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RIEMANN SURFACES, ABELIAN VARIETIES AND AUTOMORPHISMS

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1. Introduction

This a brief version of a survey conference delivered at the XII Escola de Algebra. The main objective was to give an overview of results obtained by the Chilean researchers in Complex Geometry (V. González, G. Riera and the author) in the area of Riemann surfaces and principally polarized abelian varieties with automorphisms.

2. Riemann Surfaces with Automorphisms

Let M denote a compact Riemann surface of genus g with (sufficiently large) automorphism group G. In this section we indicate how to uniformize the surface M by a hyperbolic polygon adapted to the action of G. The approach is via the uniformizations of M and M/G.

Let k be a natural number greater than one and let $n_1, n_2, \ldots, n_{k+1}$ be natural numbers such that $n_1 \leq n_2 \leq \ldots \leq n_{k+1}$ and such that

$$\sum_{j=1}^{k+1}(1 - 1/n_j) > 2 \ .$$

Let M be a compact Riemann surface of genus g ($g \geq 2$) and let G be the group of automorphisms of M. Assume the signature of the covering $M \longrightarrow M/G$ is $(0; n_1, n_2, \ldots, n_{k+1})$, with k and $n_1, n_2, \ldots, n_{k+1}$ satisfying the

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above conditions (the conditions prevent the Fuchsian group uniformizing M/G from being elementary). In this case we will say that G is a *large* group of automorphisms.

It is known that then G has a presentation of the form

$$P = \langle a_1, a_2, \ldots, a_k; a_1^{n_1} = \ldots = a_k^{n_k} = 1 = (\prod_{j=1}^{k} a_j)^{n_{k+1}} = R_1 = \ldots = R_s \rangle , \quad (1)$$

where the order of each a_j is n_j, the order of the product $(\prod_{j=1}^{k} a_j)$ is n_{k+1}, the R_j are some extra relators, and

$$2g - 2 = |G|[-2 + \sum_{j=1}^{k+1}(1 - 1/n_j)] . \quad (2)$$

Let Δ denote the unit disc in $\mathbf{C}$.

Theorem 1. *Under the above conditions, let T denote the Fuchsian group acting on Δ and uniformizing the Riemann sphere $\hat{\mathbf{C}}$ with signature $(0; n_1, \ldots, n_{k+1})$, and let Γ denote the torsion-free Fuchsian normal subgroup of T uniformizing M.*

Then there exist

i) *a fundamental domain F for T which is a strictly convex hyperbolic polygon of the form: $2k$ consecutive sides of lengths $c_1, c_1, c_2, c_2, \ldots, c_k, c_k$ and corresponding interior angles $2\pi/n_1, \theta_1, 2\pi/n_2, \theta_2, \ldots, 2\pi/n_k, \theta_k$, where $\sum_{j=1}^{k} \theta_j = 2\pi/n_{k+1}$, with consecutive sides paired by generators of T, and*

ii) *a fundamental domain D for Γ which is composed of $|G|$ copies of F and whose side-pairing transformations are given by the extra relators R_j of the presentation of G.*

2.1. Homology Bases adapted to Group Actions

In order to compute the Riemann matrices of surfaces admitting automorphisms, we need a basis for $H_1(M, \mathbf{Z})$ and the induced action of G on it.

In this section, we give a general technique to find a basis for $H_1(M, \mathbf{Z})$ on which the group G acts nicely. In the next section, we will use this type of basis to compute the Riemann period matrix for M.

With the notation of Section 2, an especially suited homology basis (not necessarily canonical) can be computed from the fundamental polygon D for the Fuchsian group Γ uniformizing M, as follows:

Let $U = \{P_1, P_2, \ldots . P_v\}$ be the set of points on $M = \Delta/\Gamma$ corresponding to the vertices of D, and let $\beta_1, \beta_2, \ldots, \beta_N$ be the curves on M corresponding to the sides of D. Considering the exact sequence

$$0 \longrightarrow H_1(M) \xrightarrow{i} H_1(M, U) \xrightarrow{\delta} H_0(U) \xrightarrow{s} H_0(M) \longrightarrow 0 ,$$

where i is the inclusion, and δ and s are defined by

$$\delta(\gamma) = \gamma(1) - \gamma(0) \ , \ s(\sum_{k=1}^{v} p_k P_k) = \sum_{k=1}^{v} p_k \ ,$$

we conclude that $2g + v = N + 1$; hence kerδ has rank $2g$ and we can find a basis $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}$ for $H_1(M, \mathbf{Z})$, where each α_j is an integral combination of the β_j's ; that is, of the sides of D.

