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LIMITS OF GRAPHS

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Abstract

We show that the variety parametrizing graphs of maps of $\mathbf{P}_1$ into $\mathbf{P}_1 \times \mathbf{P}_1$ of bidegree $(1,1)$ admits a compactification closely related to the classical construction of complete quadrics. We conjecture a similar behavior holds for higher bidegrees.

The space of harmonic maps (*i.e.*, branched minimal immersions) of S^2 into S^{2n} is described in [2] as a union of moduli spaces labelled by the harmonic degree $d \geq 0$. It was shown in [6] that their study may be reduced to that of the moduli space $\mathcal{M}_d$ of pairs of meromorphic functions of degree d with the same ramification divisor. Since the space of meromorphic functions of degree d is not compact, neither is $\mathcal{M}_d$.

In [7] a natural compactification of $\mathcal{M}_d$ was introduced and the boundary of this compactified space was studied by considering limits of graphs of maps. The inadequacy of this natural compactification was illustrated by numerous examples regarding limits of one-parameter families of graphs. For example, some points in the boundary were found to correspond to various possible limit graphs. To resolve the ambiguity of assigning to each point p in the boundary a ***unique*** limit graph, we consider all germs of smooth curves in $\mathcal{M}_d$ emanating from p which intersect the boundary of $\mathcal{M}_d$ only at p. Let γ be such a germ. Then $\gamma - \{p\}$ parametrizes a flat family of curves in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$. Since flatness extends uniquely across the puncture, we obtain a unique limit curve associated to the germ. The limit curve so obtained depends on the choice of the parameter curve emanating from p. Intuitively, one should be able to separate these limit curves by separating the normal directions to the boundary at p. Classical

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algebro-geometric techniques tell us that this idea is captured in a suitable blow-up process.

In this paper we study a simpler compactification problem. We show that the variety parametrizing graphs of maps of $\mathbf{P}_1$ into $\mathbf{P}_1 \times \mathbf{P}_1$ of bidegree $(1,1)$ admits a compactification closely related to other classical limit problems such as complete linear maps, complete quadrics *etc*... (cf. [4] and references therein). See Prop. 3.1. We conjecture a similar behavior holds for higher bidegrees. An understanding of this simpler problem will be of great help in tackling the problem of compactifying $\mathcal{M}_d$, with the above mentioned ambiguities resolved.

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1. Preliminaries

Recall that a morphism (or holomorphic map) of degree d, $f : \mathbf{P}_1 \to \mathbf{P}_1$, is defined by $f(z) = P(z)/Q(z)$ where P and Q are polynomials of degree at most d, either P or Q has degree d, and where P and Q have no factors in common. In homogeneous form, we can write $F(z) = (F_0(z) : F_1(z))$ where F_0 and F_1 are homogeneous polynomials of degree d in the homogeneous coordinates z_0, z_1, with $\gcd(F_0, F_1) = 1$. Observe that if we relax the gcd condition, say $\gcd(F_0, F_1) = F_2$ where F_2 is a homogeneous polynomial of degree k, then F can be considered as a map of degree $d - k$, together with "extra information" encapsulated by the polynomial F_2. Note that gcd $= 1$ is an open condition. Now let

$$\varphi = (F, G) : \mathbf{P}_1 \to \mathbf{P}_1 \times \mathbf{P}_1$$

be a map of bidegree (d, d), defined by

$$\varphi(z) = \big((F_0(z) : F_1(z)), (G_0(z) : G_1(z))\big)$$

where F_0, F_1, G_0 and G_1 are homogeneous polynomials of degree d in the homogeneous coordinates z_0, z_1 with $\gcd(F_0, F_1) = \gcd(G_0, G_1) = 1$. The graph,

$\Gamma_\varphi \subset \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$, is defined by a pair of bihomogeneous equations:

$$\begin{cases} x_1 F_0(z_0, z_1) = x_0 F_1(z_0, z_1) \\ y_1 G_0(z_0, z_1) = y_0 G_1(z_0, z_1). \end{cases} \tag{1}$$

Our aim is to describe a compactification of the space of graphs. An obvious way to compactify the space of graphs is to drop the gcd condition. Namely, we include pairs (F, G) where either $\gcd(F_0, F_1) \neq 1$ or $\gcd(G_0, G_1) \neq 1$, and where equation (1) holds. Let us begin by considering two simple examples where the gcd condition is relaxed. These examples will illustrate that such a naïve compactification is inadequate. In Proposition 3.1, we will state our result in precise terms via the appropriate Hilbert scheme (see [8] for foundational material) of curves in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$.

We will adopt the convention that the variables for the first, second and third factors of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ are given by z, x and y respectively.

Example 1.1. Let $\varphi_t = (F, G)$ where $F_0(z) = z_0$, $F_1(z) = z_0 + tz_1$, and G_0 and G_1 are homogeneous polynomials of degree 1 in z_0 and z_1 (and independent of the parameter t) subject to the condition that $G_0(0:1) \neq 0$ or $G_1(0:1) \neq 0$. For $t \neq 0$ let Γ_{φ_t} denote the graph of φ_t. Setting $t = 0$, we have $\gcd(F_0, F_1) = z_0$ and the first equation in (1) becomes $x_1 z_0 = x_0 z_0$. At the point $(z_0 : z_1) = (0 : 1)$, the second equation in (1) tells us that $(y_0 : y_1) = (G_0(0:1) : G_1(0:1))$. Thus at $t = 0$ the curve described by (1) is

$$\Big((0{:}1) \times \mathbf{P}_1 \times (G_0(0{:}1) : G_1(0{:}1))\Big) \cup \Big(\{(z, (1{:}1), (G_0(z) : G_1(z))) \mid z \in \mathbf{P}_1\}\Big).$$

Observe that the first term in the union is no longer a graph. Nevertheless, we still get a curve in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$—albeit a reducible curve with two components. We may thus say that as t approaches 0 the graph φ_t approaches the above curve (the "limit graph"). Note that this example will work even if $\gcd(G_0, G_1) \neq 1$—provided that $\gcd(G_0, G_1) \neq z_0$. Indeed, equation 1 implies that the limit graph will have three components. Let us suppose that $G_0(z_0 : z_1) = G_1(z_0 : z_1) = z_1$. For $t \neq 0$ we obtain from (1) the curve

$$\Big((0:1) \times (1:1) \times \mathbf{P}_1\Big) \cup \Big(\{(z, (F_0(z) : F_1(z)), (1:1)) \mid z \in \mathbf{P}_1\}\Big).$$

