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ELLIPTIC SURFACES AND THE MORDELL-WEIL GROUP*

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Contents

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These notes, although revised somewhat after the lectures were given, represent a faithful record of the mini-course on elliptic surfaces which was delivered in August 1992 at the 12^{th} Escola de Álgebra. I am happy to express my deep gratitude to the organizers, especially Professors Avritzer and Vainsencher who were responsible for the algebraic geometry session, and to Professor Spira who was the over-all coordinator. The week we all spent in Diamantina was one of the most mathematically stimulating and personally enjoyable I can remember, and I was honored to have been a part of the program.

1 Elliptic Curves, Elliptic Surfaces, and Classification Questions

1.1 Curves and Elliptic Curves

Compact complex curves are usually classified, at least initially, by the topological genus g, which can be any natural number: $g \geq 0$. However this is not the coarsest classification possible. In fact, they can be classified in the broadest way into three classes, which arise in several ways. Possibly the easiest way to see the three classes is to classify a compact complex curve by its universal covering space, which is then a simply connected (but not necessarily compact) Riemann surface. The only simply connected Riemann surfaces, up to analytic isomorphism, are the Riemann sphere, the complex plane, and the unit disc in the complex plane. These three possibilities for the universal cover give the three broad classes.

These three broad classes are determined by the genus. If a curve has genus 0, then it is isomorphic to the Riemann sphere. If a curve has genus 1, then its universal cover is the complex plane. If a curve has genus at least 2, then its universal cover is the disc.

A second way of seeing this classification is by studying the tangent bundle of the curve. For the Riemann sphere with genus 0, the tangent bundle has positive degree (degree 2). For a curve of genus 1, the tangent bundle is trivial (and has degree 0). For curves of genus 2 or more, the tangent bundle has negative degree (equal to $2-2g$).

In these lectures I will concentrate on curves of genus 1 and in particular 1-dimensional families of such curves. Recall that a curve of genus 1 can be made into a group; this is inherited from the group law (of addition) on the universal cover $\mathbf{C}$. Taking the group law into account, we have the following terminology.

An *elliptic curve* $(E, 0)$ over a field K is a complete curve E of genus one defined over K, together with a given point 0 defined over K. One is often sloppy and refers to the elliptic curve as E alone, suppressing the given point in the notation.

Note that we have called the chosen point 0; it is usually taken to be the origin of the group law on the K-rational points of the elliptic curve.

There are several quite common ways that elliptic curves arise in nature.

Example 1.1.1

Fix a non-real complex number τ, and denote by $\Lambda(\tau)$ the subgroup of $\mathbf{C}$ generated by 1 and τ:

$$\Lambda(\tau) = \mathbf{Z} \oplus \mathbf{Z}\tau.$$

Then $E = \mathbf{C}/\Lambda(\tau)$ is an elliptic curve over $\mathbf{C}$. The chosen point is of course the class of 0.

Example 1.1.2

Let K be any field, and let $E \subset \mathbf{P}^2_K$ be a smooth cubic curve with a given flex point p defined over K. Then (E, p) is an elliptic curve over K.

Example 1.1.3

Let K be any field of characteristic unequal to 2, and let E be the double cover of $\mathbf{P}^1_K$ branched over ∞ and three other distinct points, which as a divisor of degree 3 on $\mathbf{P}^1_K$ is defined over K. Then $(E, 0)$ is an elliptic curve over K, where 0 is the point over ∞. Such an elliptic curve can always be written as $y^2 = f(x)$, where f is a polynomial of degree 3 in $K[x]$.

