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RINGS WITH POLYNOMIAL IDENTITIES AN ELEMENTARY INTRODUCTION

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Abstract

These notes form an elementary introduction to the theory of rings satisfying a polynomial identity. No background in non-commutative ring theory is assumed; the only prerequisite is familiarity with the fundamental concepts of commutative algebra.

Contents.

Rings satisfying a polynomial identity, PI-rings for short, play a prominent role among the non-commutative rings, since they are "non-commutative enough" to exhibit interesting, non-commutative behavior, but still not too far away from commutative rings, so that they have a very rich structure theory. An indication of the close connection between PI-rings and commutative rings is a theorem of AMITSUR (3.7 below) which says that the "good" PI-rings are obtained as subrings of matrix rings over commutative rings: To be precise, if R is a PI-ring without nilpotent ideals, then there is a commutative ring C and some integer n such that R embeds into the $n \times n$-matrices over C.

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These notes were written as a supplement to a series of lectures given at the XII Brazilian Algebra Meeting, held during the first week of August of 1992 in Diamantina, Minas Gerais, Brazil. The intended audience were algebraists not familiar with non-commutative ring theory. In fact, the only prerequisite required was a good understanding of the fundamental concepts of commutative algebra. This has consequences for the presentation of the material: For example, the concept of primitivity is not treated in the main body of the text, only in an appendix to Section 2. Thus we state KAPLANSKY's theorem in Section 3 only for simple rings, relegating its actual statement and a detailed outline of its proof to the appendix of that section. My reasoning here is that after all, one of the statements in KAPLANSKY's theorem is that the primitive PI-rings are just the simple PI-rings, making it possible (on an elementary level) to study PI-rings without the concept of primitivity. In another direction, the concept of AZUMAYA algebra is not being covered, resulting in the omission of the important theorem of ARTIN and PROCESI. Many other interesting topics had to be excluded, for example chain conditions for PI-rings, and SCHELTER's work about affine PI-rings. Also, proofs are often only sketched and sometimes completely omitted. These remarks make it clear that after reading these notes, anyone interested in PI-theory should consult one of the excellent texts on PI-rings listed in the bibliography (see also the remarks preceding the bibliography).

No attempt has been made to give a history of the theory of PI-rings, although I have tried to give credit where due. I owe much to the textbooks and survey articles from which I learned the material, and a knowledgeable reader will doubtlessly note the connections.

Finally, I would like to thank the organizers of the XII Brazilian Algebra Meeting, in particular Michel Spira and Dan Avritzer, for their support, and for making this conference a great success.

Conventions. All rings have a unit element 1. If S is a subring of a ring R, then R and S have the same unit element. Homomorphisms of rings preserve unit elements. Ideals are always two-sided ideals (i.e., closed under multiplication by elements of the ring both from the left and from the right). All modules are left modules: If M is a left R-module, we let R act on M from the left, i.e., the product of $r \in R$ and $x \in M$ is rx, not xr.

1. Definitions and Examples

1.1. Denote by $F_\infty = \mathbf{Z}\{X_1, X_2, \ldots\}$ the free polynomial ring in the countably many non-commuting variables X_1, X_2, ... with coefficients in $\mathbf{Z}$. We assume that the coefficients commute with the variables, but that the variables do not commute among themselves. Thus $X_1 3 = 3X_1$, but $X_1X_2X_1 \neq (X_1)^2X_2$.

Definition 1.2 Let R be a ring, and let $f(X_1, \ldots, X_n) \in F_\infty$ be a polynomial such that at least one of its monomials of highest degree has ± 1 as a coefficient. If $f(r_1, \ldots, r_n) = 0$ for all choices of $r_1, \ldots, r_n \in R$, then f is called an *identity* or *polynomial identity* of R. If R has a polynomial identity, then R is called a *PI-ring*.

1.3. The definition of a polynomial identity is somewhat technical in order to avoid certain trivialities. For example, every non-commutative algebra over a field of characteristic two satisfies the polynomial $2X_1$. In order to exclude such trivial identities, one has to require that at least one of the monomials occuring in a polynomial identity has ± 1 as a coefficient. When working with a polynomial identity, it is sometimes necessary to work with monomials of highest degree; for that reason it is useful to require that some monomial of highest degree has coefficient ± 1.

1.4. Nearly every text on PI-rings has a different definition of what is a polynomial identity. But in the end, this does not make any difference: Any reasonable definition of PI-ring implies that every PI-ring satisfies some power of the "standard polynomial"; we will come back to this in 1.7 below.

Now let us look at some examples.

1.5. *Commutative rings are PI-rings:* They satisfy the identity $XY - YX = 0$.

1.6. *Subrings and homomorphic images of PI-rings are PI-rings.* This follows immediately from the definition.

1.7. *Any ring which is a finite module over its center is a PI-ring.* In fact, if a ring R is generated as a module over its center C by $n-1$ elements, then R satisfies the *n-th standard polynomial*

$$S_n(X_1,\ldots,X_n) = \sum_{\pi} \mathrm{sign}(\pi) X_{\pi(1)} \cdots X_{\pi(n)},$$

where π runs over all permutations in the n-th symmetric group. The identity in 1.5 is $S_2(X,Y)$. Note that S_n is multilinear and alternating in n variables. Thus $S_n(x_1,\ldots,x_n) = 0$ if two of the x_i are equal.

Now say that $R = \sum_{i=1}^{n-1} Cr_i$ for some generators $r_i \in R$. Let $x_1,\ldots,x_n$ be n arbitrary elements of R. Then $x_j = \sum c_{i,j} r_i$ for $c_{i,j} \in C$. Because of the multilinearity of S_n, $S_n(x_1,\ldots,x_n)$ can be written as a sum with coefficients in C of evaluations of S_n of the form $S_n(r_{i_1},\ldots,r_{i_n})$. Since there are only $n-1$ distinct generators r_i, two of the r_{i_j} must be equal. Thus $S_n(r_{i_1},\ldots,r_{i_n}) = 0$. Consequently, R satisfies S_n.

As we have just seen, any ring which is a finite module over its center satisfies some standard polynomial. This is not true for all PI-rings, but nearly so: AMITSUR showed that every PI-ring satisfies some power of some standard polynomial, i.e., it satisfies $(S_n(X_1,\ldots,X_n))^m$ for some integers $m, n \geq 1$ (see [C$_2$, §12.6, Exercise (4)]).

1.8. *If C is any commutative ring, then $M_n(C)$, the ring of $n \times n$-matrices over C, is a PI-ring.* In fact, it satisfies S_{n^2+1}, since $M_n(C)$ is a free module of rank n^2 over its center C. The famous theorem of AMITSUR and LEVITZKI says that $M_n(C)$ satisfies actually S_{2n} (see [C$_2$, §12.5, Theorem 11]).

Matrix rings will be the most important PI-rings for our purposes since any "good" PI-ring can be embedded into a matrix ring over some commutative ring. To be precise, "good" means any PI-ring which does not have non-zero nilpotent ideals. We will outline a proof of this important result of AMITSUR at the end of Section 3 (Theorem 3.7).

1.9. It is easy to see that $M_n(C)$ does not satisfy S_k for $k < 2n$. Denote by $e_{i,j}$ the *matrix unit* with a 1 in the i,j-entry and zeroes everywhere else. Then $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$. Consider the first k elements of the $2n-1$ matrix units

$$e_{1,1},\ e_{1,2},\ e_{2,2},\ e_{2,3},\ e_{3,3},\ldots,\ e_{n-1,n-1},\ e_{n-1,n},\ e_{n,n}\,.$$

Multiplied together in the order in which they appear in this list, their product is some non-zero matrix unit $e_{1,j}$, but their product in any other order is zero. Thus the evaluation of S_k at these k matrix units is $e_{1,j} \neq 0$. Consequently, $M_n(C)$ does not satisfy S_k for $k < 2n$.