In view of the first term, this is not a graph. However, as we shall see in Lemma 2.1, it has the "correct" Hilbert polynomial. Now at $t=0$ we obtain the curve

$$\big((0:1)\times \mathbf{P}_1 \times (1:1)\big) \cup \big((1:0)\times (1:1)\times \mathbf{P}_1\big) \cup \big(\mathbf{P}_1\times (1:1)\times (1:1)\big),$$

which also has the correct Hilbert polynomial by Lemma 2.1. Note that in this example, at $t=0$, we have $\gcd(F_0,F_1,G_0,G_1)=1$. Thus the acceptance of such non-graph curves to our space of graphs does not create any real problems, in the precise sense that the extended family of curves remains flat.

Example 1.2. Let φ_t be as in Example 1.1 except that $\gcd(G_0,G_1)=z_0$, *i.e.*, $G_0(z_0,z_1)=\alpha z_0$ and $G_1(z_0,z_1)=\beta z_0$, where $(\alpha:\beta)\in\mathbf{P}_1$. From (1) we obtain

$$\begin{cases} x_1 z_0 &= x_0(z_0+tz_1)\\ y_1\alpha z_0 &= y_0\beta z_0. \end{cases} \tag{2}$$

In this case if we simply set $t=0$ we would have

$$\begin{cases} x_1 z_0 &= x_0 z_0\\ y_1\alpha z_0 &= y_0\beta z_0. \end{cases}$$

Here $\gcd(F_0,F_1,G_0,G_1)=z_0$. We could thus be led to say that as t approaches 0, Γ_{φ_t} approaches

$$\big((0{:}1)\times\mathbf{P}_1\times\mathbf{P}_1\big) \cup \big(\mathbf{P}_1\times(1{:}1)\times(\alpha{:}\beta)\big).$$

Note that the $(0:1)\times\mathbf{P}_1\times\mathbf{P}_1$ term above is of dimension two. Since the least we should expect from a "good" notion of limit in the present setting is that it preserve dimension, this naïve procedure must be refined: limits of equations may not correspond to limits of curves.

This example illustrates that in the naïve compactification, it is precisely along the subset of the boundary where $\gcd(F_0,F_1,G_0,G_1)\neq 1$ that we need to perform modifications in order to obtain the desired decent behaviour, *i.e.*, a complete, flat family. We will explore this further in section 2.

Continuing with Example 1.2, we now discuss how one can obtain the "legitimate" limit of a 1-parameter family of graphs from a 1-parameter family of equations as above. Set

$$S=\mathbf{C}[x_0,x_1,y_0,y_1,z_0,z_1].$$

For each constant t, let $I_t = (F_t, G) \subset S$ denote the ideal generated by $F_t = z_0(x_1 - x_0) - tx_0z_1$ and $G = z_0(y_1\alpha - y_0\beta)$. Then the graph described by the equations (2) is just the variety $V(I_t)$ for $t \neq 0$. Observe that

$$(x_1 - x_0)G - (y_1\alpha - y_0\beta)F_t = tx_0z_1(y_1\alpha - y_0\beta).$$

Let $I_t' = (F_t, G, x_0z_1(y_1\alpha - y_0\beta))$. Off $t = 0$ we have $V(I_t) = V(I_t')$. At $t = 0$, we get $I_0' = (z_0(x_1 - x_0), z_0(y_1\alpha - y_0\beta), x_0z_1(y_1\alpha - y_0\beta))$. From the two charts $z_0 \neq 0$ and $z_1 \neq 0$ we may convince ourselves that a better candidate for the limit graph as $t \to 0$ is given by

$$V(I_0') = \big(\mathbf{P}_1 \times (1{:}1) \times (\alpha{:}\beta)\big) \cup \big((0{:}1) \times (0{:}1) \times \mathbf{P}_1\big) \cup \big((0{:}1) \times \mathbf{P}_1 \times (\alpha{:}\beta)\big),$$

which is indeed a curve—with three irreducible components.

The algebraic process for producing I_t' from I_t is usually referred to as *saturation*. One thinks of t as a new variable and one enlarges I_t by including all polynomials $f \in S[t]$ such that $t^n f \in I_t$ for some n. The enlarged ideal I_t' will eventually have the property that

(i) t is not a zero divisor mod. I_t' and

(ii) $I_t = I_t'$ at each $t \neq 0$.

The first turns out to be the condition of flatness, the algebraic translation of the notion of continuity for a family of varieties, (cf. [3], Prop. 9.7, p. 257). Condition (ii) tells us that the families $V(I_t)$ and $V(I_t')$ coincide off $t = 0$. At $t = 0$, $V(I_0)$ presents a jump of dimension, whereas $V(I_0')$ is not only of the right dimension but, in fact, it has the same Hilbert polynomial as all the other $V(I_t)$, $t \neq 0$. This is a useful criterion for flatness. See Prop. 3.1 below.

We recall from [3] (*loc. cit.*) that for a smooth curve γ and a point $0 \in \gamma$, if S is a scheme and $W \subset (\gamma - \{0\}) \times S$ is a closed subscheme flat over $\gamma - \{0\}$, there exists a unique closed subscheme $W' \subset \gamma \times S$ flat over γ such that W' restricted over $\gamma - \{0\}$ coincides with W. In fact, one takes for W' the closure of W in $\gamma \times S$. We refer to W_0' as the *limit* of the family $\{W_t\}_{t \in (\gamma - \{0\})}$ as $t \to 0$ and we write $W_0' = \lim_{t \to 0} W_t$. Thus, in the examples above, the reader may check that $V(I_t') \subset \mathbf{C} \times \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ is the closure of $V(I_t) \cap ((\mathbf{C} - \{0\}) \times \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1)$.

