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A CROSSED EMBEDDING OF GROUPS AND THE COMPUTATION OF CERTAIN INVARIANTS OF FINITE SOLVABLE GROUPS

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The aim of this note is to announce some results relating a group theoretical construction with others, eventually coming from topological needs, which involve relations among commutators.

Let us first consider the following elementary situation resulting in the dihedral group of order eight. Suppose we are given two copies $\langle a \,|\, a^2 = 1 \rangle$ and $\langle b \,|\, b^2 = 1 \rangle$ of the cyclic group C_2 and let us introduce in the free product $C_2 * C_2$ the only condition that a commutator $[a, b]$ ($= a^{-1}b^{-1}ab$) is central. This is equivalent, in the present situation, to require the relations

$$[a, b]^a = [a^a, b^b] = [a, b]^b,$$

so that if N denotes the normal closure in $C_2 * C_2$ of all expressions $[a, b]^a \cdot [a^a, b^b]^{-1}$ and $[a, b]^b \cdot [a^a, b^b]^{-1}$, then obviously we have $C_2 * C_2/N \cong D_4$, the dihedral group of order 8, i.e.

$$D_4 \cong \langle\, a, b \mid a^2 = 1,\ b^2 = 1,\ [a, b]^a = [a^a, b^b] = [a, b]^b \,\rangle.$$

The idea is to generalize the above construction by taking for point of departure two copies of an arbitrary finite solvable group G. It is well known that such a group has a subnormal series $G = G_o \rhd G_1 \rhd \ldots \rhd G_i \rhd \ldots \rhd G_n = \{1\}$ with G_{i-1}/G_i cyclic, say of order r_i for $1 \leq i \leq n$. Thus from a generator a_n of the cyclic group $G_{n-1} \cong G_{n-1}/G_n$, we can find a system of generators

$\{a_1, \dots, a_n\}$ of G $(= G_o)$ satisfying the following relations

$$(I) \begin{cases} a_i^{r_i} = w_{ii}(a_{i+1}, \dots, a_n), & 1 \leq i \leq n-1; \\ a_n^{r_n} = 1; \\ a_i^{a_j} = w_{ij}(a_{j+1}, \dots, a_n), & 1 \leq j < i \leq n. \end{cases}$$

A n-tuple $(a_1, \dots, a_n)$ in the above conditions is called an *AG-system*[1] for G and relations (I) are called *power-conjugates relations*, to which we refer in the sequel as $G(a)$-relations to make mention to the generators $a_{i's}$ of G $(= G(a))$.

Now given a finite solvable group $G(a)$ with an AG-system $(a_1, \dots, a_n)$, we can take an isomorphic copy $G(b)$ of G with a corresponding AG-system $(b_1, \dots, b_n)$, where $a_i \leftrightarrow b_i$, $1 \leq i \leq n$, and define the presentation

$$\delta(G) := \langle\, a_1, \dots, a_n, b_1, \dots, b_n \mid G(a)-\text{relations}, G(b)-\text{relations}, [a_i, b_j]^{a_k} = [a_i^{a_k}, b_j^{b_k}] = [a_i, b_j]^{b_k}, 1 \leq i, j, k \leq n \,\rangle.$$

We thus have

Theorem 1. (cf. [4, 5])

(i) *$\delta(G)$ defines a finite solvable group such that $l(G) \leq l(\delta(G)) \leq l(G) + 1$ (here $l(G)$ stands for the derived length of G);*

(ii) *If G is a nilpotent group, of class c, then $\delta(G)$ is nilpotent, of class $c+1$;*

(iii) *If G is a p-group with $|G| = p^n$ and $|G'| = p^m$ then $|\delta(G)|$ divides $p^{n^2+2n-mn}$.*

A more general construction in this line is easely obtained by considering two arbitrary isomorphic distinct groups G, G^φ, where φ stands for an isomorphism $\varphi : g \mapsto g^\varphi$, $\forall\, g \in G$, and by defining

$$\mathcal{V}(G) := \langle\, G, G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^{k^\varphi}, \forall\, g, h, k \in G \rangle,$$

i.e. $\mathcal{V}(G)$ is the quotient of the free product $G * G^\varphi$ by its normal subgroup generated by all expressions $[g, h^\varphi]^k \cdot [g^k, (h^k)^\varphi]^{-1}$ and $[g, h^\varphi]^{k^\varphi} \cdot [g^k, (h^k)^\varphi]^{-1}$ for

[1] Jürgensen [2] used the terminology *AG-system* to designate that we are dealling with a system for solvable groups (*Auflösbare Gruppen*)

g, h, $k \in G$. In this more general context we have also

Theorem 2. (cf. [4]).

(i) *If G is a finite group (resp. solvable of derived length l, resp. nilpotent of class c), then $\mathcal{V}(G)$ is finite (resp. solvable of derived length at most $l+1$, resp. nilpotent of class $c+1$);*

(ii) *If G is a p-group with $|G| = p^n$ and $|G'| = p^m$ then $|\mathcal{V}(G)|$ divides $p^{n^2+2n-mn}$.*

In spite of its eventual intrinsic interest, one may as well ask whether such a construction brings along any special feature. The answer is yes, certainly. The first aim in introducing those defining relations for $\mathcal{V}(G)$ was a tentative to find an inherent representation of the *non-abelian tensor square* $G \otimes G$ of a group G, a construction which has certain interest in topology (cf. Brown/Loday [1]), by commutators. Fortunately we hit home: the subgroup $\Upsilon(G) := [G, G^{\varphi}]$ of $\mathcal{V}(G)$ is isomorphic to $G \otimes G$ (cf. Rocco [4]).