The advantage of this basis α is that its intersection matrix C and the action of the generators of G on α can be computed easily from the polygon D, thus obtaining a $2g \times 2g$ integral matrix A for each generator of G.

3. Riemann Matrices of Riemann Surfaces with Automorphisms

The standard theory for the Jacobian of a Riemann surface involves considering a canonical basis of homology and the dual basis of abelian differentials.

H. Weyl [9] and, later, C. L. Siegel [7] found a different construction which, though equivalent to this standard theory, is more suitable for the study of automorphisms of surfaces. Since this presentation is not as well known, we outline the main ideas.

Let $\alpha_1, \ldots, \alpha_{2g}$ be any basis of closed curves for $H_1(M, \mathbf{Z})$. The intersection product defines a skew-symmetric integral valued matrix, $C = (c_{jk})$, whose entries are the intersection numbers

$$c_{jk} = -(\alpha_j \cdot \alpha_k) \ . \tag{3}$$

A basis for $H^{1,0}(M, \mathbf{Z})$ as a <u>real</u> vector space, $d\omega_1, \ldots, d\omega_{2g}$, is said to be dual to the given basis of curves if

$$\Re \int_{\alpha_k} d\omega_j = c_{jk} \ . \tag{4}$$

Riemann's bilinear relations imply that the real matrix $S = (s_{jk})$ with entries

$$\Im \int_{\alpha_k} d\omega_j = s_{jk} \tag{5}$$

is positive definite and symmetric.

Multiplication by $\sqrt{-1}$ in the real vector space of holomorphic differentials is represented, in terms of the basis $(d\omega_j)$, by a $2g \times 2g$ real matrix R such that $R^2 = -I$, and, in addition, one has the fundamental relation

$$R = C^{-1} S \ . \tag{6}$$

In other words, in the vector space $\mathbf{R}^{2g}$ the matrix R defines a complex structure and the matrices C and S define a hermitian product via the formula

$$\langle x, y \rangle = x' S y - i\, x' C y \ ,$$

where ' denotes the transpose.

In this context, any automorphism of M gives rise to a $2g \times 2g$ integral matrix A (corresponding to the action induced by the automorphism on the given homology basis) such that $AR = RA$ and $A^{-1} = C^{-1} A' C$.

For the case of a canonical homology basis, the relation between the standard complex theory (with normalized period matrix $Z = X + iY$, $Z' = Z$, $\Im Z > 0$) and this real theory (with "period matrix" the pair (R, C)) is given as follows:

$$C = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \tag{7}$$

$$R = \begin{pmatrix} XY^{-1} & Y + XY^{-1}X \\ -Y^{-1} & -Y^{-1}X \end{pmatrix} \tag{8}$$

$$Z = R_{21}^{-1}(R_{22} - iI) \ . \tag{9}$$

Now we have all the ingredients for our

Algorithm. *In order to compute the period matrix of a Riemann surface M with large group of automorphisms G,*

i) *Find a suitable presentation for G,*

ii) *Find an adapted fundamental domain D for G (as in Theorem 1),*

iii) *Find an adapted homology basis from D (as in Section 2.1.),*

iv) *Find the intersection matrix C from D,*

v) *Find the action of the generators of G on the adapted basis,*

vi) *Find R commuting with the generators, and such that $R^2 = -I$ and CR is positive definite.*

4. Abelian Varieties with Automorphisms

In order to generalize the above results to principally polarized abelian varieties, we need

Definition 1. *Let R be a real $2g \times 2g$ matrix such that $R^2 = -I$ and let C be an integral skew-symmetric $2g \times 2g$ matrix of determinant one such that CR is* *positive definite. Then the triple* $(\mathbf{R}^{2g}/\mathbf{Z}^{2g}, R, C)$ *is called a (real) principally polarized abelian variety of dimension* g.

