Matemática Contemporânea, Vol. 7, 1–17
http://doi.org/10.21711/231766361994/rmc71

HILBERT POLYNOMIALS OF SMOOTH SURFACES IN P^4. COMMENTS.

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1 Introduction.

Consider C a smooth projective curve. It can always be embedded in $\mathbf{P}_3$ (the projective complex space of dimension 3). More generally, a smooth variety of dimension n can always be embedded in $\mathbf{P}_{2n+1}$, via any embedding and general projections.

The smooth curves which can be embedded in $\mathbf{P}_2$ are essentially well known. They belong to certain subvarieties of the moduli varieties, which have been extensively studied.

For surfaces, the picture is certainly not as clear. As mentioned above, any smooth surface can be embedded in $\mathbf{P}_5$.

The smooth surfaces which can be embedded in $\mathbf{P}_3$ are well described. An important necessary condition is that linear and numerical equivalences on such a surface are alike, hence the surface has to be regular. Families of smooth space surfaces have been, in many ways, studied more precisely, but we do not intend to develop this theme here.

One has no criterium to decide when a given surface is embeddable in $\mathbf{P}_4$. Severi proved that "the natural" way to do it (projection) was, as in the case of $\mathbf{P}_3$, unpracticable, by establishing the following celebrated result :

Theorem 1.1 *(Severi) : If $S \subset \mathbf{P}_5$ is a smooth surface generating $\mathbf{P}_5$ (i.e. not contained in a hyperplane), the general projection of S to $\mathbf{P}_4$ is not an* *isomorphism, except if S is a Veronese surface ($\mathbf{P}_2$ embedded in $\mathbf{P}_5$ by the complete linear system of conics).*

Severi's conclusion was that a surface is naturally embedded in $\mathbf{P}_4$ with singularities. Hence for the Italian school, an important part of the information concerning a surface in $\mathbf{P}_4$ is contained in the singularities carried by this surface. Surprisingly, there was very little interest in this school for the very natural special case of an empty singular locus.

There are many interesting smooth surfaces in $\mathbf{P}_4$. Complete intersections of 2 hypersurfaces are of course easy to construct. As all complete intersections, they share the most remarkable property of space surfaces : Linear and numerical equivalences are alike and a hyperplane section is not divisible.

More generally, if E is a vector bundle (say of rank r) on $\mathbf{P}_4$, and if s_i, with $i \in [1, r-1]$, is a section, general enough, of some twist $E(n_i)$, then

$$coker[\oplus_i O_{\mathbf{P}_4}(-n_i) \overset{(s_1,\dots,s_{r-1})}{\longrightarrow} E] = \mathcal{J} \otimes O_{\mathbf{P}_4}(c_1(E) + \sum n_i),$$

where $\mathcal{J}$ is the sheaf of ideals of a smooth surface. All smooth surfaces in $\mathbf{P}_4$ can be obtained in this way. Unfortunatly, undecomposable vector bundles on $\mathbf{P}_4$ are even more mysterious than surfaces in $\mathbf{P}_4$.

Surfaces in $\mathbf{P}_4$ can also be constructed from a given surface, using linkage ("liaison") or endomorphisms of $\mathbf{P}_4$. These constructions apparently more geometric than the preceding are in fact of the same nature. Nevertheless, they produce, naturally, curves on the constructed surface, hence they give information on the Picard group of this surface.

Now, if we want to embed a smooth surface S in $\mathbf{P}_4$, we must find on the surface a "small" complete linear system (small because projective dimension 4 is small, on a surface, and complete because of Severi's theorem) which is also "positive" enough (very ample!).

A natural way to study this problem is to start with a surface equiped with a precisely described Picard group, for example a rational surface obtained by blowing up points in the plane. We can choose among many linear systems if the Picard group is large. Let's blow up many points in the plane and try to find such a linear system. R.Hartshorne realized that if the points are in general position and if there are many points, it is, in most cases, impossible to embed the surface in $\mathbf{P}_4$ so that exceptional divisors become lines. He was bold and conjectured that there should be few rational surfaces in $\mathbf{P}_4$, perhaps only a finite number of families. This conjecture proved to be true.

