

ALGORITHMIC ASPECTS OF CHARACTER THEORY

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Characters of finite groups play an important rôle in many applications in Algebra and Number Theory. Both the construction of character tables (ordinary ones, which describe the complex irreducible representations, and modular Brauer character tables, which describe the irreducible representations over finite fields) and their applications usually involve a large amount of computations. In order to deal with these efficiently various algorithms have been designed and implemented in the Computer Algebra System GAP (“Groups, Algorithms, Programming”, cf. [7]) which has been developed in Aachen for the last couple of years.

One of the main motivations for Computer Algebra is the search for experimental data to support general conjectures in the theory (or even to find patterns which might help to formulate such conjectures) or otherwise to find counterexamples. As examples of the latter, where the character theoretic part of GAP and in particular its large library of character tables turned out to be very useful, the following problems were considered.

Question 1. (Ferguson, Isaacs [2]) Is every multiple of a primitive character of a finite group also primitive?

Here a primitive character of a finite group is one which, by definition, is not induced from any character of a proper subgroup.

Question 2. (Janusz [5]) Is it true that for every irreducible (complex) character χ of a finite group G there is a character φ of a proper subgroup H such

that

$$(\chi_H, \varphi) = 1?$$

The second question was posed again by various authors in the last 25 years, since it is of relevance for many applications; e.g. it came up lately again in the work of J Ritter and S. Sehgal on units in group rings over cyclotomic integers.

Both questions have positive answers for solvable groups. For the first one this is the main content of [2]. For the second one, this is obvious by Clifford's theorem (cf. [4]) choosing H to be a normal subgroup of G of prime index. Then all constituents of χ_H occur with multiplicity one. Also the answer to both questions is "yes" for many small non-solvable groups.

But in general the answer to both questions is "no". The sporadic groups J_2 , Ru , and Suz each have an irreducible primitive character (which is χ_{20} , χ_{33} , and χ_{36} , respectively in the notation of the ATLAS [1]) multiples of which (in fact, doubles) are imprimitive. As for the second question, the first counterexample was found by O. Bonten (Aachen), it was the sporadic simple group J_4 . Another example, which I found recently, is the Lyons group Ly .

Thus both questions provide instances of the phenomenon where one is likely to be misled to false conjectures by just looking at solvable groups or groups of relatively small order.

For answering the questions above it was essential to have (for some interesting examples) not only the character table of the group G one is dealing with (as it is contained in the ATLAS for the examples above) but also the character tables of some or all maximal subgroups of G . This has proved to be extremely useful also for many other applications. One such application is to the Inverse Problem of Galois Theory (cf. e.g. [6]). Another one is the computation of decomposition numbers and Brauer character tables using the computer package MOC [3] developed in Aachen; in this case the characters of maximal subgroups are useful because they give rise to projective characters (characters of projective modules) upon induction under suitable conditions. But there are numerous other applications, too. Because of all this, work is under way in Aachen to compute the character tables of the known maximal subgroups of the

sporadic simple groups. At present the work is complete for roughly two third of the sporadic simple groups. The tables are generally accessible via GAP.

References

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