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CHAIN RINGS AND DISTRIBUTIVE RINGS

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1. Introduction

Throughout this paper every ring has an identity element. A ring R is said to be a right chain ring if the lattice of right ideals of R is linearly ordered by set inclusion. This is equivalent to the following condition: for any a, $b \in R$ we have either $aR \subseteq bR$ or $bR \subseteq aR$.

A left chain ring is defined similarly. A chain ring is a ring which is both: a right and a left chain ring. These rings are obvious generalizations of commutative valuation domains as well as of division rings.

On the other hand, a ring R is said to be a right distributive ring (right D-ring for short) if the lattice of right ideals is distributive. This means that for all right ideals A, B and C of R we have $A \cap (B + C) = (A \cap B) + (A \cap C)$ (equivalently, $A + (B \cap C) = (A + B) \cap (A + C)$). Left D-rings and D-rings are defined similarly as above. The commutative D-domains are the Prüfer domains.

It is clear that a right chain ring is a right D-ring.

Right chain rings and right D-rings have extensively been studied in the last 30 years. Concerning right chain rings the reader can consult [BBT] and the literature quoted therein (do not forget [P]). The study of right D-rings should begin by [St]. Next we may quote [B]. In this paper the author proved that if R is a either a right Noetherian ring or a domain, then R is a right D-ring if and only if for every maximal right ideal M of R the ring of fractions R_M is defined and is a right chain ring. There are also several recent papers which we want to quote ([FT_1], [FT_2], [M], [MP]).

In this survey we want to present an interesting open question raised 30 years ago by E. Posner [P].

2. Main Question.

An ideal (when we say ideal we mean two-sided ideal) P of R is said to be prime if $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, for ideals A and B of R. Equivalently, for ideals I and K of $\overline{R} = R/P$ we have $IK = 0$ implies either $I = 0$ or $K = 0$. In this case $\overline{R}$ is said to be a prime ring.

Also, an ideal P of R is said to be completely prime if $ab \in P$ implies either $a \in P$ or $b \in P$, for elements a, b in R. A domain is a ring in which the ideal (0) is completely prime.

If R is a commutative ring, then the above definitions are equivalent. In general, every completely prime ideal is prime but the converse is not true. To see this it is enough to consider the ideal (0) in a matrix ring over a field.

The question implicitely raised in the paper by Posner [P] is the following:

Question 1: Do there exist prime ideals in right chain rings which are not completely prime?

In 1983 Dubrovin [D] gave an example of a right chain ring which is prime but has nilpotent elements. This example would answer the above question (consider the ideal (0)), but there is a gap in his construction and the example is wrong as it was proved in 1990 (see [Sc]). So the question is still open.

On the other hand, there are two recent papers on right D-rings ([M], [MP]) in which an interesting condition is introduced. We say that a right D-ring R satisfies (MP) if:

(MP) There exists a completely prime ideal Q of R contained in the Jacobson radical $J(R)$.

Let us point out that (MP) is automatically satisfied for right chain rings.

We observe in our papers [FT_1] and [FT_2] that when (MP) is assumed the behaviour of the prime ideals of R contained in Q is quite similar to that of prime ideals in right chain rings. In particular (and this was already proved by Mazurek [M]), the prime radical $L(R)$ of R is prime in this case.

Thus we may state question 1 in the following more general form.

Question 2. Let R be a right D-ring which has a completely prime ideal Q contained in $J(R)$ and let P be a prime ideal with $P \subseteq Q$. Is necessarily P completely prime?

In particular, the following is also interesting:

Question 3. Let R be a right D-ring which satisfies (MP). Is the prime radical L(R) necessarily completely prime?

Of course, the above questions are also still open. We want to present some results which we obtained in [FT_1] and [FT_2].

Remark. (added in proof) After this survey was finished I was informed that there is a recent result by Dubrovin showing that there exists a right chain ring having prime ideals which are not completely prime. This result is not published yet.

3. Results

If S is a subset of R, the right annihilator of S is defined as $r(S) = \{x \in R : Sx = 0\}$. Then $r(S)$ is a right ideal. If S contains just one element a we simply write r(a).

A right ideal I of R is said to be a right annihilator (resp. principal right annihilator) if there exists a subset S (resp. an element $a \in R$) such that $I = r(S)$ (resp. $I = r(a)$).

We say that R satisfies a.c.c. on right annihilators if every ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ of right annihilators stabilizes. We define similarly a.c.c. on principal right annihilators, descending chain condition (d.c.c.) on right annihilators and d.c.c. on principal right annihilators.

Left annihilators and a.c.c. and d.c.c. conditions on left annihilators are defined obviously.

Coming back to the paper by Posner [P]. He claimed in his Theorem 2 that if R is a right chain ring which satisfies either a.c.c. or d.c.c. on (right or left) annihilators, then the prime radical $L(R)$ is completely prime. But there is a gap in the proof and the result is actually proved only for prime right chain rings as well as for right Noetherian right chain rings. The problem is that he does not prove that the annihilator chain conditions are inherited by $R/L(R)$ and this fact is not trivial at all.

When I went to Germany to visit G. Törner we started by trying to find a proof of the Posner assertion. We suceeded by proving the following more general result.

Theorem 1. ([FT$_1$], Theorem). *Let R be a right D-ring which satisfies (MP) and either a.c.c. or d.c.c. on principal right annihilators. Then the prime radical of R equals the right singular ideal of R and is completely prime and nilpotent.*

The right singular ideal of a ring R is defined as the set of all the elements $a \in R$ such that $r(a)$ is an essential right ideal ([G], p.30-36).

We also proved some other results concerning Questions 2 and 3 in [FT$_2$].

We say that R is algebraic (resp. almost integral) over its center C if for every $a \in R$ there exists a non-zero polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in C[x]$ (resp. which has at least one coefficient a_i which is invertible, for $1 \leq i \leq n$) with $f(a) = 0$.

Theorem 2. ([FT$_2$], Theorem 4.1). *Let R be a prime right D-ring which is algebraic over the center. Then R is a domain.*

Theorem 3. ([FT$_2$], Theorem 4.3). *Let R be a right D-ring which is almost integral over the center. Then every one-sided prime ideal of R is two-sided and completely prime.*

Finally, we proved also the following.

Theorem 4. ([FT_2], Theorem 5.1). *Let R be a right D-ring which satisfies (MP). If the index of the subgroup $U \cap C$ in U is finite, then the prime radical of R is completely prime (here U is the group of units of R).*

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