In fact we will see that except for a finite number of components of the Hilbert scheme of smooth surfaces in $\mathbf{P}_4$, all surfaces are of general type.

This surprising statement does not really describe how little we understand our problem.

It seems to suggest that one knows necessary geometric conditions to impose on a surface to be able to embed it in $\mathbf{P}_4$. After all, surfaces which are not of general type do, often, carry special families of curves. One would like to believe that these special families cannot be embedded in $\mathbf{P}_4$. No such result was obtained, but this interesting path was tried by several specialists (Alexander, Aure, Okonek, Ranestad, Serrano,...). Hoping to find an indication, they constructed, using geometric methods, many families of smooth (rational, K3, elliptic, bielliptic) surfaces in $\mathbf{P}_4$, carrying beautiful families of curves. With the celebrated abelian surfaces of Comessati and Horrocks-Mumford, they form a an exciting but disparate set in which it seems difficult to find a unity, hence to find a reason for which these surfaces should be exceptions to a natural theorem.

Another optimistic analysis would follow a more algebraic line. One believes nowadays that the syzygies of the graded ideal of an embedded variety should, in some sense, reflect the intrinsic geometric properties of this variety. Now, obviously, if a surface is embedded with codimension 2 the syzygies of the graded ideal of the surface should be "small" ("short"). This could imply that the Picard group of the generic surface in the corresponding component of the Hilbert Scheme should also be "small"...! The syzygies path was particularly followed by Decker, Ein and Schreyer (and Popescu). They produced an elegant algorithm to construct surfaces in $\mathbf{P}_4$, via there syzygies bundles. They recovered in this way many interesting known surfaces (in $\mathbf{P}_4$) and discovered new families. It is rather exciting to see that this algorithm has, up to now, failed to produce surfaces not of general type of degree more than 15.

With both approaches, one is immediatly impressed by the difficulty to construct irregular smooth surfaces embedded in $\mathbf{P}_4$. Hence an irritating question:

Is there a uniform bound for the irregularity of smooth surfaces in $\mathbf{P}_4$?

This problem has been floating around for years, with no serious clue in any direction, except for the fact that one does'nt know any smooth surface in $\mathbf{P}_4$ with irregularity larger than 2.

One can construct infinitely many families of surfaces of general type, in $\mathbf{P}_4$, with irregularity 1 and 2 (using endomorphisms of $\mathbf{P}_4$). Remarkably enough, the general hyperplane section of a surface contained in one of these families is linearly complete, except, once more, for a finite number of families. Hence the two (obviously related) following questions:

Is there a uniform bound for the embedding dimension of a general hyperplane section of a smooth surface in $\mathbf{P}_4$?

Are smooth surfaces in $\mathbf{P}_4$ whose general hyperplane section is not linearly complete contained in a finite number of families of the Hilbert Scheme?

Unfortunatly, after considering with excitment these exciting schemes, one has to go back to the unpleasant reality and particularly to more primitive means. The method we propose is to study the Hilbert Polynomials of smooth surfaces embedded in $\mathbf{P}_4$, with a special interest for their constant term (which we recall is intrinsic). It is too much to hope that surfaces in $\mathbf{P}_4$ can be characterized by their Hilbert polynomial, but we believe that there are serious constraints on this polynomial. We intend to explain some of these constraints here.

A few years ago, we proved with Ellingsrud, ([8]), that given an integer χ

the smooth surfaces S such that $\chi(O_S) = \chi$ are contained in a finite number of components of the Hilbert scheme of smooth surfaces in $\mathbf{P}_4$.

As an obvious consequence we see that the degrees of such surfaces are bounded. To stress the meaning of this result for newcomers, one should think of curves with a given genus which quite obviously can be embedded in $\mathbf{P}_3$ with all large enough degrees.

Two other consequences are less obvious, but nevertheless easy.

