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THE UNIT GROUP OF INTEGRAL GROUP RINGS: GENERATORS OF SUBGROUPS OF FINITE INDEX.

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Let $\mathbf{Z}G$ be the integral group ring of a finite group G, that is

$$\mathbf{Z}G = \{\sum_{g\in G} z_g g \mid z_g \in \mathbf{Z},\ g \in G\},$$

with operations defined by

$$\left(\sum_{g\in G} z_g g\right) + \left(\sum_{g\in G} s_g g\right) = \sum_{g\in G}(z_g + s_g)g;$$

$$\left(\sum_{g\in G} z_g g\right) \cdot \left(\sum_{h\in G} s_h h\right) = \sum_{x\in G}\left(\sum_{g,h\in G,\ gh=x} z_g s_h\right) x.$$

In particular

$$(z_g g)(s_h h) = (z_g s_h)gh.$$

As the only information in the group ring comes from the group G, the integral group ring is a natural object for interplay between group theory and ring theory.

In this lecture we look at the unit group $\mathcal{U}(\mathbf{Z}G)$, that is

$$\mathcal{U}(\mathbf{Z}G) = \{\alpha \in \mathbf{Z}G \mid \alpha\beta = 1 = \beta\alpha, \text{ for some } \beta \in \mathbf{Z}G\}.$$

One of the reasons for the interest in the unit group is that sometimes knowledge about its structure (such as having a torsion-free normal complement) enables one to prove the isomorphism theorem, that is

$$\mathbf{Z}G \cong \mathbf{Z}H \text{ implies } G \cong H.$$

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Clearly, every element $\pm g$, $g \in G$, is a unit in $\mathbf{Z}G$. These are called the trivial units.

We deal with the commutative and non-commutative case seperatly.

Commutative Groups

Let us first look at an example (see Example 15.6, page 270, in [9]). Let A be the cyclic group of order 8; so $A =< a \mid a^8 = 1 >$. Then $\mathcal{U}(\mathbf{Z}A) = \pm A \times < a^6 + 2a^5 + a^4 - a^2 - a - 1 >$. Hence, modulo the trivial units, $\mathcal{U}(\mathbf{Z}A)$ is a free group of rank 1.

In 1940, Higman determined the strucutre of the unit group for any abelian group. For a proof we refer to [15].

Theorem 1 *If A is a finite abelian group, then*

$$\mathcal{U}(\mathbf{Z}A) = \pm A \times F,$$

where F is a free abelian group of rank

$$\rho(F) = \frac{1}{2}\left(|A| + n_2 - 2c + 1\right),$$

where n_2 is the number of elements of order 2 in A, and c is the number of cyclic subgroups of A.

In the example, $|A| = 8$, $n_2 = 1$, $c = 4$ and thus the rank of the free abelian part is indeed 1.

One can now determine when the free abelian part is trivial, that is all units are trivial. This, ultimately, comes down to determine when the ring of cyclotomic integers $\mathbf{Z}[\xi]$, ξ a primitive root of unity, has no units of infinite order. The answer to this comes from the Dirichlet Unit Theorem. Hence one obtains the following result of Higman (cf. [15]). We also include the answer for non-commutative groups.

Theorem 2 *1. Let A be a finite abelian group. Then $\mathcal{U}(\mathbf{Z}A) = \pm A$ if and only if A has exponent $1, 2, 3, 4$ or 6.*

2. Let G be a non-abelian finite group. Then $\mathcal{U}(\mathbf{Z}G) = \pm G$ if and only if $G = K_8 \times E$, where K_8 is the quaternion group of order 8, and E is an elementary abelian 2-group.

So in the commutative case one has a complete description of the structure of the unit group. However, one would now also want to have a description of the generators of the free abelian part of the unit group. However, this is a very difficult problem for it is closely related to a desription of the fundamental units of the unit group of the cyclotomic integers.

Therefore, as a compromise, one would like to have a description of generators of a subgroup of finite index in the unit group of an integral group ring. Such a description has been found by H.Bass in the commutative case. For this let us recall a construction of some units, called Bass cyclic units, in integral group rings of finite cyclic groups (and thus in every finite group).

Let G be a finite group of order m. Let $a \in G$ and $n = |<a>|$. Also let ξ be a primitive root of unity of order n, and $1 < i < n$ with $(i, n) = 1$. Then

$$\frac{\xi^i - 1}{\xi - 1} = 1 + \xi + \cdots + \xi^{i-1}$$

has an inverse in $\mathbf{Z}[\xi]$, namely

$$\frac{\xi - 1}{\xi^i - 1} = \frac{\xi^{ik} - 1}{\xi^i - 1} = 1 + \xi^i + \cdots + \xi^{i(k-1)},$$

where $ik \equiv 1 \ (mod\ n)$. Take $v = 1 + a + \cdots + a^{i-1} \in \mathbf{Z} < a >$. It is well-known that

$$\mathbf{Q} < a > \cong \oplus_{d|n} \mathbf{Q}[\xi^d].$$

Set $M = \oplus_{d|n} \mathbf{Z}[\xi^d]$; then $\mathbf{Z} < a > \subseteq M$ and the projection of v in every component of M is a unit except when $d = n$, and then the projection is i. Since $(i, n) = 1$ and $\varphi(n)|\varphi(m)$, $i^{\varphi(m)} = 1 + ln$, for some $l \in \mathbf{Z}$. So we modify v to

$$u = (1 + a + \cdots + a^{i-1})^{\varphi(m)} - l\hat{a},$$

where $\hat{a} = 1 + a + \cdots + a^{n-1}$. Then the projection of u in every component of M is a unit. Hence u is a unit in M. Since $u \in \mathbf{Z}G$ it follows that u is a unit in $\mathbf{Z}G$. All the units of the type u are called the Bass cyclic units.