Actually, the only property of S_k which we used in the previous paragraph is that S_k is a multilinear polynomial of degree k. A slight modification of the argument shows that $M_n(C)$ does not satisfy any multilinear polynomial of degree $< 2n$. We will use this fact later on repeatedly.

1.10. *Any ring R which is integral of bounded degree n over its center C is a PI-ring.* Every $x \in R$ satisfies by assumption a monic integral equation of degree $\leq n$ over C. Multiplying such an equation by a suitable power of x, we may assume that x satisfies an equation of the form

$$x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n = 0 \qquad (c_i \in C)\,.$$

Given any $y \in R$, we can form the commutator of y with this equation and obtain

$$[x^n, y] + c_1[x^{n-1}, y] + \cdots + c_{n-1}[x, y] = 0\,.$$

(Here for any elements a and b, their commutator is $[a,b] = ab - ba$.) It follows that the n commutators $[x^n, y]$, $[x^{n-1}, y]$, ..., $[x, y]$ are linearly dependent over C, the center of R. Since the standard polynomial S_n is multilinear and alternating, it follows that

$$S_n([X^n, Y], [X^{n-1}, Y], \ldots, [X, Y])$$

is an identity for R.

1.11. *Tensor products of PI-rings are PI-rings.* To be precise, the tensor product should be taken over a central subring. This is a deep theorem of

REGEV (see [R_1, Theorem 6.1.1]). In the special case that one of the two rings is commutative, the proof of this result is much easier: It is then an easy consequence of the following lemma which we will also use frequently in other situations.

Lemma 1.12 *Let R be a PI-ring. Then R satisfies some multilinear polynomial identity.*

Sketch of the Proof Say R satisfies the polynomial identity $f(X_1, \ldots, X_n)$. We will "multilinearize" f to obtain a multilinear identity whose degree is less or equal to the degree of f. If there is some monomial of f in which X_1 occurs more than once, replace f by $g(X_1, \ldots, X_{n+1}) = f(X_1 + X_{n+1}, X_2, \ldots, X_n) - f(X_1, X_2, \ldots, X_n) - f(X_{n+1}, X_2, \ldots, X_n)$. One can convince oneself that the total degree of g is less or equal to the total degree of f, that every coefficient of g is a coefficient of f (up to sign), that g is non-zero, that g is a polynomial identity of R, and that X_1 and X_{n+1} occur in g with smaller degree than the degree of X_1 in f. Repeating this process, one arrives at a polynomial identity $h(X_1, \ldots, X_r)$ in which every X_i occurs in every monomial at most once. Fix a monomial m of h of highest degree and with coefficient ± 1. Say $X_1, \ldots, X_s$ occur in m, and $X_{s+1}, \ldots, X_r$ do not occur in m. Replacing $h(X_1, \ldots, X_r)$ by $h(X_1, \ldots, X_s, 0, \ldots, 0)$, we may assume that all X_i occur in m. Replacing h by $h(X_1, X_2, \ldots, X_r) - h(0, X_2, \ldots, X_r)$, we may assume that X_1 occurs in every monomial of h exactly once. Repeating this step also for the other X_i, we finally obtain a polynomial identity for R in which every variable occurs in every monomial exactly once. Such a polynomial identity is multilinear. □

Corollary 1.13 *Let R be a PI-ring. Let C be a central subring of R, and let L be a commutative ring containing C. Then $R \otimes_C L$ is a PI-ring. In fact, $R \otimes_C L$ satisfies every multilinear polynomial satisfied by R.*

Proof: By the lemma, R satisfies some multilinear polynomial identity, say $f(X_1, \ldots, X_n)$. Let $x_1, \ldots, x_n \in R \otimes_C L$. Write $x_i = \sum_j r_{i,j} \otimes l_{i,j}$ with $r_{i,j} \in R$

and $l_{i,j} \in L$. Then

$$f(x_1, \ldots, x_n) = \sum_{j_1, \ldots, j_n} f(r_{1,j_1}, \ldots, r_{n,j_n}) \otimes l_{1,j_1} \cdots l_{n,j_n} = 0\,. \qquad \square$$

Actually, this corollary holds in somewhat greater generality. Say S is an overring of a ring R such that $S = RL$ for some subring L of S which is contained in the center of S. Such a ring S is called a *central extension* of R. The argument used in the proof of the corollary shows that *a central extension of a PI-ring is a PI-ring.*

1.14. After having seen so many PI-rings, it is only fair to say that there are many more rings which are not PI-rings. Important examples of rings which do not satisfy polynomial identities are free algebras (like F_∞), the Weyl algebras $\mathbf{A}_n(\mathbf{C})$, and the enveloping algebras of most Lie algebras.

2. Some Elementary Notions from the Theory of Non-Commutative Rings

2.1. *Prime Rings and Ideals.* Before we go on, we have to introduce some terminology. Let R be a ring, and I an ideal of R. The ring R is *prime* if products of non-zero ideals of R are non-zero. Unlike for commutative rings, this does not imply that products of non-zero elements of R are non-zero, see 2.4. In case R has no zero-divisors, R is called a *domain*. This latter notion, however, is too strong for our purposes; too many interesting non-commutative rings are prime but not domains. In non-commutative ring theory, prime rings play the main role, not domains like in commutative algebra.

There is a characterization of prime rings using elements: One can check that R is prime iff whenever $aRb = 0$ for some elements $a, b \in R$, either $a = 0$ or $b = 0$. From this it follows that the center of a prime ring is a (commutative) domain. In fact, it follows that *non-zero central elements of a prime ring are not zero divisors*: If a is central and $ab = 0$, then $aRb = Rab = 0$, so that $b = 0$.

The ideal I of R is called *prime* if R/I is prime. Again, note that this does not imply that if I contains a product of two elements then I contains one of the factors. This is only true in the special case that R/I is a domain.

2.2. *Semiprime Rings and Ideals.* The ring R is called *semiprime* if it has no nilpotent ideals. (An ideal N of R is nilpotent if for some integer k, $N^k = 0$, i.e., for all $x_1, \ldots, x_k \in N$, the product $x_1 \cdots x_k$ is zero.) The ideal I is *semiprime* if R/I is semiprime. One can prove that R is semiprime iff the intersection of all prime ideals of R is zero (see [C_2, 12.2, Thm. 3]). On the level of ideals this means that I is semiprime iff I is the intersection of all prime ideals in which it is contained. Note that a commutative ring is semiprime iff it is reduced, i.e., iff it does not contain non-zero nilpotent elements.

2.3. *Simple Rings and Maximal Ideals.* The ring R is called *simple* if the only (two-sided) ideals of R are 0 and R itself. Note that a commutative simple ring is a field. An ideal I is called *maximal* if R/I is simple. Note that simple rings and maximal ideals are prime.

The center C of a simple ring R is a (commutative) field: Since R is prime, C is a domain. And if $0 \neq x \in C$, then $xR = RxR$ is a non-zero (two-sided) ideal of R, and thus equal to R. Consequently, x is invertible in R, and one checks easily that $x^{-1} \in C$. However, not all elements of R need be invertible, see 2.4.

Let us demonstrate these definitions with an example.

Example 2.4 *The maximal, prime and semiprime ideals of $M_n(C)$.*

Let C be a commutative ring, and let $R = M_n(C)$ be the ring of $n \times n$-matrices over C. By 1.8, R is a PI-ring. We view C as a subring of R via the embedding as scalar matrices. Note that C is the center of R. *The ideals of $R = M_n(C)$ are all of the form $I = M_n(I')$, where I' is an ideal of C.* In fact, $I' = I \cap C$. (Let I_0 be the subset of elements of C which are entries of matrices in I. Clearly $I \subseteq M_n(I_0)$. Using multiplication by matrix units (see 1.9), one checks that equality holds. From this one deduces that $I_0 = I \cap C$.)

