Matemática Contemporânea, Vol. 6, 219–224
http://doi.org/10.21711/231766361994/rmc614

MINIMAL POLYNOMIAL IDENTITIES OF BERNSTEIN ALGEBRAS A COMPUTER APPROACH

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1. Bernstein Algebras.

A Bernstein algebra is a pair (A, ω), where A is a commutative nonassociative algebra over a field K and $\omega : A \to K$ is a nonzero algebra homomorphism such that $x^2x^2 = \omega(x)^2x^2$ (for all $x \in A$). It is well-known that if (A, ω) is a Bernstein algebra then the homomorphism ω is unique. See the papers [3,4,5,6,11].

In nonassociative algebra the research is concentrated mainly in the study of varieties of algebras, i.e., the study of algebras that satisfy a set of polynomial identities. The class of Bernstein algebras is not a variety of algebras since it is not true that every subalgebra of a Bernstein algebra is a Bernstein algebra. Thus an interesting problem is to find the minimal variety of algebras containing Bernstein algebras. In other words, the problem is to find the minimal identities, i.e., lowest degree polynomial identities satisfied by any Bernstein algebra which are not consequence of commutativity.

Let (A, ω) be a Bernstein algebra. If a, b, c are elements of A the associator (a, b, c) is defined by $(a, b, c) = ab.c - a.bc$. For all $x, y \in A$ we have

$$(x^2x^2, y, x^2) = \omega(x^2)(x^2, y, x^2) = 0$$

and thus A satisfies an identity of degree 7. This shows that the degree of minimal identities is ≤ 7. In [13] we proved that the degree is 6 and that the minimal identities are:

$$(y, x^2x^2, x) - 2(yx^2, x, x^2) + 2y(x^2, x^2, x) + 2(x, yx.x, x^2) = 0, \tag{1}$$

$$(x^2x^2, y, x) - 2(x^2, y, x)x^2 = 0. \tag{2}$$

The Bernstein algebra (A,ω) is called exceptional if it satisfies $(xy)(xy) = \omega(xy)xy$ (for all $x,y \in A$) and normal when satisfies $x^2y = \omega(x)xy$ (for all $x,y \in A$). Exceptional Bernstein algebras satisfy only one minimal identity of degree 5 which is

$$\begin{aligned}(ae,bd,c) &+ (be,cd,a) + (ce,ad,b)+\\ (a,bd,ce) &+ (b,cd,ae) + (c,ad,be)+\\ (a,e,b).cd &+ (b,e,c).ad + (c,e,a).bd = 0.\end{aligned} \tag{3}$$

The minimal identities of normal Bernstein algebras have degree 4 and are the following:

$$x^2x^2 = x^3x, \tag{4}$$

$$3yx^2.x = 2(yx.x)x + yx^3, \tag{5}$$

$$2xy.xy = x^2y.y + y^2x.x, \tag{6}$$

$$(x,yz,t) + (z,yt,x) + (t,yx,z) = 0. \tag{7}$$

Identity (6) implies (4). Identities (5), (6), (7) are independent.

Nonassociative algebras that satisfy the identity (4) or (5) were studied extensively (see, for instance, Albert [1] and Hentzel-Peresi [10]). Bernstein algebras which satisfy (4) have been characterized by Walcher [9] and those satisfying (5) by Correa [12].

The problem of finding the minimal identities of Bernstein algebras, although simple from the theoretical standpoint, is complicated from the computational point of view. To solve the problem we have to handle 34 equations of degree 6. This is done by using the computational method described below.

2. Computational Method.

This method was introduced by Hentzel [7] in 1977 and consists in converting equations into matrices which can be manipulated quickly and easily on the computer. In [13] the method was slightly modified in order to increase its efficiency and to make the conversion of equations into matrices and vice-versa much clear. The method is general and can be applied to solve other problems in algebra.

We denote by S_n the symmetric group and by FS_n the group algebra. When the characteristic of the field F is zero, FS_n is semisimple and then it is isomorphic to a direct sum of matrix algebras. A map that takes FS_n into one of this summands is called an irreducible representation of FS_n.

Clifton [8] gives a procedure which calculates an irreducible representation. For each permutation π the procedure calculates a matrix A_π. The representation is then given by the map $\pi \to A_I^{-1}A_\pi$ (where I denotes the identity permutation).

In what follows we use the map $\pi \to A_\pi$, which is not a representation since $A_{\pi\sigma} = A_\pi A_I^{-1}A_\sigma$, but gives a correspondence between permutations and matrices good enough for our purposes.

Let $x_1, \ldots, x_n$ be indeterminates. Let $f(x_1, \ldots, x_n)$ be an expression where each indeterminate x_i appears once in each term. Supose that the terms of the expression can be classified in c diffeent types which we denote by $T_1, \ldots, T_c$. Thus we may represent $f(x_1, \ldots, x_n)$ as the direct sum

$$f_1(x_1, \ldots, x_n) \oplus f_2(x_1, \ldots, x_n) \oplus \ldots \oplus f_c(x_1, \ldots, x_n)$$

where in the expression $f_i(x_1, \ldots, x_n)$ appear only terms of type T_i. Considering how the positions of the variables $x_1, x_2, \ldots, x_n$ are changed we identify $f_i(x_1, \ldots, x_n)$ with the element $\sum_{\pi \in S_n} \alpha_\pi^{(i)}\pi$ of FS_n.