(i) There is an integer χ_0 such that $\chi(O_S) \geq \chi_0$ for all smooth surfaces in $\mathbf{P}_4$.

(ii) There is an integer d_0 such that $d^o(S) \leq d_0$ for all smooth surfaces in $\mathbf{P}_4$ which are not of general type.

Of course one would like to find the sharp bounds χ_0 and d_0. If $\chi_0 = 0$ seems to be a reasonable conjecture, there is no serious hint concerning d_0. Recently Braun and Floystad proved $d_0 \leq 243$ ([4]). All known smooth surfaces in $\mathbf{P}_4$ which are not of general type are of degree less than or equal to 15 (see [7]). There is still a long way to go.

We come back to the study of Hilbert Polynomials of surfaces in $\mathbf{P}_4$, in these notes. Not really to announce new results, since we do not believe that we have any worth speaking of, but to describe the true (and easy) nature of our proof (with Ellingsrud). We hope it can help for a better understanding of this problem, and perhaps to sharpen our information about χ_0 and d_0. One surprising aspect of this proof is that it relies heavily on the classification of space curves (we have seen that curves sit naturally in the space whereas we agree with Severi in thinking that surfaces are not naturally in $\mathbf{P}_4$). It is entirely numerical and the self intersection of a surface in $\mathbf{P}_4$ (the second chern class of its normal bundle) is the only useful fact concerning the embedding.

Our renewed interest for this study is due to the recent work of Braun and Floystad ([4]), as well as the generalisation of our result to solids in $\mathbf{P}_5$, obtained by Braun, Ottaviani, Schneider and Schreyer ([5]).

The comments presented here are a report on a series of discussions with Geir Ellingsrud, motivated by our desire to understand these works. The presentation of the consequences of our key Lemma is essentially the same as in our previous paper ([E.P.]), but our understanding of the proof of this Lemma is now modified. He is certainly a coauthor of any reasonable mathematical statement (assuming there are some), but he should not be considered responsible for the "foggy" aspect of these comments. I have also discussed this topic freely and extensively with W. Decker, F. Schreyer and F. Zak. I appreciate their generosity. All members of the Europroj group working on surfaces in $\mathbf{P}_4$ have also shared their ideas freely with me. This was done in a useful and pleasant atmosphere.

To conclude, I would like to thank the organizers of the XII Escola de Algebra in Diamantina, for giving me the opportunity to present these comments.

2 The key Lemma and its consequences.

Let σ be a positive integer. Consider the polynomial with rational coefficients

$$P_\sigma(T) = (T^3/6\sigma^2) + ((\sigma - 5)T^2/4\sigma) + ((2\sigma^2 - 15\sigma + 35)T/12).$$

An obvious computation shows the following result :

Proposition 2.1 : *If a surface S in $\mathbf{P}_4$ is the complete intersection of an hypersurface of degree σ and another hypersurface, then*

$$\chi(O_S) = 1 + h^2(O_S) = P_\sigma(d),$$

where d is the degree of S.

A less obvious analysis, to which we will come back later, proves the following statement :

Theorem 2.2 : *If a surface S lies in a reduced irreducible hypersurface Σ of degree σ of $\mathbf{P}_4$, then $\chi(O_S) \leq 1 + h^2(O_S) \leq P_\sigma(d)$, where d is the degree of S.*

Furthermore $1+h^2(O_S) = P_\sigma(d)$ *if and only if* S *is the complete intersection of* Σ *and another hypersurface.*

In this section we intend to justify our interest for the following key Lemma, by stating and proving its main corollaries :

Lemma 2.3 *(key Lemma) : For each integer* σ *there exist polynomials* Q_σ *and* Q'_σ*, with rational coefficients, of respective degrees 1 and 2, such that :*

If Σ *is a reduced irreducible hypersurface, of degree* σ*, in* $\mathbf{P}_4$ *and if* S *is a smooth surface, of degree d, lying in* Σ *and not contained in the singular locus of* Σ*, then*

(i) if the singularities of Σ *are isolated*

$$P_\sigma(d) - Q_\sigma(d) \leq \chi(O_S) \leq 1 + h^2(O_S) \leq P_\sigma(d),$$

(ii) if the singularities of Σ *are not isolated*

$$P_\sigma(d) - Q'_\sigma(d) \leq \chi(O_S) \leq 1 + h^2(O_S) \leq P_\sigma(d).$$

This apparently boring Lemma has three interesting consequences.