It follows that R is simple iff its center C is simple, i.e., iff C is a field. If I and J are ideals of R, then $IJ = M_n(I \cap C) \cdot M_n(J \cap C) = M_n((I \cap C)(J \cap C))$. It follows that R is prime iff C is prime, i.e., iff C is a domain. And R is semiprime iff C is, i.e., iff C is reduced. Translating these observations to ideals, we see that the ideal I of R is maximal, prime, or semiprime iff the ideal $I \cap C$ of C is. Finally, note that if C is a field, then R is a simple (and hence prime) ring which contains zero-divisors (if $n \geq 2$). □

Appendix to Section 2: Primitive Rings

2.5. *Primitive Rings and Ideals.* The notion of primitivity is not as important in PI-theory as in general non-commutative ring theory, because the primitive PI-rings are just the simple PI-rings: This is one particular consequence of KAPLANSKY's theorem (3.8 below, not the elementary version 3.1). It is needed, however, if one wants to understand how PI-theory ties in with general ring theory, and in particular, if one wants to get a better understanding of Kaplansky's theorem and its proof. In this short appendix, we discuss primitive rings. In an appendix to the next section, we will then give a more complete proof of KAPLANSKY's theorem, using the material presented here.

A module is called *simple* or *irreducible* if it does not have proper non-zero submodules. A ring R is called *primitive* if it has a faithful simple module. And R is called *semiprimitive* if the intersection of all primitive ideals is zero. As usual, an ideal I of R is *(semi)primitive* if R/I is. A word of caution: In some of the literature, the term semisimple is used instead of semiprimitive.

Note that *a commutative primitive ring C is a field*: If x is a non-zero element of a faithful simple C-module M, then Cx is a non-zero submodule, so $Cx = M$. Suppose that for some $c \in C$, $cx = 0$. Since C is commutative, $0 = Ccx = cCx = cM$. Since M is faithful, this implies that $c = 0$. Thus $C \approx Cx = M$ as C-modules. Hence C is simple as a C-module, i.e., the only ideals of C are 0 and C itself. Consequently, C is a field.

It is easy to see that *primitive rings are prime*: Suppose that a primitive ring

R has ideals I and J such that $IJ = 0$. Let M be a faithful simple R-module. If $J \neq 0$, then $JM \neq 0$ since M is faithful. Thus $JM = M$ since M is simple. Then $IM = I(JM) = (IJ)M = 0 \cdot M = 0$, so that $I = 0$ since M is faithful.

Simple rings are primitive: If m is a maximal left ideal of a simple ring R, then $M = R/m$ is a non-zero simple (left) R-module. The set I of all elements of R which annihilate M is a two-sided ideal. Since $1 \notin I$, $I \neq R$. Thus $I = 0$, and M is a faithful R-module.

2.6. The following diagram summarizes the connections between primitivity and the notions defined earlier. For a ring R (or ideal I, respectively),

$$\begin{array}{ccccc} \text{simple (maximal)} & \Longrightarrow & \text{primitive} & \Longrightarrow & \text{prime} \\ & & \Downarrow & & \Downarrow \\ & & \text{semiprimitive} & \Longrightarrow & \text{semiprime} \end{array}$$

As already mentioned, primitive PI-rings are simple, see KAPLANSKY's theorem (3.8). But even for commutative rings (and thus also for PI-rings), all other implications are strict.

3. Kaplansky's Theorem

In this section, we outline the proof of KAPLANSKY's theorem:

Theorem 3.1 (Kaplansky; Elementary Version) *If R is a simple PI-ring, then $R \approx M_t(D)$ for some division algebra D which is a finite module over the center of R.*

This is an elementary version of KAPLANSKY's theorem. Using the material on primitive rings presented in the appendix to the previous section, we will study the "real" theorem of KAPLANSKY in an appendix of this section.

Note that a simple PI-ring is in particular Artinian (i.e, it satisfies the descending chain condition for left and right ideals) since it is a finite dimensional vector space over a field (namely its center).

Sketch of the Proof of Kaplansky's Theorem. Since R is a simple ring, an important result from non-commutative ring theory (the CHEVALLEY-JACOBSON density theorem) asserts the existence of a division ring D such that

R is either isomorphic to $M_t(D)$ for some natural number t, or for each natural number t there is a subring of R mapping homomorphically onto $M_t(D)$. Let us assume that the latter is true. We will show that this leads to a contradiction.

Denote by K the center of D. Note that R has also a subring mapping homomorphically onto $M_t(K)$. By Lemma 1.12, R satisfies a multilinear polynomial identity f. Also $M_t(K)$ satisfies f. It follows by 1.9 that the degree of f is greater or equal to $2t$. Since this is true for all natural numbers t, this is a contradiction. Hence $R \approx M_t(D)$.

It remains to be shown that D (and thus R) is a finite module over its center C. This is done by a second application of the CHEVALLEY-JACOBSON density theorem. Using the notion of primitivity, we will outline the argument in the appendix to this section. □

3.2. *Facts about Finite Dimensional Simple Algebras.* As KAPLANSKY's theorem indicates, simple rings which are finite dimensional over their centers play an important role in PI-theory. Such rings are often called finite dimensional simple rings (or algebras). Let us record some facts for later reference. Note first that if R is a simple ring which is finite dimensional over its center, then R is a PI-ring by 1.7.

Theorem 3.3 *Let R be a simple ring which is finite dimensional over its center C. Then the dimension of R over C is a perfect square, say $\dim_C R = n^2$. The integer n is called the degree of R. By KAPLANSKY's theorem, $R \approx M_t(D)$ for some division ring D which is finite dimensional over the center of R. Let L be a maximal subfield of D. Then L contains C, and $R \otimes_C L \approx M_n(L)$ is isomorphic to the ring of $n \times n$ matrices over L.*

Any field extension K of C such that $R \otimes_C K \approx M_n(K)$ is called a *splitting field* for R.

During the proof of KAPLANSKY's "real" theorem in the appendix of this section, we will see quite a bit of evidence that this theorem is true. Good references for a complete proof are [Fe] or [H].

We will use the following lemma frequently.

Lemma 3.4 *Let R be a simple PI-ring satisfying a multilinear polynomial identity of degree d. Then $d \geq 2n$, where n denotes the degree of R.*

Proof: Note that by KAPLANSKY's theorem 3.1, R is finite dimensional over its center C, so that Theorem 3.3 applies. Let L be a splitting field of R. Then $R \otimes_C L \approx M_n(L)$. By Corollary 1.13, also $M_n(L)$ satisfies f. But by 1.9, $M_n(L)$ does not satisfy any multilinear identity of degree $< 2n$. It follows that $d \geq 2n$. □

3.5. Using KAPLANSKY's theorem, we will now see that every semiprime PI-ring embeds into a matrix ring over a commutative ring, a fact which we already mentioned in 1.8. For this, we need a very useful theorem; for a proof, see [C_2, Prop. 8 in §12.5 and Cor. to Prop. 6 in §12.6].

Theorem 3.6 (Amitsur) *Let R be a semiprime PI-ring. Let t be a central indeterminate over R (i.e, t commutes with the elements of R). Then the intersection of all maximal ideals of the polynomial ring $R[t]$ is zero.*

Note that $R[t]$ is a PI-ring: Denoting by C the center or R, one sees that $R[t] \approx R \otimes_C C[t]$, so that Corollary 1.13 applies.

(Actually, the proof of this theorem consists of showing that $R[t]$ is semiprimitive, i.e, the intersection of all primitive ideals is zero. Using the "real" theorem of KAPLANSKY, it follows that the intersection of all maximal ideals of $R[t]$ is zero.)