We can generalize the above example to the following limit equations:

$$\begin{cases} x_1 F_0 D = x_0 F = F_1 D \\ y_1 G_0 D = y_0 G_1 D, \end{cases} \tag{3}$$

where D, F_i, G_i denote homogeneous polynomials in z_0, z_1, with $gcd(F_0, F_1, G_0, G_1) = 1$. Set $\deg(D) = \delta, \deg(F_i) = \deg(G_i) = d - \delta$, $F^0 = x_1 F_0 - x_0 F_1^0$ and $G^0 = y_1 G_0 - y_0 G_1$. Then the graph of the map (F^0, G^0), $\Gamma_{(F^0, G^0)}$, is of tri-degree $(1, d - \delta, d - \delta)$. The variety obtained from (3) is then the union of $\Gamma_{(F^0, G^0)}$ with $\bigcup r_i \times \mathbf{P}_1 \times \mathbf{P}_1$, the r_i denoting the roots of D. By taking 1-parameter families and a saturation of ideals as before, we would expect to "complete" the above picture by adding to each $r_i \times \mathbf{P}_1 \times \mathbf{P}_1$ an appropriate curve, γ_{r_i}, such that $\Gamma_{(F^0, G^0)} \cup \bigcup_i \gamma_{r_i}$ makes a one dimensional subscheme of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ with the "right" Hilbert polynomial $h(t) = (2d + 1)t + 1$ with respect to the ample line bundle $\mathcal{O}(1, 1, 1)$. See Lemma 2.1 below.

Instead of merely taking specific 1-parameter limits we will try a global approach. We will start from a suitable parameter space, $\mathbf{\Sigma}_d$, (cf. 5) and dominate the component of **Hilb** of curves in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ containing the graphs by modifications of $\mathbf{\Sigma}_d$. The technique we employ is suggested by the winning strategy of a similar case for codimension 2 complete intersection $(2, 2)$ in $\mathbf{P}_n$ (cf. [1]). Perhaps, much too optimistically, it may shed some light on a more serious question as to whether a smooth specialization of a family of g.c.i. curves in $\mathbf{P}_3$ may not be a g.c.i. (cf. [5] in characteristic p, [9] in higher dimension).

2. The space of graphs

Let S_x^d denote the space of homogeneous polynomials of degree d in the variables x_0, x_1. Set

$$\begin{cases} F := x_1 F_0 - x_0 F_1 \\ G := y_1 G_0 - y_0 G, \end{cases} \tag{4}$$

where $F_i, G_i \in S_z^d$. Then the graph defined by equations (1) is just the subvariety of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ of the ideal of S generated by F and G. Note that at the beginning of §1 we used the pair (F,G) to indicate a holomorphic map of bidegree (d,d). This pair was later expressed in terms of F_0, F_1, G_0 and G_1, with some gcd conditions. We will abuse notation and sometimes refer to the polynomials F, G in (4) and the pair (F,G) as "maps". Notice that F is an element of $S_z^d S_x^1$ and likewise $G \in S_z^d S_y^1$. Conversely, any element of $S_z^d S_x^1$ can be written in the form $x_1 F_0 - x_0 F_1$ for suitable $F_i \in S_z^d$.

For a vector space V, we let $\mathbf{P}(V)$ denote the projective space of lines through the origin in V. For simplicity of expression, we will often abuse notation as well as language, denoting a nonzero F and the corresponding point in projective space by the same letter. It should be clear from the context whether we are referring to the polynomial or its equivalence class.

We also note, once and for all, that since F is of degree 1 in the x's and G is of degree 1 in the y's, $\gcd(F,G)$ must be a purely $z-$factor common to all F_0, F_1, G_0, G_1.

We define

$$\boldsymbol{\Sigma}_d := \mathbf{P}(S_z^d S_x^1) \times \mathbf{P}(S_z^d S_y^1). \tag{5}$$

For $i = 1, \ldots, d$, let

$$\begin{aligned}&\boldsymbol{\Sigma}_{d,i} := \\ &\{(F,G) \in \boldsymbol{\Sigma}_d \mid F = DF^0,\ G = DG^0 \text{ with } D \in S_z^i, (F^0, G^0) \in \boldsymbol{\Sigma}_{d-i}\}.\end{aligned} \tag{6}$$

The space of graphs of maps of $\mathbf{P}_1$ to $\mathbf{P}_1 \times \mathbf{P}_1$ of bidegree (d,d) is the open subset of $\boldsymbol{\Sigma}_d$ consisting of pairs (F,G) such that F and G are irreducible (and hence $\gcd(F,G) = 1$). Now consider the open subset, $\mathring{\boldsymbol{\Sigma}}_d$, of $\boldsymbol{\Sigma}_d$ consisting of pairs (F,G) such that $\gcd(F,G) = 1$. Observe that $\mathring{\boldsymbol{\Sigma}}_d$ contains non-graph elements: in Example 1.1, for $t = 0$, we have $(F,G) \in \mathring{\boldsymbol{\Sigma}}_d$, but the corresponding curve in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ has two components, one of which is of tri-degree $(0,1,0)$ thereby making the curve a non-graph. Nevertheless, every element in $\mathring{\boldsymbol{\Sigma}}_d$ corresponds to a *curve* in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$. We will abuse language and refer to the curve corresponding to a pair $(F,G) \in \mathring{\boldsymbol{\Sigma}}_d$ as the "graph" of (F,G) and, as before, denote it by $\Gamma_{(F,G)}$.

Lemma 2.1.

1. *For each* $(F,G) \in \overset{\circ}{\Sigma}_d$ *, the Hilbert polynomial of the curve* $\Gamma_{(F,G)} \subset \mathbf{P}_1{\times}\mathbf{P}_1{\times}\mathbf{P}_1$ *with respect to the ample sheaf* $\mathcal{O}(1,1,1)$ *is equal to* $(2d+1)t+1$.

2. *The family of curves* $\{\Gamma_{(F,G)} \mid (F,G) \in \overset{\circ}{\Sigma}_d \}$ *is flat.*

3. *The induced map* $\Gamma : \overset{\circ}{\Sigma}_d \rightarrow \mathbf{Hilb}^{(2d+1)t+1}$ *defined by* $(F,G) \mapsto \Gamma_{(F,G)}$ *is a monomorphism. In particular, for* $(F_1,G_1) \neq (F_2,G_2)$ *we have* $\Gamma_{(F_1,G_1)} \neq \Gamma_{(F_2,G_2)}$.