Example 1.1.4

Consider the curve given by the equation $y^2 = x^3 + Ax + B$, where A and B are in a field K. This is the famous *Weierstrass equation*, and if $\Delta = 4A^3 + 27B^2$ is not zero in K, then this equation defines a smooth curve E, of genus one, with a single point 0 at infinity. $(E, 0)$ is an elliptic curve over K. This equation is the one relating the Weierstrass $\mathcal{P}$-function and its derivative (up to some constants), for an elliptic curve as in Example 1.1.1. In general, any elliptic curve over a field of characteristic unequal to 2 or 3 can be defined by a Weierstrass equation; one requires $\Delta \neq 0$ for E to be smooth.

Elliptic curves over algebraically closed fields are themselves classified by a single number, the so-called J-invariant. For a curve given by an Weierstrass equation $y^2 = x^3 + Ax + B$, the J-invariant is

$$J = J(A, B) = \frac{4A^3}{\Delta} = \frac{4A^3}{4A^3 + 27B^2}.$$

Two elliptic curves over a field K are isomorphic over the algebraic closure of K if and only if they have the same J-invariant.

There are many references for the theory of elliptic curves; two recent books are [Sil] and [Hu].

1.2 Surfaces and Elliptic Surfaces

Moving to the next dimension, that is, studying compact complex *surfaces* having complex dimension two, involves a quantum level leap in complexity (pardon the pun). The classification of surfaces is still much studied, but it is fair to say that the analogue of the three-class description for curves is well-known, and this is called the Enriques-Kodaira classification.

The generalization from the curve cases does not use universal coverings (not anywhere near enough is known about simply connected complex surfaces) but takes the tangent bundle approach. In fact, things are generally dualized a bit and the cotangent bundle of 1-forms Ω^1 (which has rank 2 for a surface) and its second exterior power $\bigwedge^2 \Omega^1 = \Omega^2$ of 2-forms are used. The bundle Ω^2 has rank one, and is therefore somewhat easier to work with and is in fact preferred: it is called the *canonical bundle* of the surface, and is denoted by K.

On curves, rank one bundles have degrees, and this was used to put curves into three classes (by the sign of the degree). On surfaces, rank one bundles do not have degrees, so something a bit more complicated must be introduced: the Kodaira number of the bundle. The Kodaira number of any line bundle L measures the growth rate of the dimension of the vector space $H^0(L^n)$ of sections of tensor powers of the bundle, and for a surface and its canonical bundle, this growth rate is denoted by κ.

The same kind of definition can be made for compact complex varieties of any dimension d (using growth rates for sections of tensor powers of the bundle of d-forms Ω^d). In other words: the dimension of $H^0(K^n)$ grows like n^κ. (If this vector space is empty for all n, we set $\kappa = -\infty$.) In general, κ can take on the values $-\infty, 0, 1, \ldots, d$, where d is the dimension of the complex manifold.

For a complex curve, we have

$$\begin{aligned}
&\kappa = -\infty \text{ if } H^0(K^n) = 0 \text{ for all } n.\\
&\kappa = 0 \text{ if } \dim H^0(K^n) \text{ is bounded for all } n.\\
&\kappa = 1 \text{ if } \dim H^0(K^n) \text{ grows linearly with } n.
\end{aligned}$$

and for a complex surface, we have:

$$\begin{aligned}
&\kappa = -\infty \text{ if } H^0(K^n) = 0 \text{ for all } n.\\
&\kappa = 0 \text{ if } \dim H^0(K^n) \text{ is bounded for all } n.\\
&\kappa = 1 \text{ if } \dim H^0(K^n) \text{ grows linearly with } n.\\
&\kappa = 2 \text{ if } \dim H^0(K^n) \text{ grows quadratically with } n.
\end{aligned}$$

Notice that for curves ($d = 1$) the Kodaira dimension gives the three broad classes mentioned above; The Riemann sphere is the only curve with $\kappa = -\infty$, elliptic curves have $\kappa = 0$, and curves of genus at least 2 have $\kappa = 1$.