Theorem 3.7 (Amitsur) *Let R be a semiprime PI-ring. Then for some integer n and some commutative ring C, R embeds into $M_n(C)$. Thus all semiprime PI-rings are obtained as subrings of matrix rings over commutative rings.*

Proof: The ring R embeds into the polynomial ring $R[t]$. By Theorem 3.6, the intersection of all maximal ideals of $R[t]$ is zero. Replacing R by the polynomial ring $R[t]$, we may thus assume that the intersection of all maximal ideals is zero.

Say R satisfies a multilinear polynomial identity f of degree d. Denote by $\{M_\alpha\}$ the family of maximal ideals of R. Fix a maximal ideal M_α. As seen in 3.3 and 3.4, the simple PI-ring R/M_α embeds into some matrix algebra $M_{n_\alpha}(C_\alpha)$, where C_α is some field and $d \geq 2n_\alpha$. Note that the kernel of the composition $R \to R/M_\alpha \subseteq M_{n_\alpha}(C_\alpha)$ is the ideal M_α. The maps $R \to M_{n_\alpha}(C_\alpha)$ induce a map $\Phi\colon R \to \prod_\alpha M_{n_\alpha}(C_\alpha)$. Since the kernel of $R \to M_{n_\alpha}(C_\alpha)$ is just M_α, and since the intersection of all maximal ideals is zero, it follows that Φ is injective.

Choose a maximal number n_0 among the n_α, and let $n = (n_0)!$. Note that if $m \mid n$, then $m \times m$-matrices embed "block-wise" into $n \times n$-matrices. For example, 2×2-matrices imbed into 4×4-matrices via the function

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

It follows that $M_{n_\alpha}(C_\alpha)$ embeds into $M_n(C_\alpha)$. Consequently R embeds into $\prod_\alpha M_n(C_\alpha) \approx M_n(\prod_\alpha C_\alpha) = M_n(C)$, where $C = \prod_\alpha C_\alpha$ is a commutative ring (actually, a product of (commutative) fields). □

Appendix to Section 3

Using the notion of primitivity (introduced in the appendix to Sectio 2), we now state and sketch the proof of

Theorem 3.8 (Kaplansky) *If R is a primitive PI-ring, then R is a simple algebra finite dimensional over its center. In fact, $R \approx M_t(D)$ for some division algebra D which is a finite module over the center of R.*

As seen in 2.5, simple rings are always primitive. One assertion of KAPLANSKY's theorem is that for PI-rings, the notions "primitive" and "simple" coincide. On the level of ideals, this means that the primitive ideals of a PI-ring are just the maximal ideals.

The proof of this result needs quite a bit of the general theory of non-commutative rings. But I included it in this appendix because it shows very nicely how the assumption that a ring satisfies a polynomial identity can be used to deduce something about the structure of R. I tried to present the proof in such a way that even a reader unfamiliar with non-commutative ring theory can understand it, if he (or she) is willing to take some of the background material for granted.

We have now to take a closer look at the CHEVALLEY-JACOBSON density theorem. It says that if R is a primitive ring and M a faithful irreducible R-module, then R is a "dense ring of linear transformations on M." This has the following consequence. Denote by D the ring of R-linear endomorphisms of M. By Schur's lemma, D is a division ring. One can deduce from the density theorem that R is either isomorphic to $M_t(D)$ for some natural number t, or for each natural number t there is a subring of R mapping homomorphically onto $M_t(D)$. These facts can be found, e.g., in [R_1, 1.5.2 and 1.5.5].

Sketch of the Proof of Kaplansky's Theorem. As in the proof of the elementary version of KAPLANSKY's theorem, one sees that $R \approx M_t(D)$, where $D = \mathrm{End}_R(M)$ is the ring of R-linear endomorphisms of the faithful simple R-module M. It is easy to check that a matrix ring over a division ring is simple (cf. 2.4). Thus R is simple. It remains to be shown that R is a finite module over its center C.

Denote by $\mathrm{End}(M)$ the set of $\mathbf{Z}$-linear endomorphisms of the faithful irreducible R-module M. Via multiplication, each element of R gives rise to a $\mathbf{Z}$-linear endomorphism of M. This induces a ring homomorphism $R \to \mathrm{End}(M)$. Since M is a faithful R-module, this map is injective, and we identify R with a subring of $\mathrm{End}(M)$. The division algebra D (which is by definition the set of R-linear endomorphisms of M) is also a subring of $\mathrm{End}(M)$.

Recall from 2.5 that the center C of the simple ring R is a field. Let L be a maximal subfield of D. Note that L contains C. Now consider the subring RL of $\mathrm{End}(M)$. Since RL contains R, M is also a faithful irreducible RL-module.

Thus RL is a primitive ring. What are the RL-linear endomorphisms of M? They certainly commute with R, so belong to D. But they also commute with L, which is a maximal subfield of D. Thus they must belong to L. So $\text{End}_{RL}(M) = L$. We now apply the density theorem to the faithful irreducible module M of the primitive ring RL, and deduce that either $RL \approx M_k(L)$ for some k, or that for each k, there is a subring of RL which maps homomorphically onto $M_k(L)$. Since RL is a homomorphic image of $R \otimes_C L$, it is a PI-ring by Corollary 1.13. The argument used in the sketch of the proof of the elementary version of KAPLANSKY's theorem shows that the second possibility cannot occur. Thus $RL \approx M_k(L)$ for some k.

Now a standard result from the theory of simple rings shows that $RL \approx R \otimes_C L$. (The reason is as follows: A theorem says that if one tensors a simple ring over its center with an extension field of the center, the tensor product is a simple ring. Thus $R \otimes_C L$ is a simple ring. The ring homomorphism from $R \otimes L$ onto RL given by multiplication is non-zero, so that its kernel is not the whole ring. Thus the kernel is zero, and the map is an isomorphism.)

It follows that $\dim_C R = \dim_L R \otimes_C L = \dim_L RL = \dim_L M_k(L) = k^2 < \infty$. This concludes the proof of the theorem. □

Note that modulo some background material, we also proved Theorem 3.3.

4. Central Polynomials and Posner's Theorem

We begin with a definition.

Definition 4.1 Let R be a PI-ring. Let $f(X_1, \ldots, X_n) \in F_\infty$ be a polynomial *without* constant term, and which is *not* a polynomial identity for R. If for all choices of $r_1, \ldots, r_n \in R$, $f(r_1, \ldots, r_n)$ is a central element of R, then f is called a *central polynomial* for R.

As in the definition of polynomial identity, the definition of central polynomial is somewhat technical in order to exclude certain trivial cases: If f is a polynomial identity of R, then both f and $1+f$ take on central values on R, but for trivial reasons. These polynomials would not give us any new information.

Example 4.2 *A central polynomial for $M_2(C)$.*

Let C be a commutative ring. If x and y are 2×2-matrices over C, then $xy - yx$ is a matrix of trace zero. A simple calculation shows that the square of a trace zero matrix is a scalar matrix, so belongs to the center of $M_2(C)$. Thus all evaluations of the polynomial $(XY - YX)^2$ lie in the center of $M_2(C)$. This polynomial does not have a constant term, and one checks easily that it does not vanish identically on $M_2(C)$. It follows that it is a central polynomial for $M_2(C)$. □

It is very difficult to exhibit other central polynomials. That they exist in abundance was only discovered in the early seventies. The first central polynomials for $n \times n$-matrices were found by FORMANEK and RAZMYSLOV. It is no exaggeration to say that central polynomials revolutionized PI-theory: They allowed the discovery of many new results, and also gave rise to new and better proofs for many of the earlier theorems about PI-rings.