Proof. Since the graph, γ, of a pair $(F,G) \in \overset{\circ}{\Sigma}_d$ is a complete intersection in $\mathbf{P}_1{\times}\mathbf{P}_1{\times}\mathbf{P}_1$, its Hilbert polynomial with respect to $\mathcal{O}(1,1,1)$ is easily computed via the Koszul resolution,

$$0 \leftarrow \mathcal{O}_\gamma \leftarrow \mathcal{O} \overset{(F,G)}{\leftarrow} \mathcal{O}(-d,-1,0) \oplus \mathcal{O}(-d,0,-1) \leftarrow \mathcal{O}(-2d,-1,-1) \leftarrow 0$$

and checked to be $h(t) := (2d+1)t+1$. Since a family parametrized by an integral scheme is flat if and only if the Hilbert polynomial is the same for all members ([3] p. 261), it follows that $(F,G) \mapsto \Gamma_{(F,G)}$ indeed defines a morphism

$$\Gamma : \overset{\circ}{\Sigma}_d \rightarrow \mathbf{Hilb}^{(2d+1)t+1}.$$

The assertion that Γ is in fact a *mono*morphism means the following:

> Let T be a scheme and for $i = 1,2$ let $t_i : T \rightarrow \overset{\circ}{\Sigma}_d$ be maps such that $\Gamma \circ t_1 = \Gamma \circ t_2$. Then $t_1 = t_2$.

Since $t_1 = t_2$ holds if their restrictions to open subsets covering T are equal, we may assume $T = \mathrm{Spec}(R)$. Recall that, for a vector space V, there is a natural bijection

$$\{T \rightarrow \mathbf{P}(V)\} \leftrightarrow \{L = \text{locally split rank 1 } R\text{-submodule of } V \otimes R\},$$

(cf. [8] pp. 1–3). Shrinking T if needed to trivialize L, we may assume that the so called $R-$valued point $t : T \rightarrow \mathbf{P}(V)$ is in fact given by an element in a free

R–basis of $V \otimes R$. Presently, let R be a local ring and for $i = 1,2$ let

$$\begin{cases} F_i = F_{i0}(z)x_0 + F_{i1}(z)x_1, \\ G_i = G_{i0}(z)y_0 + G_{i1}(z)y_1 \end{cases}$$

be in the polynomial ring $R[x_0, x_1, y_0, y_1, z_0, z_1]$. Assume that the coefficients of F_1 (resp. F_2) generate R (this hypothesis means that F_1 defines an R-valued point of $\mathbf{P}(S_z^d S_x^1)$ because it generates a split R–submodule of $S_z^d S_x^1 \otimes R)$ and likewise for G_i. We now assume that the tri-homogeneous ideals $I_1 = (F_1, G_1)$ and $I_2 = (F_2, G_2)$ define the same closed subscheme of $(\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1)_R$. It follows from Serre's correspondence between homogeneous modules and sheaves on a projective space that there is some n such that $x_i^n I_j \subset I_k$ for $i = 0,1$; $j,k = 1,2$ (and similarly with y's in place of the x's ...). Without loss of generality, we may assume a coefficient of $F_{21}(z)$ is a unit in R. We have a relation $x_0^n F_1 = AF_2 + BG_2$, whence (setting y's equal to 0) a relation $x_0^n F_1 = AF_2$. Setting $x_0 = 0$ in the last equation we obtain $0 = A((0,1),z)F_{21}(z)x_1$. Since $F_{21}(z)x_1$ is not a zero divisor, we deduce a relation $F_1 = AF_2$. Comparing contents (= ideal of coefficients) on both sides, it follows that A must be a unit in R. We have proven the equality of R-submodules, $(F_1) = (F_2)$ and likewise for the G's.

□

Remark 2.2. Let $(F,G) \in \mathring{\mathbf{\Sigma}}_d$ and let $\gamma = \Gamma_{(F,G)}$. From the diagram of tangent bundles sequence,

$$\begin{array}{ccccc} T_\gamma & \to & (T_{\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1})_{|\gamma} & \to & N = N_{\gamma/\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1} \\ \| & & \| & & \\ \mathcal{O}(2) & \to & \mathcal{O}(2) \oplus \mathcal{O}(2d) \oplus \mathcal{O}(2d) & & \end{array}$$

we have $H^0(N) = S_z^{2d} \oplus S_z^{2d}$, $h^0(N) = 4d + 2$ and $h^1(N) = 0$. Thus

$$\dim_\gamma \mathbf{Hilb}^{(2d+1)t+1} = 4d + 2 = \dim \mathbf{\Sigma}_d.$$

This implies that $\mathbf{\Sigma}_d$ is a "good" approximation to $\mathbf{Hilb}^{(2d+1)t+1}$ (cf. [8] pp. 10-5, 10-8, Remark (10.4)(iii)).

Remark 2.3. In the case of maps from $\mathbf{P}_1$ to $\mathbf{P}_1{\times}\mathbf{P}_1$, of bidegree (1,0), the space of graphs is just

$$\{(F,G) \in \mathbf{P}(S^1_z S^1_x) \times \mathbf{P}(S^1_y) \mid F \text{ is irreducible}\}.$$

For each $(F,G) \in \mathbf{P}(S^1_z S^1_x) \times \mathbf{P}(S^1_y)$, it is easily checked by a Koszul resolution that the Hilbert polynomial of its graph $\Gamma_{(F,G)}$ is $h(t) = 2t+1$. Thus the (complete) family of curves, $\{\Gamma_{(F,G)} \mid (F,G) \in \mathbf{P}(S^1_z S^1_x) \times \mathbf{P}(S^1_y)\}$, is flat. Similarly, for maps from $\mathbf{P}_1$ to $\mathbf{P}_1{\times}\mathbf{P}_1$, of bidegree (0,1), we obtain a complete, flat family of curves, $\{\Gamma_{(F,G)} \mid (F,G) \in \mathbf{P}(S^1_x) \times \mathbf{P}(S^1_z S^1_y)\}$.

The reason why we consider maps from $\mathbf{P}_1$ to $\mathbf{P}_1{\times}\mathbf{P}_1$, of bidegree (d,d) rather than the more general bidegree (m,n) maps is that the intricacies involved in studying the boundary of the corresponding space of graphs are the same as that in the bidegree (d,d) case. Moreover, the indexation of the various spaces used in our construction is vastly simplified in the bidegree (d,d) case.