Now a striking thing happens in higher dimensions. If a variety X has κ strictly between 1 and its dimension d, then there is a fibration structure, namely a rational map $f : X \to Y$, where Y has dimension equal to κ, and the fibers of f have $\kappa = 0$. Thus to classify varieties, broadly speaking, one must concentrate on the following problems:

a. Classify varieties with $\kappa = -\infty$.

b. Classify varieties with $\kappa = 0$.

c. Classify varieties with $\kappa = d$.

d. Classify fibrations of $\kappa = 0$ varieties.

Since the only $\kappa = 0$ curves are the genus one curves, fibrations of genus one curves play a central role in classifying varieties of dimension at least two. In dimension equal to two, a complex surface with a fibration of genus one curves is called an *elliptic surface*. All $\kappa = 1$ surfaces are elliptic surfaces. Let us be more precise:

Definition 1.2.1 *An* elliptic surface *is a complex surface X together with a holomorphic map $\pi : X \to C$ from X to a smooth curve C such that the general fiber of π is a smooth connected curve of genus one.*

Note that we have not said that the general fiber of π is an elliptic curve, which might strike you as more logical. This would imply that there is given in each fiber a chosen point, which would mean that a cross-section for the map π would be given. This is considered a special (though fundamental) case and we just agree to abuse the language in this way.

An elliptic surface $\pi : X \to C$ is *smooth* if X is a smooth surface. It is *relatively minimal*, or a *minimal elliptic* surface if X is smooth and there are no (-1)-curves in the fibers of π.

We will call a curve on X *vertical* if it lies in a fiber of π. We will otherwise call it *horizontal*. Thus a minimal elliptic surface has no vertical (-1)-curves. It may well have horizontal ones, however, and therefore not be a minimal surface in the sense of surface theory; it will only be "minimal elliptic".

We say $\pi : X \to C$ is an elliptic surface *with section S*, or simply *with section*, if a section $s : C \to X$ of π is given; the image of s is the curve S on X.

We say that X is an elliptic surface *over* C if we wish to specify that C is the base curve. For example, an elliptic surface over $\mathbf{P}^1$ has a smooth rational base curve.

Finally, note that any elliptic surface X over a curve C can be viewed as a curve of genus one defined over the function field $K(C)$. There is in fact a 1-1 correspondence between these two notions, if one sticks to relatively minimal elliptic surfaces. In this correspondence, $K(C)$-rational points of the genus one curve correspond to sections of the elliptic surface.

1.3 How Elliptic Surfaces are Classified

Jacobian Surfaces

Elliptic surfaces are themselves classified in several ways. Probably the easiest class of elliptic surfaces to understand (and the most important for the classification) is the class of elliptic surfaces with a chosen section S_0.

Such a surface arises from any elliptic surface X over C by considering the elliptic surface as a curve E of genus one over $K(C)$, then taking the Jacobian curve A of E, which is defined to be the group of divisors on E of degree 0. A has a natural $K(C)$-rational point (namely the 0 divisor), and so the corresponding elliptic surface to A has a natural section. For this reason elliptic surfaces with a section are called *Jacobian* surfaces. Given a section, that section is always used as the zero of the group of sections.

Jacobian surfaces can always be written with a Weierstrass equation

$$y^2 = x^3 + Ax^2 + B$$

where A and B are functions on the base curve. More precisely, there is a natural line bundle $\mathbf{L}$ on C, called the Weierstrass bundle, and A is a section of $\mathbf{L}^4$, and B is a section of $\mathbf{L}^6$. The bundle $\mathbf{L}$ is the conormal bundle of the zero-section S_0. The discriminant Δ is then a section of $\mathbf{L}^{12}$.

Multiple Fibers, Local Obstructions, and the Tate-Shafarevich Group

Given the good situation with understanding Jacobian surfaces, (the existence of a nice equation like the Weierstrass equation makes life much easier!)

one is led to ask: how far is the general elliptic surface from a Jacobian? This is of course equivalent to asking: what does it take for an elliptic surface to have a section?