An automorphism of a (real) principally polarized abelian variety of dimension g *is given by a* $2g \times 2g$ *integral matrix* A *such that* $AR = RA$ *and* $A^{-1} = C^{-1}A'C$.

Remark. There is a bijection between the real principally polarized abelian varieties with $C = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ and the complex principally polarized abelian varieties, given by: to the real p.p.a.v. $(\mathbf{R}^{2g}/\mathbf{Z}^{2g}, R, C)$ we associate the complex p.p.a.v. given by the complex vector space $\mathbf{R}^{2g}$ with complex structure given by R, the lattice generated by the column vectors of I and Z, with Z as in (9) and the hermitian product $\langle x, y\rangle = x'CRy - i\, x'Cy$.

We finish this presentation with two examples that (hopefully) illustrates the theory.

Example 1. *Bring's Curve (c.f. [4].)*

It is known that there is a unique curve of genus four, Bring's curve, admitting the symmetric group S_5 as a group of automorphisms (see Figure 1).

One can use the methods given in this paper to find the action induced by S_5 on a canonical basis of homology; in this way one obtains a faithful representation of S_5 in the symplectic group $Sp(8, \mathbf{Z})$, with image SpS_5.

One can also embed $\Gamma_0(5)$ in $Sp(8, \mathbf{Z})$ (with image $Sp\Gamma_0(5)$) by sending $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$, where $m, n, p, q \in \mathbf{Z}$, $mq - pn = 1$, and $5/p$ to $\begin{pmatrix} mI & nc \\ pc^{-1} & qI \end{pmatrix}$,

where $c = \begin{pmatrix} 4 & 1 & -1 & 1 \\ 1 & 4 & 1 & -1 \\ -1 & 1 & 4 & 1 \\ 1 & -1 & 1 & 4 \end{pmatrix}$.

By studying the action of SpS_5 and of $Sp\Gamma_0(5)$ on the Siegel space of dimension four, we obtain