3. Compactifying

We wish to resolve the indeterminacies of the rational map (2.1)

$$\Gamma : \mathbf{\Sigma}_d \cdots \rightarrow \mathbf{Hilb}^{(2d+1)t+1}.$$

Of course one knows from general principles that the indeterminacies of any rational map may be solved by a sequence of blowing ups along smooth centres (at least in char. 0). The point here is to make the resolution explicit so as to shed light on the boundary. We shall consider only the case $d = 1$ in detail. The main result is the following.

Proposition 3.1. *Let $\mathbf{\Sigma}^1_d$ be the blowup of $\mathbf{\Sigma}_d$ along $\mathbf{\Sigma}_{d,d}$.*

1. *The rational map $\Gamma : \mathbf{\Sigma}_1 \cdots \rightarrow \mathbf{Hilb}^{3t+1}$ fails to be defined precisely along $\mathbf{\Sigma}_{1,1}$ and*

2. *Γ extends to a morphism $\mathbf{\Sigma}^1_1 \rightarrow \mathbf{Hilb}^{3t+1}$.*

Some preliminary results will be stated for general d. Our first task is to exhibit $\mathbf{\Sigma}_{d,d}$ (see (6)) as the locus where a certain map of locally free $\mathcal{O}_{\Sigma_d}$–modules drops rank.

Lemma 3.2. *Let $\mathcal{O}_{zx}(-1) \to S^d_z S^1_x \otimes \mathcal{O}_{\mathbf{P}(S^d_z S^1_x)}$ denote the tautological line subbundle over the projective space $\mathbf{P}(S^d_z S^1_x)$ and likewise for $\mathcal{O}_{zy}(-1) \to S^d_z S^1_y \otimes \mathcal{O}_{\mathbf{P}(S^d_z S^1_y)}$. Tensoring the first by $S^1_y \otimes \mathcal{O}_{\Sigma_d}$ and likewise the second by $S^1_x \otimes \mathcal{O}_{\Sigma_d}$ and taking direct sums defines the map of locally free $\mathcal{O}_{\Sigma_d}$–modules,*

$$\begin{array}{rcl} \mu : \mathcal{O}_{zx}(-1) \otimes S^1_y \oplus \mathcal{O}_{zy}(-1) \otimes S^1_x & \longrightarrow & S^d_z S^1_x S^1_y \otimes \mathcal{O}_{\Sigma_d} \\ (F \otimes A, G \otimes B) & \longmapsto & AF + BG. \end{array}$$

Then $\overset{4}{\wedge}\mu$ vanishes precisely along $\mathbf{\Sigma}_{d,d}$ (see (5)). *Furthermore we have a natural identification $\mathbf{\Sigma}_{d,d} \cong \mathbf{P}(S^d_z) \times \mathbf{P}(S^1_x) \times \mathbf{P}(S^1_y)$.*

Proof. Let Z denote the scheme of zeros of $\overset{4}{\wedge}\mu$. Let Z' be the image of the map $\iota : \mathbf{P}(S^d_z) \times \mathbf{P}(S^1_x) \times \mathbf{P}(S^1_y) \to \mathbf{\Sigma}_d$ defined by $(D, F^0, G^0) \mapsto (DF^0, DG^0)$. One checks easily that ι is a closed embedding. The existence of a non-trivial relation $AF + BG = 0$ with $(F, G) \in \mathbf{\Sigma}_d$, $A \in S^1_y$, $B \in S^1_x$ is quickly seen to be equivalent to $\gcd(F, G) = D$ for some $D \in \mathbf{P}(S^d_z)$. This proves $Z = Z'$ at least set-theoretically. To complete the verification, it suffices to show that Z is smooth. For this, we argue with a local matrix representation of μ. First notice that the group $\mathbf{G} = \mathbf{GL}(S^1_z) \times \mathbf{GL}(S^1_x) \times \mathbf{GL}(S^1_y)$ acts naturally on $\mathbf{\Sigma}_d$ and $\mathbf{P}(S^d_z) \times \mathbf{P}(S^1_x) \times \mathbf{P}(S^1_y)$. Clearly, the map ι is $\mathbf{G}$–equivariant and μ is $\mathbf{G}$–invariant. If $\mathrm{sing}(Z)$ were non-empty, it would contain a closed orbit. The sole closed $\mathbf{G}$–orbit in $\mathbf{P}(S^d_z) \times \mathbf{P}(S^1_x) \times \mathbf{P}(S^1_y)$ can be checked to be the orbit of (z_0^d, x_0, y_0). Write

$$\begin{cases} F = x_0(z_0^d + a_1 z_0^{d-1} z_1 + \cdots + a_d z_1^d) + x_1(a_{d+1} z_0^d + \cdots + a_{2d+1} z_1^d) \\ G = y_0(z_0^d + b_1 z_0^{d-1} z_1 + \cdots + b_d z_1^d) + y_1(b_{d+1} z_0^d + \cdots + b_{2d+1} z_1^d). \end{cases}$$

Thus, F is a local basis of $\mathcal{O}_{zx}(-1)$, the a's denote affine coordinates and likewise for G and b's. Multiplying F (resp. G) by y_0, y_1 (resp. x_0, x_1) we find a local representation of μ by a $4 \times (4d+4)$ matrix,

$$\begin{array}{ccccccccc}
z_0^d x_0 y_0, & z_0^{d-1} z_1 x_0 y_0, & \ldots z_1^d x_0 y_0, & z_0^d x_1 y_0, & \ldots z_1^d x_1 y_0, & z_0^d x_0 y_1, & \ldots z_1^d x_0 y_1, & z_0^d x_1 y_1, & \ldots z_1^d x_1 y_1 \\
\hline
1 & a_1 & \ldots a_d & a_{d+1} & \ldots a_{2d+1} & 0 & \ldots 0 & 0 & \ldots 0 \\
0 & 0 & \ldots 0 & 0 & \ldots 0 & 1 & \ldots a_d & a_{d+1} & \ldots a_{2d+1} \\
1 & b_1 & \ldots b_d & 0 & \ldots 0 & b_{d+1} & \ldots b_{2d+1} & 0 & \ldots 0 \\
0 & 0 & \ldots 0 & 1 & \ldots b_d & 0 & \ldots 0 & b_{d+1} & \ldots b_{2d+1}
\end{array}$$

where the top row of zxy's indicates the ordered basis we have chosen for $S_z^d S_x^1 S_y^1$. Performing row operations modulo the square of the ideal generated by the a, b's, we obtain a matrix with the third row presenting the nonzero entries,

$$b_1 - a_1, \ldots, b_d - a_d, -a_{d+2}, \ldots, -a_{2d+1}, b_{d+2}, \ldots, b_{2d+1}.$$

This yields a total of $3d = \dim \Sigma_d - (d+2)$ elements in the ideal of 4×4 minors of the matrix that have independent differentials at the origin, as desired.