Firstly, if an elliptic surface has a section S, then no fiber can be multiple; every fiber must have at least one vertical curve of multiplicity one, through which the section passes. Conversely, if a fiber has a multiplicity one component, then at least locally one can find a section.

Hence the local obstruction to finding sections of elliptic surfaces (or of curve fibrations in general) is the existence of multiple fibers. Now multiple fibers can be created on an elliptic surface by a process called the *logarithmic transformation.* Moreover the inverse of this operation will revert multiple fibers to fibers without multiplicity. This operation is quite analytic, and may change an algebraic surface into a non-algebraic one; but in any case it can be performed at will, and is rather straightforward. So one can always take an elliptic surface, and if there are any multiple fibers, perform inverse logarithmic transformations, to arrive at an elliptic surface with no multiple fibers.

Such a surface with no multiple fibers has therefore no local obstructions to finding sections. Hence the only obstructions must be global in nature: is it possible to patch together the local sections into a global section? This is a cohomological-type problem, and is measured by the Tate-Shafarevich group.

So a somewhat satisfactory answer to the question of whether a given elliptic surface X is a Jacobian is to answer the following:

a. Are there multiple fibers for X? (If so, X is *not* a Jacobian.)

b. If there are no multiple fibers for X, does X represent the zero element in the Tate-Shafarevich group? If so, X is a Jacobian. (If not, not!)

In some special cases of interest the Tate-Shafarevich group is zero, in particular for rational elliptic surfaces. Therefore in these cases X is a Jacobian if and only if X has no multiple fibers.

A good discussion of these matters can be found in [Sf].

The J-map

An important invariant associated to an elliptic surface is it J-map. Recall that any genus one curve E has a J-value, which classifies the isomorphism type of E. Therefore if $f : X \to C$ is an elliptic surface, the varying elliptic curves $f^{-1}(c)$ for $c \in C$ give varying J-values $J(c) := J(f^{-1}(c))$. This gives a map, called the J-map, from the points of C over which the fibers of f are smooth, to the complex plane $\mathbf{C}$. This map naturally extends to a holomorphic map from all of C to the projective line $\mathbf{P}^1$, which is also called the J-map.

The J-map gives an alternate way of classifying elliptic surfaces. A surface and its Jacobian will have the same J-map, since an elliptic curve E and its Jacobian A are isomorphic over the algebraic closure of the defining field. Hence the J-map is only useful for distinguishing between Jacobian surfaces. Two non-isomorphic surfaces can have the same J-map, unfortunately. If t is any function on the base curve C, then the two Jacobian surfaces defined by the two Weierstrass equations

$$y^2 = x^3 + Ax^2 + B$$

and

$$y^2 = x^3 + t^2Ax^2 + t^3B$$

will have the same J-map. On the other hand they will not be isomorphic if t has any simple zeroes. Fortunately, this is the only way two Jacobians can have the same J-map, so actually the J-function is a rather good classifying tool.

Numerical Invariants

Finally I want to simply list the most common numerical invariants for a smooth minimal Jacobian elliptic surface $f : X \to C$, with section S. Let $\mathbf{L}$ be the conormal bundle to S, viewed as a line bundle on C, let d be its degree, and let g be the genus of C.

Lemma 1.3.1

a) *X is a product surface (isomorphic to $C \times E$ for some elliptic curve E) if and only if $\mathbf{L}$ is the trivial line bundle $\mathcal{O}_C$.*

b) The canonical class K_X *of the surface* X *is* $f^*(K_C + \mathbf{L})$. *In particular,* $K_X^2 = 0$, *and* K_X *is numerically equivalent to* $d + 2g - 2$ *fibers of* f.

c) The irregularity q *of* X *is equal to the genus* g *of* C *if* X *is not a product.*

d) The holomorphic Euler-Poincare characteristic $\chi = \chi(\mathcal{O}_X)$ *of the structure sheaf* $\mathcal{O}_X$ *is equal to the degree of* $\mathbf{L}$: $\chi = d$.

e) If X *is not a product, the geometric genus* p_g *of* X *is* $p_g = d + g - 1$.

f) If S *is any section of* X, *then* $S^2 = -d$.