Theorem 4.3 (Formanek, Razmyslov) *For each integer n, there exists a polynomial g_n which is a central polynomial for every ring of $n \times n$-matrices over a commutative ring. In fact, g_n can be chosen to be multilinear.*

The proof of this result is quite intricate and we omit it. One of the many references is, e.g., [C_2, §12.6].

The following useful observation is due to PROCESI.

Lemma 4.4 *A central polynomial for $n \times n$-matrices is a polynomial identity for matrices of smaller size.*

Proof: Let $k < n$, and let R be the set of all matrices in $M_n(C)$ whose last $n - k$ rows and columns are zero. We can identify R with $M_k(C)$. If g is a central polynomial for $M_n(C)$, then the values taken on by g on R are scalar matrices in $M_n(k)$, and these scalar matrices have to be zero since the last row and column of each element of R are zero. (Recall that the constant term of the central polynomial g is by definition zero.) Thus g vanishes identically on R. □

It is now fairly easy to deduce the existence of central polynomials for simple PI-rings.

Theorem 4.5 *Every simple PI-ring R admits a central polynomial. In fact, if the degree of R is n (see Theorem 3.3), then g_n is a central polynomial of R. Moreover, g_k is a polynomial identity for R for all $k > n$.*

Actually, also all prime PI-rings admit central polynomials: This is an easy consequence of POSNER's theorem (4.8 below). But we will see later on that not all PI-rings admit central polynomials (Example 4.7).

Proof: By KAPLANSKY's theorem, R is finite dimensional over its center C. Let L be a splitting field of R (which exists by Theorem 3.3). Then $R \otimes_C L \approx M_n(L)$. By 4.3 and 4.4, g_k is an identity for $M_n(L)$ and thus for R if $k > n$. Clearly, the evaluations of g_n are central in R, since they are even central in the bigger ring $M_n(L)$. It remains to show that g_n is not a polynomial identity of R. Suppose the contrary. Since g_n is multilinear, it is then by Corollary 1.13 also a polynomial identity of the central extension $M_n(L) \approx R \otimes_C L$. This is a contradiction, since g_n, being a central polynomial for $M_n(L)$, is not a polynomial identity for $M_n(L)$. □

We now see that the center of a semiprime PI-ring is very large.

Theorem 4.6 (Rowen) *Let R be a semiprime PI-ring. Then every non-zero (two-sided) ideal I of R has non-zero intersection with the center C of R, i.e., $I \cap C \neq 0$.*

As an immediate consequence of this theorem, note that a semiprime PI-ring whose center is a field is actually simple.

We remark that the proof of this theorem is very similar to the proof of AMITSUR's Theorem (3.7) which asserts that every semiprime ring embeds into a matrix ring over a commutative field.

Proof: Let t be a central indeterminate over R. Consider the polynomial ring $R[t]$. Since R is a semiprime PI-ring, Theorem 3.6 says that the intersection of all maximal ideals of $R[t]$ is zero. By Corollary 1.13, also $R[t]$ is a PI-ring. And if I is a non-zero ideal of R, then $I[t]$, the set of all polynomials with coefficients in I, is an ideal of $R[t]$. Suppose that $I[t]$ contains a polynomial which belongs to the center of $R[T]$. One checks easily that the coefficients of this polynomial are central in R. Thus I contains non-zero central elements of R. In order to prove the theorem, we may thus replace R by $R[t]$. Hence we may assume that the intersection of all maximal ideals of R is zero.

Now let I be a non-zero ideal of R. If M is a maximal ideal of R, then R/M is simple, so that the image of I in R/M is either zero or all of R/M. Since $I \neq 0$, the image of I in some R/M is non-zero. By Theorem 4.5, there is for each maximal ideal M an integer n_M such that g_{n_M} is a central polynomial for R/M. Using the fact that R satisfies some multilinear polynomial of some degree d, one checks that the integers n_M are bounded above: Also R/M satisfies this multilinear polynomial, so that $d \geq 2n_M$ by Lemma 3.4.

Choose among the n_M an integer n_0 maximal with respect to the property that the image of I in R/M is non-zero. Denote the corresponding maximal ideal by M_0. Now choose any elements in I such that the evaluation c of g_{n_0} at these elements has non-zero image in R/M_0. Clearly $c \neq 0$. Since g_{n_0} has zero constant term, c belongs to I. We will show that c is central in R.

Let M be a maximal ideal. We show first that the image $\bar{c}$ of c in R/M is central. If the image $\bar{I}$ of I is zero in R/M, then $\bar{c} = 0$, so that $\bar{c}$ is central. Now assume that $\bar{I} \neq 0$. Then by the maximality of n_0, g_{n_0} is either a central polynomial for R/M or (by Theorem 4.5) a polynomial identity for R/M. In either case, $\bar{c}$ is central in R/M.

Finally, let x be any element of R. Then modulo every maximal ideal M, $xc - cx$ is zero, since c is central in R/M. Thus $xc - cx$ belongs to every maximal ideal of R. Since the intersection over all maximal ideals is zero, it follows that $xc - cx = 0$, i.e., $xc = cx$. Thus c is central in R. □

The following example shows that not all PI-rings admit central polynomials, and that Rowen's theorem can fail for PI-rings which are not semiprime.

Example 4.7 *There is a PI-ring R which* (1) *has a non-zero ideal I whose intersection with the center of R is zero, and which* (2) *does not admit central polynomials.*

Let C be a commutative ring. Let R be the ring of upper triangular 2×2-matrices over C. One checks easily that the center of R consists of the scalar matrices and is thus isomorphic to C. Let I be the set of all matrices whose second row is zero. So

$$R = \begin{pmatrix} C & C \\ 0 & C \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} C & C \\ 0 & 0 \end{pmatrix}.$$

One verifies easily that I is an ideal of R. Since the zero matrix is the only scalar matrix in I, the intersection of I with the center or R is zero, proving (1). According to ROWEN's theorem, this means that R cannot be semiprime. And indeed, the set of all strictly upper triangular matrices

$$J = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$$

forms an ideal which is nilpotent: $J^2 = 0$.

Now suppose that f is a central polynomial for R. Consider the map $\varphi: C \to R$ given by $\varphi(a) = ae_{2,2}$. This map is a ring homomorphism which does not preserve 1: $\varphi(1_C) = e_{2,2} \neq 1_R$. Since the evaluations of f on $\varphi(C)$ are scalar matrices belonging to $\varphi(C)$, it follows that f vanishes on $\varphi(C)$. Thus f is a polynomial identity for $C \approx \varphi(C)$. Since $R/I \approx C$, it follows that all the evaluations of f on R belong to I. So the evaluations of f on R are scalar matrices belonging to I, i.e., they are zero. Consequently, f is a polynomial identity for R, in contradiction to f being a central polynomial for R. This proves (2). □

As a corollary to ROWEN's theorem, we obtain a strong form of POSNER's theorem.

Theorem 4.8 (Posner) *Let R be a prime PI-ring. Then the center C of R is a domain. Denote by S the set of non-zero elements of C. Then RS^{-1} is a simple Artinian PI-ring. In fact, $RS^{-1} \approx M_t(D)$ for some division ring D which is finite dimensional over its center.*

The ring RS^{-1} is called the *total ring of fractions of R*. Note that the center of D is equal to the center of RS^{-1}, which is nothing but CS^{-1}, the field of fractions of the center C of R.