$\square$

The proof of the above Lemma shows that $4 \geq \text{rank}(\mu) \geq 3$ and that $\text{rank}(\mu) < 4$ precisely along $\Sigma_{d,d}$. We thus have a rational map

$$\rho : \Sigma_d \cdots \to \mathrm{Gr}(4, S_z^d S_x^1 S_y^1) \tag{7}$$

given by $\text{image}(\mu_{|(F,G)}) = \text{span}(y_0 F, y_1 F, x_0 G, x_1 G)$ for $(F, G) \in \Sigma_d - \Sigma_{d,d}$.

We include a proof of the Lemma below for lack of a reference.

Lemma 3.3. *Let X be a scheme and let $\psi : E \to F$ be a generically surjective map of locally free $\mathcal{O}_X$–modules. Put $r = $ rank F and let Y be the scheme of zeros of $\overset{r}{\wedge}\psi$. Let X' be the blowup of X along Y. Let $\pi : G^r(E) \to X$ be the Grassmann bundle over X with tautological exact sequence $0 \to \mathcal{S} \to \pi^* E \to \mathcal{Q} \to 0$, where $\mathcal{Q}$ denotes the universal locally free quotient sheaf of rank $= r$. Then*

1. *The restriction of ψ over $X - Y$ defines a section of the structure map $\pi_{|X-Y}$;*

2. *X' embeds in $G^r(E)$ as the closure of the image of the aforementioned section;*

3. $\psi_{|X'}$ *factors as* $E_{|X'} \to \mathcal{Q}_{|X'} \to F_{|X'}$.

Proof. Set $E' := \overset{r}{\wedge} E \otimes \overset{r}{\wedge} F^*$ (* indicating dual). The ideal I of Y is the image of the natural map, $\psi' := \overset{r}{\wedge}\psi \otimes \overset{r}{\wedge} F^* : E' \to \mathcal{O}_X$. It induces a surjection of sheaves of graded $\mathcal{O}_X$–algebras, $Sym_\bullet E' \to \oplus I^n$, whence a closed embedding ι : $X' = Proj(\oplus I^n) \subset Proj(Sym_\bullet E') = G^1(E') = G^1(\overset{r}{\wedge} E)$ of schemes over X. By construction, the tautological 1–quotient of E' over $G^1(E')$ restricts to X' as the composition $E' \otimes \mathcal{O}_{X'} \to I \otimes \mathcal{O}_{X'} \to \mathcal{O}_{X'}(1)$, the latter being equal to the invertible exceptional ideal. We further remark that, off the exceptional divisor, this composition coincides with ψ'. On the other hand, since $\psi_{|X-Y}$ is surjective, we obtain a section $\sigma : X - Y \to G^r(E)$. Composing it with the Plücker embedding $G^r(E) \subset G^1(\overset{r}{\wedge} E)$, we obtain that the tautological 1–quotient of $\overset{r}{\wedge} E$ restricts to $X - Y$ as $\overset{r}{\wedge}\psi$. Taking into account the identification $G^1(E') = G^1(\overset{r}{\wedge} E)$, we see that $\iota = \sigma$ holds over $X - Y$. Hence $\iota_{|X-Y}$ factors through $G^r(E)$. The hypothesis on ψ says $X - Y$ is schematically dense in X. Consequently, ι also factors through $G^r(E)$. Finally, since $\mathcal{S}_{|X'} \to E_{|X'} \to F_{|X'}$ is zero over $X - Y$, it vanishes everywhere, thereby producing the factorization stated for $\psi_{|X'}$.

□

Lemma 3.4. *Notation as in Prop. (3.1), we have:*

1. $\mathbf{\Sigma}^1_d$ embeds into $\mathbf{\Sigma}_d \times \mathbf{Gr}(4, S^d_z S^1_x S^1_y)$ as the closure of the graph of ρ (7);

2. *ρ extends to a morphism (still denoted by the same letter)*

$$\rho : \mathbf{\Sigma}^1_d \to \mathbf{Gr}(4, S^d_z S^1_x S^1_y);$$

3. *for all $\eta \in \mathbf{\Sigma}^1_d$ lying over $(F, G) \in \mathbf{\Sigma}_d$, the 4-dimensional vector space $\rho(\eta)$ contains the image of $\mu_{|(F,G)}$.*

Proof. The assertions follow from the previous Lemma by taking ψ as the dual of μ in the notation of Lemma 3.2, $X = \mathbf{\Sigma}_d$, and noticing that $G^4((S^d_z S^1_x S^1_y \otimes \mathcal{O}_X)^*) = \mathbf{\Sigma}_d \times \mathbf{Gr}(4, S^d_z S^1_x S^1_y)$.

Remark 3.5. It follows from the above that any point in Σ_d^1 can be written as a triplet (F, G, w), where (F, G) is in Σ_d and $w \in \mathbf{Gr}(4, S_z^d S_x^1 S_y^1)$ is a subspace of $S_z^d S_x^1 S_y^1$ of dimension 4 containing the span of $y_i F, x_j G$. The next result gives a description of the exceptional divisor of Σ_d^1. It also informs us what the subspace w may look like.