It is not hard to make computations for the plurigenera of X, and from this to place Jacobian surfaces surfaces in the over-all classification of surfaces. The result is the following.

Lemma 1.3.2

a) Let $g = 0$. *Then* X *is*
a product $E \times \mathbf{P}^1$ *if* $d = 0$,
a rational surface if $d = 1$,
a $K3$ *surface if* $d = 2$, *and*
a properly elliptic surface (i.e., $\kappa = 1$*) if* $d \geq 3$.

b) Let $g = 1$. *Then* X *is*
an abelian surface (a product) if $\mathbf{L} \cong \mathcal{O}_C$.
a hyperelliptic ("bielliptic" in Beauville's notation, see[B2]) surface if $\mathbf{L}$ *is torsion of order* 2, 3, 4, *or* 6, *and*
a properly elliptic surface if $d \geq 1$.
In case X *is hyperelliptic, the order of* K_X *is that of* $\mathbf{L}$.

c) Let $g \geq 2$. *Then* X *is a properly elliptic surface.*

The theory of elliptic surfaces has received many treatments, starting with Kodaira's papers [K1], [K2]. For general surface theory one can consult [B2], [BPV], or [SF]. More information on the Kodaira number κ can be found in the text by Iitaka [Ii], and general classification of surfaces and elliptic surfaces (especially in characteristic p) can be found in [BMu]. The reader may also wish to consult Dolgachev-Cossec [CsD] and the notes from lectures given by the author in Pisa in 1988 [M5]. This really only scratches the "surface"!

2 Singular Fibers and Sections of Elliptic Surfaces

Let $f : X \to C$ be a smooth elliptic surface over a curve C. Over all but finitely many points c of C, the fiber $f^{-1}(c)$ will be a smooth elliptic curve. Those fibers which are not smooth are called *singular fibers*, and for elliptic fibrations these are quite well understood.

One usually restricts oneself to singular fibers on minimal elliptic surfaces, since one can make any fiber look singular by blowing up a point in it.

If X is a smooth minimal Jacobian surface, with Weierstrass equation

$$y^2 = x^3 + Ax^2 + B,$$

then a point $c \in C$ has a singular fiber above it if and only if the discriminant function $\Delta(c) = 4A(c)^3 + 27B(c)^2$ is zero. Using the global description of A, B, and Δ, we see that since Δ is a section of $\mathbf{L}^{12}$, there are $12 \cdot \deg(\mathbf{L})$ singular fibers (counted properly).

2.1 Semistable Fibers

If one looks at the Weierstrass equation $y^2 = x^3 + Ax^2 + B$ and considers A and B as complex numbers, then one has essentially three types of phenomena:

a. $\Delta = 4A^3 + 27B^2 \neq 0$. In this case the curve is a smooth elliptic curve.

b. $\Delta = 0$ but A and B are not both 0. In this case the curve is a singular cubic curve, with an ordinary node.

c. A and B are both 0. In this case the curve is an cuspidal cubic curve.

The fibers of Jacobian elliptic surfaces are broadly classified into the above three types. Of course almost all fibers are smooth fibers. A fiber is called *semistable* if, in the Weierstrass equation, not both A and B are zero. Semistable fibers have a nice classification, which is fairly easy to describe. There is one type for every natural number $n \geq 0$, with $n = 0$ being the smooth fiber case. In general the semistable fiber types are denoted by I_n.