Using this identification, we represent an expression $f(x_1, \ldots, x_n)$ as an element of $FS_n \oplus \ldots \oplus FS_n$. Given the expressions $f^{(1)}(x_1, \ldots, x_n), \ldots, f^{(k+1)}(x_1, \ldots, x_n)$, let L be the FS_n-module generated by the first k expressions. Thus the expression $f^{(k+1)}$ is a consequence of $f^{(1)}, \ldots, f^{(k)}$ if and only if $f^{(k+1)} \in L$. The map $\pi \to A_\pi$ which takes FS_n to the algebra $M_f(F)$ of all $f \times f$ matrices induces the map

$$P : FS_n \oplus \ldots \oplus FS_n \to M_f(F) \oplus \ldots \oplus M_f(F)$$

which takes L to $P(L)$. Note that an element of $P(L)$ is a direct sum of c $f \times f$

matrices. It follows that $f^{(k+1)} \in L$ if and only if $P(f^{(k+1)}) \in P(L)$ for all maps P determined by all maps $\pi \to A_\pi$.

We thus have an efficient method to perform calculations with equations. For each map $\pi \to A_\pi$, the equation $f^{(i)}(x_1, \ldots, x_n) = 0$ is represented by a direct sum of $f \times f$ matrices. Considering all equations $f^{(1)}(x_1, \ldots, x_n) = 0, \ldots, f^{(k+1)}(x_1, \ldots, x_n) = 0$ we obtain a block matrix. We calculate the rank of this matrix including and not including the rows that come from the equation $f^{(k+1)}(x_1, \ldots, x_n) = 0$. If the rank is the same for all maps $\pi \to A_\pi$ then the $k+1$-th equation is a consequence of the first k equations.

Professor Irvin Roy Hentzel of Iowa State University created a software called CRUNCH which calculates the block matrix and reduces it to its row canonical form. The input is a set of equations and the output is the row canonical form for each map $\pi \to A_\pi$.

We used the method to search for degree five polynomial identities which hold for all Bernstein algebras but is not consequence of commutativity. We found none. We then searched for identities of degree 6.

Using the equation

$$\begin{aligned} g(a,b,c,d) &= 2\{ab.cd + ac.bd + ad.bc\} - \{\omega(ab)cd + \omega(cd)ab + \omega(ac)bd + \\ &\quad \omega(bd)ac + \omega(ad)bc + \omega(bc)ad\} = 0 \end{aligned}$$

(the linearized form of equations $x^2x^2 - \omega(x)^2x^2 = 0$) we obtain 8 equations of degree 6:

$$\begin{array}{ll} g(a,b,c,d)e.f = 0, & g(a,b,c,d).ef = 0, \\ g(ae,b,c,d)f = 0, & g(ae,bf,c,d) = 0, \\ g(ae.f,b,c,d) = 0, & \omega(f)g(a,b,c,d)e = 0, \\ \omega(f)g(ae,b,c,d) = 0, & \omega(ef)g(a,b,c,d) = 0. \end{array}$$

These equations involve 11 types:

1. $\omega(R)(RR.RR)R$	2. $\omega(R)(RR.R)(RR)$	3. $\omega(RR)(RR.R)R$
4. $\omega(RR)(RR.RR)$	5. $\omega(RRR)(RR.R)$	6. $\omega(RRRR)RR$
7. $(RR.RR)R.R$	8. $(RR.RR).RR$	9. $(RR.R)R.RR$
10. $(RR.R)(RR.R)$	11. $((RR.R).RR)R$	

Using only the commutative law we get 26 equations which involve these types. For example, for type 4 we obtain:

$$\omega(ab)(cd.ef) - \omega(ba)(cd.ef) = 0,$$
$$\omega(ab)(cd.ef) - \omega(ab)(dc.ef) = 0,$$
$$\omega(ab)(cd.ef) - \omega(ab)(ef.cd) = 0.$$

We have 34 equations of degree 6. For each map $\pi \longrightarrow A_\pi$, we use these equations to create a matrix which contains 34×11 $f \times f$ blocks. The first $26f$ rows of this matrix contain the equations from the commutative law. We reduce these $26f$ rows to row canonical form. These reduced rows represent all equations implied by commutativity. Then we reduce the $34f$ rows to row canonical form. The additional stairstep ones which appear in this last row canonical form represent all the equations satisfied by all Bernstein algebras (and that are not implied by the commutative law).

We look for additional stairstep ones which are located under the types not involving the homomorphism ω, i.e., types 7, 8, 9, 10 and 11. Finally we verify that these stairstep ones are determined by identities (1) and (2).

A similar procedure was used to obtain equations (3) - (7).

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