Lemma 3.6. *Fix nonzero polynomials $F = DF^0, G = DG^0$ with $D \in S_z^d$, $F^0 \in S_x^1$, $G^0 \in S_y^1$. Denote by V the quotient of the vector space,*

$$(S_z^d/\langle D\rangle) \otimes S_y^1 \oplus (S_z^d/\langle D\rangle) \otimes S_x^1$$

by the subspace consisting of pairs (A, B) of the form $A = G^0 C$, $B = -F^0 C$, with $C \in S_z^d/\langle D\rangle$. Then:

1. *the map*
$$\begin{array}{ccc} V & \longrightarrow & (S_z^d/\langle D\rangle) \otimes S_x^1 S_y^1 \\ (A, B) & \longmapsto & H = AF^0 + BG^0 \end{array} \tag{8}$$
is injective;

2. *if $\widetilde{H} \in S_z^d S_x^1 S_y^1$ is a representative of any nonzero H as above, the subspace of $S_z^d S_x^1 S_y^1$ spanned by $(y_0 F, y_1 F, x_0 G, x_1 G, \widetilde{H})$ is independent of the choice of the representative and is of dimension 4;*

3. *the fibre of the exceptional divisor of Σ_d^1 above (F, G) is isomorphic to $\mathbf{P}(V)$.*

Proof. We may assume (changing variables, *i.e.*, via the group action) that $F^0 = x_0$ and $G^0 = y_0$. Pick $A \in S_z^d S_y^1$, $B \in S_z^d S_x^1$. Let

$$\widetilde{H} = Ax_0 + By_0. \tag{9}$$

Suppose $\widetilde{H}$ represents the zero class in $(S_z^d/\langle D\rangle) \otimes S_x^1 S_y^1$. Then we may write $\widetilde{H} = \lambda D$ for some $\lambda \in S_x^1 S_y^1$. Say $\lambda = \alpha x_0 + \beta y_0 + \kappa x_1 y_1$, where $\alpha \in S_y^1$, $\beta \in S_x^1$ and κ is a constant. Since there is no $x_1 y_1$ term in $\widetilde{H}$, therefore $\kappa = 0$. From

$$Ax_0 + By_0 = (\alpha x_0 + \beta y_0)D \tag{10}$$

we get $A - \alpha D = Cy_0$, $B - \beta D = -Cx_0$ for some $C \in S_z^d$, whence (A, B) represents the zero class in V.

Let H be a nonzero class in the image of of the map in (8). By item *1* just proved, there exists a unique nonzero class v in V such that H is the image of v. Let (A, B) be a representative of v. Let $\widetilde{H}$ be as in equation (9). Let $\alpha \in S_y^1$, $\beta \in S_x^1$ and $C \in S_z^d$. Put $A' = \alpha D + Cy_0$, $B' = \beta D - Cx_0$ so that (A', B') represents the zero class in V. Put $\widetilde{H}' = (A + A')x_0 + (B + B')y_0$. One checks at once that $\widetilde{H}' - \widetilde{H}$ is in the $\mathrm{span}(y_0F, y_1F, x_0G, x_1G) = \mathrm{span}(x_0y_0D, x_0y_1D, x_1y_0D)$. Moreover, if $\widetilde{H}$ were in this 3-dimensional space, we would have a relation as in (10) which is impossible since H is a nonzero class. By the same token, one checks that for any $\widetilde{H}'$ of the form (9), if $\mathrm{span}(y_0F, y_1F, x_0G, x_1G, \widetilde{H}) = \mathrm{span}(y_0F, y_1F, x_0G, x_1G, \widetilde{H}')$ then $\widetilde{H}$ and $\widetilde{H}'$ yield the same element in $\mathbf{P}(V)$. Hence, we have an injective map

$$\begin{array}{rcl} \mathbf{P}(V) & \longrightarrow & \mathbf{Gr}(4, S_z^dS_x^1S_y^1) \\ (A, B) & \longmapsto & \mathrm{span}(y_0F, y_1F, x_0G, x_1G, \widetilde{H}). \end{array}$$

We now prove *3*. Say $B = \beta_0x_0 + \beta_1x_1$ with $\beta_i \in S_z^d$. Since $(A + \beta_0y_0, \beta_1x_1)$ and (A, B) represent the same class in V, we may assume $B = \beta_1x_1$. If A is divisible by D, we may set $A = 0$ (hence $\beta_1 \neq 0 \bmod \langle D\rangle$). With these choices, we now consider the curve

$$\{\gamma_t := (F_t = x_0D + tB, G_t = y_0D - tA)\} \subset \Sigma_d.$$

We claim that $\gamma_t \notin \Sigma_{d,d}$ for $t \neq 0$. Indeed, if γ_t were in $\Sigma_{d,d}$, there would be a z-factor of degree d in $x_0D + tx_1\beta_1$. This forces $t\beta_1$ and A to be both divisible by D, contradicting the assumption that (A, B) represents a nonzero class. It follows that $\{\gamma_t\}$ lifts to a curve $\{\gamma_t'\}$ in Σ_d^1. Since $\gamma_0 = (x_0D, y_0D)$ lies in $\Sigma_{d,d}$, it follows that γ_0' is in the exceptional divisor $\Sigma_{d,d}^1$. We proceed to identify the "w"-component (cf. Remark 3.5) of γ_0' in $\mathbf{Gr}(4, S_z^dS_x^1S_y^1)$. Computing

$$y_0F_t - x_0G_t = y_0(x_0D + tB) - x_0(y_0D - tA) = t\widetilde{H}$$

shows that for all $t \neq 0$ we have

$$\widetilde{H} \in \rho(F_t, G_t) = \mathrm{span}(y_0F_t, y_1F_t, x_0G_t, x_1G_t).$$

By continuity, we also get $\widetilde{H} \in \rho(\gamma_0')$.

Since the argument above shows in fact that $\mathbf{P}(V)$ is mapped to the image of the fibre of the exceptional divisor over (F, G), the lemma follows by counting the dimension of the space V in (8): $\dim V = 2d + 2d - d = \operatorname{codim}(\mathbf{\Sigma}_{d,d}, \mathbf{\Sigma}_d)$.

□

Remarks 3.7. *1.* The reader may recognize V as the normal space to $\mathbf{\Sigma}_{d,d}$ at the point (F^0D, G^0D).

2. We also observe for later use that, for H as in (8), we have that $\gcd(y_0F, y_1F, x_0G, x_1G, H)$ is a polynomial involving no x's or y's and of z-degree at most $d-1$.