Proof: As for localization in commutative algebra, one checks that the ideals of RS^{-1} are in one-to-one correspondence with those ideals of R which do not meet S. But by ROWEN's theorem, every nonzero ideal of R has non-zero intersection with C, i.e., meets S. Thus RS^{-1} is a simple ring. Again, as for localization in commutative algebra, $RS^{-1} \approx R \otimes_C CS^{-1}$. Thus RS^{-1} is a PI-ring by Corollary 1.13. So RS^{-1} is by KAPLANSKY's theorem isomorphic to $M_t(D)$ for some division ring D which is finite dimensional over its center. Consequently, RS^{-1} is a finite dimensional vector space over its center and therefore Artinian. □

5. Examples of Prime PI-Rings

It is time to look at some examples of prime PI-rings, and to review the results of the previous sections in some concrete situations.

The following lemma is a useful criterion to verify if a ring is prime.

Lemma 5.1 *Let R be a subring of a prime ring S. If R contains a non-zero two-sided ideal I of S, then R is prime.*

Proof: Say A and B are ideals of R such that $AB = 0$. Since $I = SIS \subseteq R$ and $AI \subseteq AR = A$, it follows that $ASISB = 0$ and thus $(SAS)I(SBS) = 0$. Hence one of the three ideals SAS, I, and SBS of the prime ring S must be zero. Since by assumption I is non-zero, either A or B must be zero. □

5.2. We know that every prime PI-ring R is a subring of a matrix ring $S = M_n(C)$ over a commutative ring C. We concentrate now on a special class of PI-rings, namely those "defined component-wise." To explain this, consider the matrix units $e_{i,j} \in S$. If $r \in R$ is a matrix with entries $r_{i,j}$, then $e_{i,i}re_{j,j} = r_{i,j}e_{i,j}$ is the matrix with the same entry as r in the i,j-position and zeroes elsewhere. Informally, we will call R *defined component-wise* iff R contains for every element r also all the $e_{i,i}re_{j,j} = r_{i,j}e_{i,j}$. Such rings are very easy to work with. And although they are rather special as far as general PI-rings are concerned, they can be used to demonstrate many properties of PI-rings. We already saw a component-wise defined ring in Example 4.7. Let us look at some more intricate examples.

Example 5.3 Let A be a commutative ring, and let $A[t]$ be the commutative polynomial ring in one variable over A. Denote by (t) the ideal of $A[t]$ generated by t. Let A_1 and A_2 be subrings of A. Let $S = M_2(A[t])$ be the 2×2-matrices over $A[t]$, and let

$$R = \begin{pmatrix} A_1 + (t) & A[t] \\ (t) & A_2 + (t) \end{pmatrix}.$$

Here R is defined component-wise: A matrix of S belongs to R iff its $1,1$-entry belongs to $A_1 + (t)$, its $1,2$-entry belongs to $A[t]$, etc. One checks easily that R is closed under matrix multiplication, and that R is an additive subgroup of S. Thus R is a ring. As a subring of the PI-ring S, R is also a PI-ring. Set $I = M_2((t)) = tS$. By 2.4, I is an ideal of S and S is prime. Since I is contained in R, it follows by Lemma 5.1 that R is prime. So R is a prime PI-ring.

What is the center C of R? One checks easily that a matrix in R which commutes with all other matrices in R has to be a scalar matrix. Thus C is the set of all scalar matrices in R, i.e., $C = (A_1 \cap A_2) + (t) = (A_1 \cap A_2) + tA[t]$. □

5.4. Let us verify ROWEN's and POSNER's theorems for this example. Let $x = (a_{i,j})$ be a non-zero element of a non-zero ideal J of R. Say $a_{1,2}$, the 1,2-entry of x, is non-zero. Then I contains the non-zero scalar matrix $e_{1,1}x(te_{2,1}) + (te_{2,1})xe_{2,2} = ta_{1,2}I_2$, where I_2 denotes the 2×2 identity matrix. So in this case I contains a non-zero central element. One argues similarly if some other entry of x is non-zero. So $I \cap C \neq 0$.

The field of fractions of the center C of R is nothing but the field of fractions of $A[t]$. Denote this field by K. Since $A_1K = A_2K = (t)K = K$, it follows that $RK = M_2(K)$. So the ring of central fractions of R is a matrix ring over a field. In particular, it is simple (see 2.4) and finite dimensional over its center (which is K).

5.5. Actually it is enough to invert a single element to make R into a full matrix ring over a commutative ring: The localization $R[t^{-1}] = M_2(A[t, t^{-1}])$ is the ring of 2×2 matrices over the Laurent polynomial ring $A[t, t^{-1}]$. In particular, $R[t^{-1}]$ is a finite module over its center.

This is special case of a general phenomenon: Whenever R is a prime PI-ring, there exist central elements $c \in R$ such that $R[c^{-1}]$ is a finite module over its center (and in fact an "AZUMAYA algebra"). Moreover, one can choose c to be any non-zero evaluation of a central polynomial of R. This is a particular consequence of the theorem of ARTIN and PROCESI, one of the most important in PI-theory. For a statement and proof of it, see, e.g., [C$_2$, §12.6].

5.6. Let us verify that for some evaluation c of a central polynomial, the ring $R[c^{-1}]$ is a full matrix ring. Recall from 4.2 that $(XY - YX)^2$ is a central polynomial for 2×2 matrices. Setting $X = te_{2,1}$ and $Y = e_{1,2}$, we see that $c = t^2 = t^2 I_2$ is an evaluation of a central polynomial for R. Note that $R[c^{-1}] = R[t^{-2}] = R[t^{-1}] = M_2(A[t, t^{-1}])$ is a full matrix ring over $A[t, t^{-1}]$.

But now an example which shows among other things that not every prime PI-ring is a finite module over its center:

Example 5.7 *A prime PI-ring R such that*

(1) *R is not a finite module over its center. (In fact, R is not even integral over its center.)*

(2) *R is a finitely generated algebra over a field. (That is, R is "affine".)*

(3) *R is neither left nor right Noetherian.*

(1) Let R be as in Example 5.3, and denote by C its center. As a C-module, R is the direct sum of the four C-modules $A_1 + (t)$, $A[t]$, $A_2 + (t)$ and (t). Thus R is a finite C-module only if $A[T]$ is a finite C-module. Since $C = (A_1 \cap A_2) + tA(t)$, this is only true if A is a finite $A_1 \cap A_2$-module. It is easy to arrange that the latter fails: For example, let $A = k[x, y]$ be a polynomial ring over a field, and let $A_1 = k[x]$ and $A_2 = k[y]$. Then $A = k[x, y]$ is not finite over $A_1 \cap A_2 = k$. Explicitly, the ring R is now of form

$$R = \begin{pmatrix} k[x] + tk[x,y,t] & k[x,y,t] \\ tk[x,y,t] & k[y] + tk[x,y,t] \end{pmatrix}.$$

Note that the subring of diagonal matrices is not integral over $C = k + tk[x, y, t]$ since both $A_1 = k[x]$ and $A_2 = k[y]$ are not integral over $A_1 \cap A_2 = k$. Thus R is not integral over C.

(2) One can verify that R is generated as k-algebra by the six matrices

$$e_{1,1},\ xe_{1,1},\ e_{1,2},\ te_{2,1},\ e_{2,2},\ \text{and}\ ye_{2,2}.$$

(3) Finally, let us check that R is not Noetherian. Let I be the ideal $M_2((t))$. Then

$$R/I = \begin{pmatrix} k[x] & k[x,y] \\ 0 & k[y] \end{pmatrix}.$$

We will show that R/I is not Noetherian, which implies that also R is not Noetherian. Let V be any $k[x]$-submodule of $k[x, y]$. Then $Ve_{1,2}$ is a left ideal of R/I. Since $k[x, y]$ is not a Noetherian $k[x]$-module, this shows that R/I does not satisfy the ascending chain condition on left ideals, i.e., R/I is not left Noetherian. A similar argument using $k[y]$-submodules of $k[x, y]$ shows that R/I is also not right Noetherian. □

Note that (2) and (3) say that HILBERT's basis theorem does not generalize to PI-algebras which are finitely generated over a central subfield.