As mentioned above, a fiber of type I_0 is a smooth fiber. A fiber of type I_1 is an irreducible rational curve with a single node. A fiber of type I_2 consists of two smooth irreducible rational curves, each having self-intersection -2 on X, which meet transversally in two distinct points. (An example of this can be seen in the plane, as a conic plus a non-tangent line.) In general, a fiber of type I_n for $n \geq 3$ is a cycle of n smooth rational curves, each having self-intersection -2 on X. (An example of the I_3 fiber type is given by three non-collinear lines in the plane.)

At a fiber of type I_n, the discriminant Δ will have a zero of order n. Therefore the order of vanishing of the discriminant Δ will determine the fiber type for semistable fibers. In addition, a fiber of type I_n counts for n in the total number of singular fibers, which is $12\deg(\mathbf{L})$.

An elliptic surface is called *semistable* if all of its fibers are semi-stable, that is, if all of its fibers are of type I_n for some n.

2.2 Kodaira's List

All possible singular fibers (not just the semistable ones) have been classified by Kodaira (see [K1]). Although I won't speak of them much in these lectures, no discussion would be complete without at least mentioning the non-semistable cases. All of the possible fibers are listed in the table, using Kodaira's names.

Table 2.2.1: Kodaira's list of singular fibers of elliptic surfaces

Name	Description of Fiber
I_0	smooth elliptic curve
I_1	nodal rational curve
I_2	two smooth rational curves meeting transversally at two points
I_3	three smooth rational curves meeting in a cycle; a triangle
$I_N, N \geq 3$	N smooth rational curves meeting in a cycle
$I_N^*, N \geq 0$	$N+5$ smooth rational curves meeting with dual graph $\tilde{D}_{N+4}$
II	a cuspidal rational curve
III	two smooth rational curves meeting at one point to order 2
IV	three smooth rational curves all meeting at one point
IV^*	7 smooth rational curves meeting with dual graph $\tilde{E}_6$
III^*	8 smooth rational curves meeting with dual graph $\tilde{E}_7$
II^*	9 smooth rational curves meeting with dual graph $\tilde{E}_8$
${}_MI_N, N \geq 0$	topologically an I_N, but each curve has multiplicity M

All components of reducible fibers have self-intersection -2; the irreducible fibers have self-intersection 0, of course.

The reader can consult [K1] for Kodaira's original proof of the completeness of the list.

Examples of some of these are not hard to see with cubic curves. A fiber of type II is realized as a cuspidal cubic curve. A fiber of type III can be seen by considering a conic and a tangent line. Three collinear lines in the plane form a fiber of type IV.

2.3 Sections and the Mordell-Weil Group of Sections

We assume in this subsection that $f : X \to C$ is an elliptic surface with a given section S_0, and associated bundle $\mathbf{L}$.

Let $\mathbf{MW}(X)$ be the set of sections of f; addition, fiber by fiber, induces a group law on $\mathbf{MW}(X)$ with S_0 as the zero element. This group is called the *Mordell-Weil Group* of f (or of X).

The group law can also be described as follows. Note that $\mathbf{MW}(X)$ can be identified with the set of rational points on the generic fiber X_η: a section gives a point by restriction, and a point gives a section as its closure. The point of X_η corresponding to the zero section S_0 will be denoted by p_0. The sum in $\mathbf{MW}(X)$ is then inherited from the sum on the points of X_η, which is after all an elliptic curve over $K(C)$ and as such its points form an abelian group.

More explicitly, let S_1 and S_2 be two sections. Then the sum $S_1 \oplus S_2$ in $\mathbf{MW}(X)$ is the section S_3, where $(S_1 + S_2 - S_0)_\eta \equiv (S_3)_\eta$.

2.4 The Shioda-Tate sequence

For any divisor E on X_η, one has the summation $\sum E$, defined by adding in the group law on X_η the points of E. This gives a homomorphism

$$\sum : \mathbf{Div}(X_\eta) \to \mathbf{MW}(X),$$

which by Abel's theorem on X_η factors through $\mathbf{Pic}(X_\eta)$.