Corollary 2.4 : *Let* χ *be an integer. There exists an integer* $d(\chi)$ *such that for all smooth surfaces* $S \subset \mathbf{P}_4$ *with* $\chi(O_S) = \chi$*, one has* $d^0(S) \leq d(\chi)$.

Corollary 2.5 : *There exists an integer* d_0 *such that all smooth surfaces, of* $\mathbf{P}_4$*, of degree* $> d_0$ *are of general type.*

Corollary 2.6 : *There exists an integer* χ_0 *such that* $\chi(O_S) \geq \chi_0$ *for all smooth surfaces* S *in* $\mathbf{P}_4$.

To understand how they can be deduced from our key Lemma, let us recall some known facts.

Theorem 2.7 : *Let S be a smooth surface, with canonical class K.*

(i) If S is not of general type or birationally ruled, then $\chi(O_S) \geq 0$ and $K^2 \leq 0$.

(ii) If S is rational then $\chi(O_S) = 1$ and $K^2 \leq 9$.

(iii) If S is birationally ruled of positive genus, then $\chi(O_S) \leq 0$ and $K^2 \leq 8\chi(O_S)$.

(iv) If S is of general type, then $\chi(O_S) \geq 1$ and $K^2 \leq 9\chi(O_S)$.

This is an easy consequence of Enriques classification, except for the last inequality (Bogomolov-Miyaoka).

Theorem 2.8 *(The double points formula)* :

If S is a smooth surface in $\mathbf{P}_4$, with degree d, sectional genus π and canonical class K, one has

$$d^2 - 5d - 5(2\pi - 2) - 2K^2 + 12\chi(O_S) = 0.$$

This well known result (see [12], Example 4.1.3., p.433, for example) is equivalent to the self intersection formula for a locally complete intersection surface in $\mathbf{P}_4$:

$$c_2(N_S) = d^2,$$

where N_S is the normal bundle of S in $\mathbf{P}_4$.

Theorem 2.9 *(The bound for the genus of space curves)* :

Let C be a space curve of degree d and genus g. Assume that C is not contained in a surface of degree $< s$ or that it is contained in a reduced irreducible surface of degree s, then

$$s(2g - 2) \leq d(d + s(s - 4)).$$

Furthermore equality holds if and only if C is a complete intersection of a surface of degree s and another surface.

This result is proved in ([10]) under the assumption $s(s - 1) \leq d$. It can easily be checked that this last hypothesis is unnecessary.

Theorem 2.10 : *If the general hyperplane section of smooth surface $S \subset \mathbf{P}_4$, of degree d, is contained in a space surface of degree σ, with $(\sigma)^2 < d$, then this space surface is a hyperplane section of a hypersurface, of $\mathbf{P}_4$, containing S.*

This result of Roth ([20], p.152) can also be proved following the "Grauert-Mulich" method.

Proof of Corollary 2.4 : Let χ be a fixed integer and let S be a smooth surface in $\mathbf{P}_4$ with $\chi(O_S) = \chi$.

By the double points formula, we have

$$5(2\pi - 2) = d^2 - 5d - 2K^2 + 12\chi.$$

Using the classification of surfaces, it gives

$$5(2\pi - 2) \geq d^2 - 5d - 6\chi,$$

except when S is birationally ruled of positive genus. In that case, one has

$$5(2\pi - 2) \geq d^2 - 5d - 4\chi.$$

But for d large enough, one has

$$(d^2 - 5d - 6\chi)/5 > (d^2 + 12d)/6 \quad \text{and} \quad (d^2 - 5d - 4\chi)/5 > (d^2 + 12d)/6.$$

By the Theorem bounding the genus of space curves, this implies that a general hyperplane section of S is contained in a space surface of degree ≤ 5, and by Roth's Theorem the surface itself is contained in a hypersurface of degree ≤ 5, of $\mathbf{P}_4$.