Let us now return to the $d = 1$ case and proceed to the

Proof of (3.1). We may think of a point in $\mathbf{\Sigma}_1^1$ as a triplet (F, G, w), where w denotes a 4-dimensional subspace of $S_z^1S_x^1S_y^1$ containing the span of y_iF, x_jG. Away from the exceptional divisor, $\mathbf{\Sigma}_{1,1}^1$, we have that w is completely determined by F and G whereas on the exceptional divisor, w requires the extra generator H as in (8). Let I_w denote the ideal of S spanned by w. The map $w \mapsto I_w$ clearly restricts to the rational map Γ we had to $\mathbf{Hilb}^{3t+1}$ from Lemma 2.1, and will henceforth be denoted by the same letter.

To see that Γ cannot extend to $\mathbf{\Sigma}_{1,1}$, it suffices to exhibit one-parameter families $\gamma_{i,t}$ emanating from the same point, say $(z_0x_1, z_0y_1) \in \mathbf{\Sigma}_{1,1}$, such that the corresponding limit curves $\gamma_{1,0}$ and $\gamma_{2,0}$ are distinct points in $\mathbf{Hilb}^{3t+1}$. For instance, consider the one-parameter families,

$$\begin{aligned} \gamma_{1,t} &:= (z_0x_1 + tz_1x_0, z_0y_1) \\ \gamma_{2,t} &:= (z_0x_1, z_0y_1 + tz_1y_0). \end{aligned}$$

Proceeding as we did in (1.2), we get

$$\begin{aligned} \gamma_{1,0} = \lim\nolimits_{t\to 0}\gamma_{1,t} &= (z_0x_1, z_0y_1, z_1x_0y_1) \\ \gamma_{2,0} = \lim\nolimits_{t\to 0}\gamma_{2,t} &= (z_0x_1, z_0y_1, z_1x_1y_0) \end{aligned}$$

which are easily seen to be distinct curves in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$.

To prove assertion (2) of (3.1), it suffices to check the constancy of the Hilbert polynomial $(h(t) = 3t + 1)$. We know from Lemma (2.1) that the

Hilbert polynomial is correct on the complement, $\overset{\circ}{\mathbf{\Sigma}}_1$, of the exceptional divisor $\mathbf{\Sigma}^1_{1,1}$. Let $(F,G,H) \in \mathbf{\Sigma}^1_{1,1}$, with H as in (8). Without loss of generality we may choose coordinates such that $F = x_0z_0$ and $G = y_0z_0$. We compute a resolution for the ideal $I := (F,G,H)$ in S. Let $S^{i,j,k}$ denote $S^i_zS^j_xS^k_y$. Assuming a relation $AF + BG + CH = 0$ with $A, B, C \in S$, we may write

$$z_0(Ax_0 + By_0) = -CH.$$

Since $\gcd(z_0, H) = 1$ (cf. Remark 3.7), we obtain $C = C'z_0$ for some $C' \in S$. Substituting this into the previous equation and cancelling out z_0 we have

$$Ax_0 + By_0 = -C'H.$$

Now $H = \alpha x_0y_0 + \beta x_0y_1 + \gamma x_1y_0$ with $\alpha, \beta, \gamma \in S^1_z$. So

$$x_0(A + C'\alpha y_0 + C'\beta y_1) = -y_0(B + C'\gamma x_1)$$

and hence $B + C'\gamma x_1 = B'x_0$ for some $B' \in S$. Putting all these together we obtain

$$\begin{cases} A = -C'\alpha y_0 - C'\beta y_1 - B'y_0 \\ B = B'x_0 - C'\gamma x_1 \\ C = C'z_0. \end{cases} \tag{11}$$

thus producing the resolution,

$$0 \longrightarrow S \oplus S \longrightarrow S \oplus S \oplus S \longrightarrow I \longrightarrow 0,$$

where the first map is given by $(B', C') \mapsto (A, B, C)$ as in (11) and the second map is given by $(A, B, C) \mapsto AF + BG + CH$. Now $\dim S^{t,t,t} = (t+1)^3$. Also from

$$0 \to S^{t-1,t-1,t-1} \oplus S^{t-2,t-1,t-1} \to S^{t-1,t-1,t} \oplus S^{t-1,t,t-1} \oplus S^{t-1,t-1,t-1} \to I^{t,t,t} \to 0,$$

we obtain

$$\dim I^{t,t,t} = 2t^2(t+1) + t^3 - (t^3 + t^2(t-1)) = t^3 + 3t^2$$

and thus

$$h(t) = \dim S^{t,t,t} - \dim I^{t,t,t} = 3t + 1$$

as desired. This shows that Σ_1^1 parametrizes a flat family, thereby producing the morphism Γ into $\mathbf{Hilb}^{3t+1}$.

□

Remarks.
(1) We believe $\Gamma : \Sigma_1^1 \to \mathbf{Hilb}^{3t+1}$ is in fact an isomorphism onto its image.
(2) In a preliminary version of this note we had conjectured the following to be true:

There is a sequence of blowups along smooth centres:

$$\begin{array}{ccccccc} \Sigma_d & \leftarrow & \Sigma_d^1 & \leftarrow & \cdots & \leftarrow & \Sigma_d^d \\ & & & & & & \downarrow \\ & & & & & & \mathbf{Hilb}^{(2d+1)t+1} \end{array}$$

where the vertical arrow extends the rational map Γ defined in (2.1). However, we have checked that for $d = 2$ one needs in fact two more blowups. In this case the centres are again smooth and defined by the singularities of an appropriate multiplication map. Blowing up $\Sigma_{2,2} \subset \Sigma_2$ produces (as in the $d = 1$ case) a rank 4 locally split subbundle $A_4 \subset S_z^2 S_x^1 S_y^1$. Next, we look at the multiplication map $A_4 \otimes S_z^1 \to S_z^3 S_x^1 S_y^1$ and check that it drops rank on the proper transform $\Sigma_{2,1}^1$ of $\Sigma_{2,1}$. Blowing it up, produces Σ_2^2 endowed with a rank 8 locally split subbundle $A_8 \subset S_z^3 S_x^1 S_y^1$. In order to produce a flat family of curves in $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ completing the family of graphs of bidegree (2,2) one still needs to blowup the loci where multiplication by S_x^1 and then by S_y^1 (in either order) drops rank. It turns out that the map from the resulting parameter space to the appropriate **Hilb** contracts certain loci. In fact, it is not clear yet what kind of extra structure has been added to the boundary elements. We hope to report on this elsewhere.

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