\eta_1 + 1}{\eta_2 - 1}, \quad x_{13} = 1,$$

$$x_{21} = x_{11} + 1, \quad x_{22} = x_{12} - 1, \quad x_{23} = 1,$$

$$x_{31} = -(x_{11} + x_{12} - 1)(x_{11} + 1), \quad x_{32} = (x_{11} + x_{12} - 1)x_{12}, \quad x_{33} = -(x_{11} + x_{12}),$$

(η_1, η_2) a solution of

$$p(t_1, t_2) = 2(t_1 + 1)(t_2 + 1)(t_1 + t_2) + t_1 t_2 = 0,$$

and K is such that

$$2\eta_1\eta_2(\eta_1 - 1)(\eta_2 - 1)(\eta_1 - \eta_2) \neq 0.$$

The coincidence with the $PGL(2,K)$ representation happens when $t_2 = \dfrac{1}{t_1}$.

Apparently, making $o(Y) = m$ finite, forces $c(=$ characteristic $(K))$ to be positive. Indeed the first nontrivial cases are

m	6	7	8	9	10	11	12	13	14
c	7	3	5,7	17	3,31	43	37	3,79,131	13,41,127

For the first two groups we have the epimorphisms

$$\mathcal{VE}(2,6) \to PGL(2,7), \qquad \mathcal{VE}(2,7) \to \mathcal{U}(3,3) \times C_7,$$

and from [2,3] we know that the kernel of the first is trivial, whereas it is infinite for the second.

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