We therefore have proved that for d large enough, all smooth surfaces of degree d, with $\chi(O_S) = \chi$ are contained in an hypersurface of degree ≤ 5.

We can now use our key Lemma without thinking twice. There exists an integer $\sigma \in [1, 5]$ such that

$$P_\sigma(d) - Q'_\sigma(d) \leq \chi,$$

hence the degree of S is bounded.

Proof of Corollary 2.5 : Let $S \subset \mathbf{P}_4$ be a smooth surface which is not of general type.

If S is biratinally ruled, then $\chi(O_S) \leq 1$ and the degree of S is bounded by the preceeding Corollary.

If not, then $\chi(O_S) \geq 0$ and $K^2 \leq 0$. The double points formula gives

$$d^2 - 5d + 12\chi \leq 5(2\pi - 2).$$

This implies

$$(2\pi - 2) \geq (d^2 - 5d)/5 > (d^2 + 12d)/6,$$

for d large enough. Hence, for d large enough S is contained in an hypersurface of degree ≤ 5, and by our key Lemma there exists a $\sigma \in [1,5]$ such that

$$P_\sigma(d) - Q'_\sigma(d) \leq \chi(O_S).$$

But an obvious application of the bound of the genus gives

$$5(2\pi - 2) \leq 5d(d-3) \quad \text{hence} \quad 12\chi \leq 5d(d-3) - d^2 - 5d = 4d^2 - 10d.$$

Combining these two inequalities we find that there exists $\sigma \in [1,5]$ such that

$$P_\sigma(d) - Q'_\sigma(d) \leq (4d^2 - 10d)/12,$$

and d is bounded, since $P_\sigma - Q'_\sigma$ is a polynomial of degree 3 with positive leading coefficient.

Proof of Corollary 2.6 : It is an obvious consequence of the previous Corollary since $\chi(O_S) > 0$ for surfaces of general type.

3 Proof of the key Lemma.

This proof is based on the following, well known but never completely stated, principle :

Proposition 3.1 : *Let X be an equidimensional projective scheme of dimension n, without embedded component, contained in a reduced irreducible hypersurface $\Sigma \subset \mathbf{P}_{n+2}$, of degree σ.*

If the degree of X is $t\sigma - r$, with $0 \leq r < \sigma$, let $Y \subset \mathbf{P}_{n+2}$ be the residual of a complete intersection of a hyperplane and a hypersurface of degree r in the complete intersection of hypersurfaces of degrees t and σ.

Let $A = \sum_l A_l$ be the graded ring of the embedding $X \subset \mathbf{P}_{n+2}$ and $B = \sum_l B_l$ be the graded ring of the embedding $Y \subset \mathbf{P}_{n+2}$. Then

$$rk(A_l) \geq rk(B_l) \quad \text{and} \quad h^n(O_X(l)) \leq h^n(O_Y(l)), \quad \forall \ l.$$

If $rk(A_l) = rk(B_l)$ for some $l \geq t$, or if $h^n(O_X(l)) = h^n(O_Y(l))$ for some $l \leq \sigma + t - n - 1 - r$, then

X is, as Y, the residual of a complete intersection of a hyperplane and a hypersurface of degree r in the complete intersection of hypersurfaces of degrees t and σ.

This result is proved in ([10]) for $n = 1$. It can be extended to all $n > 0$ by an elementary induction. Use a general hyperplane section of X.

By the way, our Theorem 2.2 is an immediate consequence of this Proposition.

We will also need the following result, whose proof is an elementary chern class computation.