Owing computations the relationship between results of Theorems 1 and 2 above is desirable. In fact we have it:

Theorem 3. (cf. [5]). *If G is a finite solvable group with an AG-system $(a_1, \ldots, a_n)$ and G^{φ} has an AG-system $(b_1, \ldots, b_n)$ where $\varphi : a_i \mapsto b_i$, $1 \leq i \leq n$, then $\delta(G)$ is a presentation for $\mathcal{V}(G)$. Furthermore the subgroup $\Upsilon(G)\,(= [G, G^{\varphi}]\,)$ is generated by $\{[a_i, b_j] \mid 1 \leq i, j \leq n\}$.*

Remark 1. Given any two groups G, H, one say that H is a crossed G-module is there exist

(a) a homomorphism $\tau : G \to Aut(H)$, $g \mapsto \tau_g : H \to H$, $h \mapsto h^g$ (which fixes an action of G on H) and

(b) a homomorphism $* : H \to G$,

such that:

1) $(h_1)^{h^*} = h_1^h \,(= h^{-1}h_1h\,),\ \forall\, h,\ h_1 \in H\,;$

2) $(h^g)^* = (h^*)^g \,(= g^{-1}h^*g\,),\ \forall\, g \in G,\ \ \forall\, h\ \in H.$

It then follows from the relations of $\mathcal{V}(G)$ that $\Upsilon(G)$ is a crossed G-module, where the action of G on $\Upsilon(G)$ is given by $[g_1, g_2^{\varphi}]^g := [g_1^g, (g_2^g)^{\varphi}]$ and $*$: $\Upsilon(G) \to G$ is the *derived map* $[g_1, g_2^{\varphi}] \mapsto [g_1, g_2]$, $\forall\, g_1,\ g_2,\ g\ \in G$ (cf. Rocco [4], Lemma 2.1).

Remark 2. For any group G there exists an epimorphism $\rho\ :\ \mathcal{V}(G) \to G$ such that $g \mapsto g,\ g^{\varphi} \mapsto g,\ \forall\, g\ \in\ G$, the kernel of which being the subgroup $\Theta(G) = \langle\, g^{-1}g^{\varphi} \mid g\ \in\ G\,\rangle$. Restriction of ρ to the subgroup $\Upsilon(G)$ matches the derived map $*$. We thus get by Theorem 1 the following well known result for finite solvable groups:

Corollary 1. *If G is a finite solvable group given by an AG-system $(a_1, \ldots, a_n)$, the derived group G' is generated by $\{[a_i,\ a_j] \mid 1\ \le\ i\ \le\ n\,\}$*

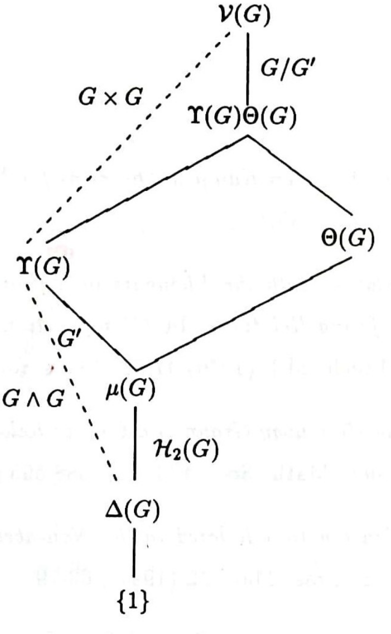
The subgroup $\mu(G) := \Upsilon(G) \cap \Theta(G)$, i.e. the kernel of the derived map $*$, is central in $\mathcal{V}(G)$ and obviously contains the subgroup $\Delta(G) := \langle\, [g, g^{\varphi}] \mid g \in G$. It holds

Proposition 1. Let G be any group. Then

(i) $\Upsilon(G)/\Delta(G) \cong G \wedge G$, the *exterior square of* G (cf. Miller [3]);

(ii) $\mu(G)/\Delta(G) \cong \mathcal{H}_2(G)$, the *second homology group* of G (cf. Rocco, [4,5]).

The diagram below summarizes some of these aspects of the structure of $\mathcal{V}(G)$.



As one sees by Proposition 1, the role of $\Delta(G)$ should require some control of it. So we can assert

Proposition 2 (cf. [5]). *Let G be any group and suppose $X = \{x_i \mid i \in I\}$ is a set of generators of G* (we take I to be a totally ordered set). *Then $\Delta(G)$ is generated by the set*

$$C := \{[x_i, x_i^{\varphi}] \mid i \in I\} \cup \{[x_i, x_j^{\varphi}] \cdot [x_j, x_i^{\varphi}] \mid i, j \in I, i < j\}.$$

The Theorem 3 and the Propositions 1 and 2 are results which also provide us with a procedure to perform computer assisted calculations with $\mathcal{V}(G)$ and those relevant invariants $G \otimes G$, $G \wedge G$ and $\mathcal{H}_2(G)$ of a finite solvable group G given by an AG-system. In particular, making use of an implementation of the *nilpotent quotient algorithm* and the facilities of the system *GAP* [6] we used this approach to construct all those such groups (giving power-commutator presentations of them as well) having as arguments G the non-abelian p-groups of order $\leq p^4$, $p = 2, 3$ (cf. [5]).

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