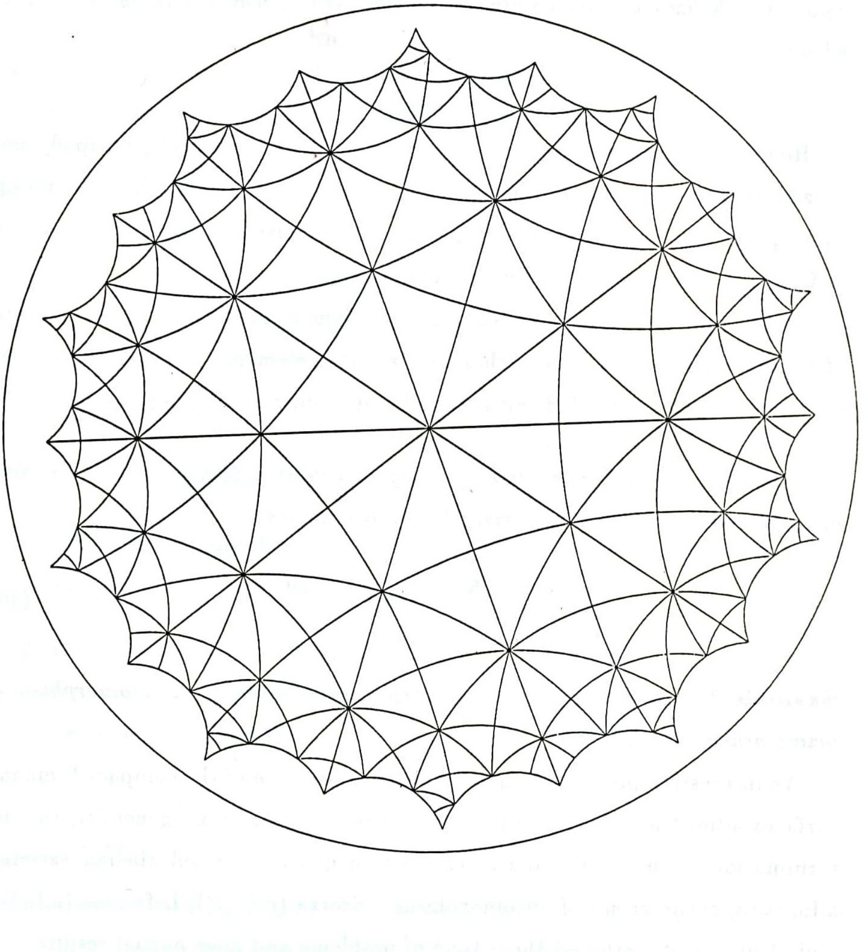

Figure 1: Bring's curve

Theorem 2. *There is a one complex parameter family – Bring's half-plane – of Riemann matrices Z in Siegel's upper half space fixed under the action of* SpS_5 *. These are the matrices of the form* $Z = \tau c$, $\tau \in \mathbf{C}, \Im\tau > 0$.

The stabilizer of Bring's half plane in $Sp(8, \mathbf{Z})$ *is the product of the groups* SpS_5*, which fixes the set pointwise, and* $Sp\Gamma_0(5)$ *, which acts by* $\tau c \rightarrow \tau^* c$*, where*

$$\tau^* = \frac{q\tau + n}{p\tau + m} \quad .$$

Hence, we obtain a pencil in the moduli space of (complex) principally polarized abelian varieties of dimension four, all of whose points admit the group S_5 as group of automorphisms; the pencil is uniformized by $\mathfrak{H}/\ \Gamma_0(5)$, where $\mathfrak{H}$ $= \mathbf{C}$, $\Im\tau > 0$ denotes the upper half plane.

At this point, we also know that only one element in this pencil is a Jacobian of a curve; the next theorem characterizes this element, and it is proven by studying the quotients of Bring's curve by appropriate subgroups of S_5.

Theorem 3. *The period matrix of Bring's curve is* $Z = \tau_0 \cdot c$*, where the class of* τ_0 *modulo* $\Gamma_0(5)$ *is characterized by the two values*

$$j(\tau_0) = -\frac{25}{2}, \quad j(5\tau_0) = -\frac{29^3 \times 5}{2^5}. \tag{10}$$

Example 2. *Riemann surfaces and Abelian varieties with an automorphism of prime order* (c.f. [5]).

An interesting problem is that of the determination of the compact Riemann surfaces admitting cyclic group of automorphisms, or, more generally, the determination of the (different classes of) principally polarized abelian varieties admitting cyclic group of automorphisms. Scorza (c.f. [6]), Lefschetz (c.f. [1]) and Weil (c.f. [8]) studied these type of problems and gave partial results.

A complete solution is given as follows:

Theorem 4. *Let* p *denote a prime number,* p *at least five. Let* L *denote the number of conformally different Riemann surfaces of genus* $g = \frac{(p-1)}{2}$ *which* *admit a cyclic group of automorphisms of order p. Let W denote the number of nonequivalent principally polarized abelian varieties of genus* $g = \frac{(p-1)}{2}$ *admitting a complex multiplication of order p. Then*

$$L = \begin{cases} (p+1)/6 & \text{if } p \equiv 2 \pmod 3 \\ (p+5)/6 & \text{if } p \equiv 1 \pmod 3 \end{cases} \quad \text{and} \quad W = h(p-1)^{-1} \sum_{\substack{d/p-1 \\ d \text{ odd}}} \phi(d)\, 2^{(p-1)/2d}$$

where h is the ideal class number and ϕ *is Euler's totient function.*

This theorem is proved using the methods described in this work. In fact, one can be very explicit and give the uniformizations of the corresponding curves, as well as their period matrices and algebraic curves (c.f. [5]).

Since for $p \geq 11$ we have $W > L$, it appears that even in these discrete cases there is a Schottky type problem: find equations in theta nulls vanishing only at Jacobian varieties of curves; this we leave as an open question.

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