Theorem 3 *(H. Bass [1]) Let A be a finite abelian group, then the group generated by the Bass cyclic units is of finite index in $\mathcal{U}(\mathbf{Z}A)$.*

Non-commutative Groups

First we consider an example. Let $S_3 = < a, b \mid a^3 = 1,\ b^2 = 1,\ ba = a^2 b >$, the symmetric group on three letters. Consider the following elements in $\mathbf{Z}S_3$:

$$\begin{aligned} u_1 &= 1 + (1-b)a(1+b) \\ u_2 &= 1 + (1-ba)a(1+ba) \\ u_3 &= 1 + (1-ba^2)a(1+ba^2) \end{aligned}$$

These elements are of the type $1 + \alpha$ with $\alpha^2 = 0$. The latter follows because for any element g of order 2 we have that $(1+g)(1-g) = 0$ (notice that this holds in any group G).

Proposition 4 *(E. Jespers and M. Parmenter [8])*

$$\mathcal{U}(\mathbf{Z}S_3) = \pm S_3 < u_1, u_2, u_3 >,$$

and $< u_1, u_2, u_3 >$ is a free group of rank 3.

There are other groups for which the unit group has been calculated. A survey of those described up to a few years ago can be found in [16]. More recently, descriptions for some groups have been obtained in [2, 4, 5, 7].

During the last few years there has been a lot of interest in computing explicit generators for a subgroup of finite index of the unit group (see for example [10, 11, 12, 13, 14]. The units that play an important role are:

1. the Bass cyclic units;

2. the bicyclic units (introduced in [12]): $1 + (1-g)h\hat{g},\ g, h \in G$;

3. the units of the type (used in [3]) $1 + \hat{g}h(1-g)$, $g, h \in G$.

It was recently shown by Jespers and Leal [3] that these three types of units generate a subgroup of finite index for a very large class of groups. Important in the method of proof is the structure of the rational group algebra. The famous Maschke and Wedderburn-Artin theorems yield

$$\mathbf{Q}G \cong \oplus_i \mathbf{Q}Ge_i \cong \oplus_i M_{n_i}(D_i),$$

where $M_{n_i}(D_i)$ is a $n_i \times n_i$-matrix ring over a division ring D_i, and e_i runs through the primitive central idempotents of the rational group algebra $\mathbf{Q}G$. If one would know what part of $\mathbf{Q}G$ (written as matrices) corresponds with the integral group ring $\mathbf{Z}G$, then several problems concerning units can be handled (see for example [2, 4, 5, 6, 7, 8]). For this one chooses a $\mathbf{Q}$-basis (written as rational group ring elements) of each $M_{n_i}(D_i)$, and represents the elements of $\mathbf{Z}Ge_i$ as matrices, and then by some elementary linear algebra determines precisely those matrices that represent integral group ring elements. In this way one can sometimes get the full unit group. However, it turns out that if one knows one non-central idempotent of each non-commutative $M_{n_i}(D_i)$, $n_i > 1$, then in some sense such elementary matrices are determined. One can then prove the following result (using some important results of Bass, Milnor, Serre, Vaserstein, Bak and Rehman on subgroups of linear groups over maximal orders).

Theorem 5 *(Jespers-Leal [3]) Let G be a finite group such that the rational group algebra $\mathbf{Q}G$ does not have simple components of the following type:*

1. *a non-commutative division algebra other than a totally definite quaternion algebra;*

2. *a 2×2-matrix ring over the rationals;*

3. *a 2×2-matrix ring over a quadratic imaginary field extension of the rationals;*

4. a 2×2-matrix ring over a noncommutative division algebra.

For every primitive central idempotent e for which $\mathbf{Q}Ge$ is not a division ring, let f be a non-central idempotent and n_f a positive integer such that $n_f f \in \mathbf{Z}G$. Then the subgroup generated by the Bass cyclic units and all the units of the form

$$1 + n_f^2 fh(1-f) \quad \textit{and} \quad 1 + n_f^2(1-f)hf,$$

$h \in G$, is of finite index in $\mathcal{U}(\mathbf{Z}G)$.

In many cases one can show that there is an idempotent of the form $\frac{1}{ord(g)}\hat{g}e$, $g \in G$. If so, one obtains very specific generators. We give a few examples obtained from [3].

Corollary 6 *Let G be a finite group of odd order. If G has no non-abelian homomorphic image which is fixed point free (for example G is a nilpotent group), then the subgroup generated by the Bass cyclic units together with the units of the form*

$$1 + (1-g)h\hat{g} \quad \textit{and} \quad 1 + \hat{g}h(1-g),$$

$g, h \in G$, is of finite index in $\mathcal{U}(\mathbf{Z}G)$.

Ritter and Sehgal in [12] showed that for nilpotent groups of odd order it is sufficient to work with the Bass cyclic units and the bicyclic units.

Corollary 7 *Let S_n be the symmetric group of degree n, and let $a = (1\ 2)$ be a transposition. If $n \geq 5$, then the units of the form*

$$1 + (1-a)h(1+a) \quad \textit{and} \quad 1 + (1+a)h(1-a),$$

$h \in S_n$, generate a subgroup of finite index in $\mathcal{U}(\mathbf{Z}S_n)$.

The number of generators for a subgroup of finite index in the case of symmetric groups is much less than the result obtained by Ritter and Sehgal in

[13], where it was shown that the subgroup generated by the Bass cyclic units together with all the bicyclic units is of finite index in $\mathcal{U}(\mathbf{Z}S_n)$.

There are many other classes of groups to which the main theorem can be applied, for example metacyclic groups and simple groups.

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