5.8. So far, we have been somewhat sloppy about left and right chain conditions. The reason is that these notions coincide for "good" PI-rings: CAUCHON's theorem asserts that if a semiprime PI-ring satisfies the ascending chain condition (acc) on two-sided ideals, it also satisfies the acc on both left and right ideals (see [R_1, 5.1.8]). Since the acc on left (or right) ideals always also implies the acc on two-sided ideals, this means that a semiprime PI-ring is left Noetherian iff it is right Noetherian! By the way, a ring which satisfies both the acc on left and right ideals is simply called Noetherian.

6. The Trace Ring of a Prime PI-Ring

6.1. A very useful construction in PI-theory is the trace ring associated to a prime PI-ring. It was introduced by SCHELTER who used it successfully to prove many interesting facts about *affine* PI-algebras. An affine PI-algebra over a field k is a PI-ring containing k in its center and which is finitely generated over k as algebra. We will not pursue this interesting subject. A good introduction to affine PI-algebras is contained in SMALL's notes [Sm] which are unfortunately difficult to obtain. See also the relevant sections in [MR], [R_1], and [R_2].

6.2. Let us come back to the trace ring. Let R be a prime PI-ring. Denote by C the center of R and by K the field of fractions of C. By POSNER's theorem, the localization RK of R is a simple PI-ring which is finite dimensional over K. Let L be a splitting field of RK (which exists by Theorem 3.3). Then $RK \otimes_K L \approx M_n(L)$ for some integer n. Viewing an element x of R as an element of $M_n(L)$, it makes sense to talk about the characteristic polynomial of x. The coefficients of this characteristic polynomial belong to L. One can show that the coefficients belong actually to K, and that they are independent of the choice of the splitting field L of RK (see, e.g., [Pi, 16.1, Proposition]). Now denote by T the commutative C-algebra generated by all the coefficients of the characteristic polynomials of all elements of R. Note that every element of R

is by construction integral over T, since it satisfies its characteristic polynomial whose coefficients belong to T.

Definition 6.3 The commutative ring T is called the *commutative trace ring* of R, and the subring $R[T]$ of RK generated by R and T is called the *trace ring* of R.

6.4. We defined the trace ring by associating to each element of R an $n \times n$ matrix over some splitting field of RK, where n is the degree of RK. In some of the literature (e.g., [MR], [R_1], and [R_2]), the trace is defined using the "regular representation of RK on itself via (right or left) multiplication," associating to each element of R an $n^2 \times n^2$ matrix over K. The resulting trace ring is not always equal to our trace ring, but shares all the important properties of the trace ring which we will discuss below.

6.5. Why is $R[T]$ called the trace ring? If $x \in R$, then the trace of x is after all only one of the coefficients of the characteristic polynomial of x. So it might be more appropriate to call $R[T]$ the characteristic ring of R, or the characteristic closure of R. The reason for the name "trace ring" is that if R is an algebra over a field k of characteristic zero, then T is generated by the traces of the elements of R. This follows from NEWTON's formulae, see, e.g., [C_1, p.179].

The following theorem summarizes the most important properties of T and $R[T]$.

Theorem 6.6 *Let R be a prime PI-ring with trace ring $R[T]$.*

(a) *$R[T]$ is a prime PI-ring.*

(b) *$R[T]$ is integral over its central subring T.*

(c) *There is a non-zero ideal of $R[T]$ which is contained in R.*

(d) *If R is Noetherian, $R[T]$ is a finite R-module. In particular, $R[T]$ is also Noetherian.*

(e) *If R is a finitely generated algebra over a central subfield k, then also T and $R[T]$ are finitely generated algebras over k. Moreover, $R[T]$ is then a finite module over T. In particular, both T and $R[T]$ are Noetherian (although R need not be).*

Some Remarks about the Proof The proofs of most of these results about the trace ring are beyond the scope of these notes. Here are just a few easy observations.

(a) Since $R[T]$ is a homomorphic image of $R \otimes_C T$, it is a PI-ring by Corollary 1.13. If A and B are ideals of $R[T]$ such that $AB = 0$, then AK and BK are ideals of the simple ring RK whose product is zero. (AK is an ideal of RK since $AK = RARKK = (RK)A(RK)$.) Thus either A or B are zero. Consequently, $R[T]$ is prime.

(b) One might be tempted to say that $R[T]$, being generated over T by integral elements, is obviously integral over T. However, this reasoning is false for non-commutative rings. But in the special case at hand ($R[T]$ being a PI-ring generated over its central subring T by a subring, namely R, all elements of which are integral over T), SHIRSHOV's theorem (see [R_1, 4.2.9]) applies and allows the deduction that $R[T]$ is indeed integral over R.

(c) This is a result of SCHELTER.

(d) Using (c), this is easy. Denote the common non-zero ideal of R and $R[T]$ by I. By ROWEN's theorem, I contains a non-zero central element c of $R[T]$. Note that $R[T]c \subseteq I \subseteq R$. So $R[T]c$ is a Noetherian (left) R-module. Since central elements in a prime ring are not zero divisors (see 2.1), the map $R[T] \to R[T]c$ given by multiplication by c is injective. Thus it is an R-module isomorphism. Hence also $R[T]$ is a Noetherian R-module.

(e) That T is finitely generated over k is a result of PROCESI. The already mentioned theorem of SHIRSHOV assures that $R[T]$ is a finite T-module. The other statements are easy consequences of these facts. □

6.7. *How to compute the trace ring.* In general, this is quite difficult. But there is an easy special case. Assume that R is a prime PI-ring whose total ring of fractions is a matrix ring over a field. To be precise, if K denotes the field of fractions of the center of R, then we require that $RK = M_n(K)$ for some n. (In general, $RK \approx M_t(D)$ for some division ring D.) Note that the center of R is just the set of scalar matrices in $M_n(K)$ which are contained in R. If L is a splitting field of RK, then $RK \otimes_K L = M_n(K) \otimes_K L \approx M_n(L)$. Here the isomorphism is induced by the inclusion of $M_n(K)$ into $M_n(L)$. So computing the characteristic polynomial of an element of R in $M_n(L)$ is the same as computing it in $RK = M_n(K)$.

If we assume additionally that K is a field of characteristic zero, then T is generated over the center C of R by all the traces of the elements of R, computed in $RK = M_n(K)$ (see 6.5). And this is quite easy if R is defined component-wise.

Let us look at the trace rings for the examples of Section 3.5.

6.8. Continuation of Example 5.3. Using the notation of that example, let

$$R = \begin{pmatrix} A_1 + (t) & A[t] \\ (t) & A_2 + (t) \end{pmatrix}.$$

Recall that the center C of R is $(A_1 \cap A_2) + (t)$. One verifies easily that $\text{trace}(R) = A_1 + A_2 + (t)$, and that $\det(R) = A_1A_2 + (t)$. Denote by B the subalgebra of A generated by A_1 and A_2. Then the commutative trace ring T of R is $B + (t) = B + tA[t]$. Consequently, the trace ring of R is

$$R[T] = \begin{pmatrix} B + (t) & A[t] \\ (t) & B + (t) \end{pmatrix}. \qquad \square$$

6.9. Continuation of Example 5.7. There we looked at the more special case that A is a commutative polynomial ring over a field k in two variables x and y, and $A_1 = k[x]$ and $A_2 = k[y]$. Now $B = A$, so that $T = k[x, y, t]$, and

$$R[T] = \begin{pmatrix} k[x,y,t] & k[x,y,t] \\ tk[x,y,t] & k[x,y,t] \end{pmatrix}. \qquad \square$$

6.10. Let us verify the results of Theorem 6.6 for these rings. First, let R and $R[T]$ be as in Example 6.8.

(a) is easy to verify.