Lemma 3.2 : *$c_2(N(-\sigma)) = d(d+\sigma(\sigma-4)) - \sigma(2\pi-2)$, where π is the sectional genus of S (the algebraic genus of an hyperplane section of S) and N the normal bundle of S in $\mathbf{P}_4$.*

Our last technical Lemma is a bit more difficult.

Lemma 3.3 : *Let S be a smooth surface contained in an hypersurface Σ, of degree σ, of $\mathbf{P}_4$, but not contained in the singular locus of Σ.*

Let N be the normal bundle of S in $\mathbf{P}_4$.

(i) If Σ is smooth along S, then $c_2(N(-\sigma)) = 0$.

(ii) If Σ has isolated singularities, then $c_2(N(-\sigma)) \leq (\sigma - 1)^4$.

(iii) If the singularities of Σ are not isolated, $c_2(N(-\sigma)) \leq (\sigma - 1)^2 d^0(S)$.

Proof : The embedding $S \subset \Sigma$ induce an homomorphism $N \to O_S(\sigma)$ whose vanishing locus is the intersection of S and the singular locus of Σ (scheme theoretically defined by the partial derivatives of the equation of Σ). This proves (i).

If Σ has isolated singularities, the singular locus of Σ is a dimension 0 scheme of degree $\leq (\sigma - 1)^4$, by Bezout's Theorem, and *(ii)* is obvious.

For *(iii)*, note that if S and the singular locus of Σ intersect in a dimension 0 scheme, this scheme is contained in the proper intersection of S and of the hypersurfaces, of degree $\sigma - 1$, defined by two general partial derivatives of the equation of Σ and we are done, by Bezout's Theorem.

If there is a singular curve of Σ in S, one has to work a bit more. Let D be this curve and Z the vanishing locus of the induced homomorphism

$$N(-\sigma) \to O_S(-D).$$

One has (by staightforward computations) :

$$c_2(N(-\sigma) = d^0(Z) + (2\sigma - 5)d^0(D) + D^2 + KD,$$

and

$$d^0(Z) \leq (\sigma - 1)^2 d^0(S) - 2(\sigma - 1)d^0(D) + D^2.$$

Combining these two inequalities and using Kodaira vanishing, we get the anouced bound.

Remark : If Σ is non singular we get

$$d(d+\sigma(\sigma-4))-\sigma(2\pi-2)=c_2(N(-\sigma))=0.$$

Using the bound of the genus for space curves, we see that in this case a hyperplane section of S is a complete intersection of the corresponding hyperplane section of Σ and another space surface. In this case S has to be the complete intersection of Σ and another hypersurface.

Proof of the key Lemma : For the sake of simplicity, we shall assume that d, the degree of S, is a multiple of σ. The proof is easier to follow in this special case, but is of the same nature in the general case.

In view of Proposition 2.1, we can obviously assume that S is not a complete intersection of Σ and another hypersurface.

We first compute $\chi(O_S((d+\sigma(\sigma-5))/\sigma)$, using Riemann-Roch and the adjonction formula for surfaces.

$$\chi(O_S((d+\sigma(\sigma-5))/\sigma)=[(d+\sigma(\sigma-5))/2\sigma][d(d+\sigma(\sigma-4))/\sigma-(2\pi-2)]+\chi(O_S)$$

$$=[(d+\sigma(\sigma-5))/\sigma][c_2(N(-\sigma)/2\sigma]+\chi(O_S).$$

Let Y be the complete intersection, in $\mathbf{P}_4$, of a hypersurface of degree σ and a hypersurface of degree d/σ. Since $O_Y((d+\sigma(\sigma-5))/\sigma)$ is the dualizing line bundle on Y, we have

$$1+h^0(O_Y((d+\sigma(\sigma-5))/\sigma))=P_\sigma(d).$$

By our Proposition 3.1, we get

$$h^0(O_S((d+\sigma(\sigma-5))/\sigma))\geq P_\sigma(d)-1.$$

Hence we find

$$1+\chi(O_S)\geq 1+h^0(O_S((d+\sigma(\sigma-5))/\sigma))-h^1(O_S((d+\sigma(\sigma-5))/\sigma))$$

$$-[(d+\sigma(\sigma-5))/\sigma][c_2(N(-\sigma)/2\sigma]$$

$$\geq P_\sigma(d) - h^1(O_S((d+\sigma(\sigma-5))/\sigma)) - [(d+\sigma(\sigma-5))/\sigma][c_2(N(-\sigma)/2\sigma].$$