(b) One readily checks that $B+(t)$, the center of $R[T]$, contains the traces and determinants of all elements of $R[T]$. So each matrix of $R[T]$ satisfies a monic equation over $T = B + (t)$, namely its characteristic polynomial. Thus $R[T]$ is integral over T.

(c) Note that $M_2((t))$ is both an ideal of R and $R[T]$.

(d) Without some special assumptions on A and the A_i, R is not Noetherian, as we saw in Example 5.7.

(e) Let us look at the situation of Example 6.9. We saw already in Example 5.7 that R is here finitely generated over k. Clearly $T = k[x, y, z]$ is a finitely generated k-algebra, and it is Noetherian. Since $R[T]$ is a finitely generated T-module with basis $e_{1,1}$, $e_{1,2}$, $te_{2,1}$, and $e_{2,2}$, it is a finitely generated k-algebra and a Noetherian T-module. In particular, $R[T]$ is Noetherian. □

It is worth noting that in the situation of Example 6.9, the prime PI-ring R is not Noetherian (as we saw in Example 5.7), although T and $R[T]$ are.

7. An Application of the Trace Ring to Non-Commutative Invariant Theory

7.1. Throughout this section, k denotes an algebraically closed field. For simplicity, we assume that $\operatorname{char} k = 0$.

In this last section, we will demonstrate the power of the trace ring construction by proving a theorem from non-commutative invariant theory. The trace ring has been used successfully to prove many results; we mentioned already SCHELTER's original applications to finitely generated PI-algebras. The particular choice of topic chosen here to exemplify the usefulness of the trace ring is due to the personal taste of the author.

7.2. *Commutative Invariant Theory.* Let us recall a famous theorem of HILBERT from commutative invariant theory. Let k be an algebraically closed field, and let R be a finitely generated commutative k-algebra. Let G be a linearly reductive group acting rationally on R by algebra automorphisms. (We will define these terms below.) Denote by R^G the fixed ring of R, i.e., the set of all elements of R invariant under the action of G. HILBERT's theorem asserts that R^G is a finitely generated k-algebra. In this section we will use the trace ring to prove a generalization of this result to certain PI-algebras.

7.3. We begin with some background material. A reader might skip the following definition and think of a "rational action of a linearly reductive group" simply as a particularly nice linear action of either $\mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$. Good references for commutative invariant theory are [Fog], [K], and [Sp].

Definition 7.4 A *linear algebraic group* G (over k) is a subgroup of some $\mathrm{GL}_n(k)$ which is closed in the ZARISKI topology. That means that G is defined as the set of zeroes of some regular functions on $\mathrm{GL}_n(k)$. Prime examples are $\mathrm{GL}_n(k)$ itself and $\mathrm{SL}_n(k)$.

Let W be a finite dimensional k-vector space. A linear representation (or action) of G on W is called *rational* if the corresponding group homomorphism $G \to \mathrm{GL}(W)$ is a morphism of affine algebraic varieties. And a linear action of G on an infinite dimensional vector space V is *rational* if V is a union of finite dimensional, G-stable subspaces W such that the induced action of G on each of these subspaces W is rational. A good example of a rational action is the action of $\mathrm{GL}_2(k)$ on the commutative polynomial ring $k[x, y]$ given by "change of coordinates."

Finally, a linear algebraic group G is called *linearly reductive* if every rational representation of G is *completely reducible*. The latter means that if V is a k-vector space with a rational G-action, then V is a direct sum of subspaces which are irreducible representations of G. Examples of linearly reductive groups are $\mathbf{G}_m = k^*$ (the multiplicative group of k), $\mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k)$. (The latter two groups are not linearly reductive in prime characteristic.)

Now let G be a linearly reductive group acting rationally on a (possibly non-commutative) k-algebra R. We will use the linear reductivity of G via the following lemma. The proof of this lemma is an easy application of the elementary properties of the so-called REYNOLDS operator (see $[V_1$,4.1]).

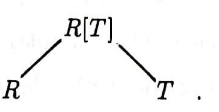
Lemma 7.5 *Let G be a linearly reductive group acting rationally on a (non-commutative) k-algebra R. Suppose that S is a G-stable subalgebra of R such that R is a Noetherian left S-module. Then R^G is a Noetherian left S^G-module. As a particular consequence, if R is (left) Noetherian, also R^G is (left) Noetherian.*

We are now ready to state the generalization of HILBERT's theorem to PI-algebras (see $[V_1,4.4]$).

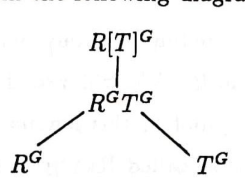
Theorem 7.6 *Let k be an algebraically closed field, and let R be a finitely generated (left) Noetherian PI-algebra over k. Let G be a linearly reductive group acting rationally on R by algebra automorphisms. Then also the fixed ring R^G is a finitely generated (left) Noetherian k-algebra.*

That R^G is (left) Noetherian is a trivial consequence of Lemma 7.5.

Idea of the Proof The proof begins with a series of reduction steps (which we omit) to the case that R is a prime ring. So assume that R is prime. Denote by T the commutative trace ring of R and by $R[T]$ the trace ring of R. Since R is Noetherian, $R[T]$ is a Noetherian R-module (Theorem 6.6(d)). And since R is a finitely generated k-algebra, both T and $R[T]$ are finitely generated over k and Noetherian, and $R[T]$ is a finite (and so Noetherian) T-module (Theorem 6.6(e)). One can check that the action of G on R extends to an action on $R[T]$ under which T is stable. Moreover, this action is rational ($[V_1, 3.4]$). So G acts rationally on the ring extensions



Taking fixed points, we obtain the following diagram:



Here R^GT^G is the subalgebra of $R[T]^G$ generated by R^G and T^G. Since $R[T]$ is a Noetherian module over both R and T, Lemma 7.5 implies that $R[T]^G$ is a Noetherian module over both R^G and T^G.

Since T is a finitely generated k-algebra, its fixed ring T^G is finitely generated by HILBERT's theorem. Now consider the subring R^GT^G of $R[T]^G$. Since $R[T]^G$ is a finite T^G-module, R^GT^G is a finite T^G-module. In particular, R^GT^G is Noetherian, and is finitely generated over k. Since $R[T]^G$ is finite over R^G, R^GT^G is also finite over R^G.

To summarize, we showed that R^GT^G is a finitely generated Noetherian k-algebra which is a finite module over R^G. In fact, R^GT^G is generated over R^G by central elements (namely T^G). In this situation one can use a non-commutative version (due to MONTGOMERY and SMALL [MS]) of the so-called ARTIN-TATE lemma from commutative algebra (see, e.g., [AM, 7.8]) to conclude that R^G is also finitely generated over k. □

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Since these notes form a rather elementary introduction to the theory of PI-rings, these references are only meant to help the beginning reader to pursue the subject further. More detailed lists of references concerning the theory of PI-rings can be found in [R_1] and [For].

The monographs of Formanek [For], Jacobson [J], Procesi [Pr], Rowen [R_1], and Small [Sm] deal exclusively with PI-rings. The standard reference for PI-rings is

Rowen's book [R_1]. Procesi's book [Pr] is very valuable although it essentially predates central polynomials. Good survey articles are Formanek's lectures [For] and Small's notes [Sm] (the latter are unfortunately difficult to obtain). An excellent introduction to PI-rings which does not assume much background in non-commutative ring theory is contained in the last chapter of Cohn's textbook [C_2]. A student with more knowledge of ring theory might prefer the more advanced introductions contained in the ring theory books by McConnell and Robson [MR] and Rowen [R_2]. The classic of Herstein [H] and Felzenszwalb's book [Fe] are good references for the facts about finite dimensional simple algebras which we used without proof.

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