To conclude the proof of our key Lemma, we need to bound

$$h^1(O_S((d+\sigma(\sigma-5))/\sigma))$$

according to the assertion. This can be done easily, " à la Castelnuovo".

Let H be a general hyperplane and $C = H \cap S$. There is an obvious exact sequence

$$H^0(O_S((d+\sigma(\sigma-5))/\sigma)) \to H^0(O_S((d+\sigma(\sigma-4))/\sigma)) \to H^0(O_C((d+\sigma(\sigma-4))/\sigma))$$

$$\to H^1(O_S((d+\sigma(\sigma-5))/\sigma)) \to H^1(O_S((d+\sigma(\sigma-4))/\sigma)) \to H^1(O_C((d+\sigma(\sigma-4))/\sigma)).$$

Now by our Proposition 3.1, we know that if $H^1(O_C((d+\sigma(\sigma-4))/\sigma)) \neq 0$, then C is a complete intersection (of $\Sigma \cap H$ and ...), hence S is a complete intersection of Σ and another hypersurface.

We can therefore assume $H^1(O_C(n) = 0$ for $n \geq (d+\sigma(\sigma-4))/\sigma)$.

If $\mathcal{I}$ is the sheaf of ideals of C in $\mathbf{P}_3$, it is clear that

$$rk(coker[H^0(O_S(n)) \to H^0(O_C(n))]) \leq h^1(\mathcal{I}(n)),$$

for all n.

Combining these informations in a straightforward way, one is left with the following inequality :

$$h^1(O_S((d+\sigma(\sigma-5))/\sigma)) \leq \sum_{n \geq (d+\sigma(\sigma-4))/\sigma)} h^1(\mathcal{I}(n)).$$

Let us be artless, once more, to bound $h^1(\mathcal{I}(n))$ for $n \geq (d+\sigma(\sigma-4))/\sigma)$. Since $H^1(O_C(n)) = 0$ for $n \geq (d+\sigma(\sigma-4))/\sigma)$, we have

$$H^0(O_C(n)) = nd + 1 - \pi \quad \text{for} \quad n \geq (d+\sigma(\sigma-4))/\sigma).$$

But if A is the graded ring of the embedding $C \subset \mathbf{P}_3$, we know, by Proposition 3.1, that

$$rk(A_n) \geq nd - (d+\sigma(\sigma-4))/2\sigma),$$

since $(d+\sigma(\sigma-4))/2\sigma)+1$ is the genus of a complete intersection (in $\mathbf{P}_3$) of surfaces of degrees σ and d/σ. We find therefore

$$h^1(\mathcal{I}(n)) \leq (d+\sigma(\sigma-4))/2\sigma) - (\pi - 1) = c_2(N(-\sigma))/2\sigma.$$

It is well known (see [11] for example), that $H^1(\mathcal{I}(n)) = 0$ for $n \geq d-2$. Hence we have proved

$$h^1(O_S((d+\sigma(\sigma-5))/\sigma)) \leq (d-3-(d+\sigma(\sigma-5))/\sigma))c_2(N(-\sigma))/2\sigma,$$

and this is enough for our key Lemma. When the singularities of Σ are isolated, as well as when they are not.

Combining all our inequalities, we proved

$$1+\chi(O_S) \geq P_\sigma(d) - (d-3)c_2(N(-\sigma))/2\sigma.$$

Since $c_2(N(-\sigma))$ is bounded independantly of d when the singularities of Σ are isolated, and by $(\sigma-1)^2 d$ when they are not, the Lemma is established.

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