Let $\mathbf{NS}(X)$ be the Neron-Severi group of divisor classes on X modulo homological equivalence. This is a finitely generated abelian group (it is the image of the Picard group $\mathbf{Pic}(X)$ in the homology group $H^2(X, \mathbf{Z})$). There is a natural map r from $\mathbf{NS}(X)$ to the Picard group of the generic fiber X_η, induced by restriction of divisors. Composition of this map $r : \mathbf{NS}(X) \to \mathbf{Pic}(X_\eta)$ with the above summation map $\sum$ gives a homomorphism β from $\mathbf{NS}(X)$ to $\mathbf{MW}(X) : \beta(\overline{D}) = \overline{\sum D_\eta} =$ the class of the closure of $\sum(D_\eta)$.

The following is called the Shioda-Tate exact sequence.

Theorem 2.4.1 *Let $A \subset \mathbf{NS}(X)$ be the subgroup generated by the class of the zero section S_0 and the vertical classes. Then the sequence*

$$0 \to A \xrightarrow{\alpha} \mathbf{NS}(X) \xrightarrow{\beta} \mathbf{MW}(X) \to 0$$

is exact, where α is the inclusion.

Proof: Clearly β is surjective: if S is any section in $\mathbf{MW}(X)$, the class of S in $\mathbf{NS}(X)$ goes to S under β. Of course α is injective, and since $\beta(S_0) = S_0$

which is the zero of $\mathbf{MW}(X)$, and $\beta(V) = 0$ for any vertical V, we have that $\beta \circ \alpha = 0$. Let $D \in \ker(\beta)$. Then $\sum(D_\eta) = p_0$; therefore $D_\eta - \deg(D_\eta)p_0$ is linearly equivalent to 0 on X_η by Abel's Theorem. Note that $\deg(D_\eta) = (D\cdot F)$, where F is the class of a fiber of f. Hence we have that $D - (D \cdot F)S_0$ restricts to 0 on X_η. Therefore $D - (D \cdot F)S_0$ is linearly equivalent to a vertical divisor V; hence $D = (D \cdot F)S_0 + V$ as classes in $\mathbf{NS}(X)$, so $D \in A$. □

Corollary 2.4.2 *The Mordell-Weil group* $\mathbf{MW}(X)$ *is a finitely generated abelian group.*

Denote by R the sublattice of A generated by vertical components not meeting S_0. R is a direct sum of root lattices of types A_N, D_N, E_6, E_7, and E_8. In particular, R is an even negative definite lattice, and

$$\mathbf{rank}(R) = \sum_{c\in\Delta}(\#\text{ of components of } X_c - 1),$$

where $\Delta \subset C$ is the discriminant locus and for $c \in C$, X_c denotes the fiber $f^{-1}(c)$ over c. We will denote this local number by "r_c"; hence we have $\operatorname{rank}(R) = \sum r_c$.

Note that the sublattice U of A generated by S_0 and the fiber F is a rank two unimodular sublattice, with R as its perpendicular space. Therefore

$$A = < S_0, F > \oplus R = U \oplus R. \tag{2.4.3}$$

In particular, $\operatorname{rank}(A) = 2 + \operatorname{rank}(R)$. This gives the following corollary, see [Shd1]. Denote by ρ the rank of $\mathbf{NS}(X)$, which is the Picard number of X.

Corollary 2.4.4 *(The Shioda-Tate formula)*

$$\rho = 2 + \sum_{c\in\Delta} r_c + \mathbf{rank}(\mathbf{MW}(X)).$$

Since U is unimodular, it splits off $\mathbf{NS}(X)$ also, giving the exact sequence

$$0 \to R \to U^\perp \to \mathbf{MW}(X) \to 0 \tag{2.4.5}$$

where $U^\perp$ is the perpendicular space to U in $\mathbf{NS}(X)$; this version of the Shioda-Tate exact sequence is sometimes useful.

In addition to information about the rank of the Mordell-Weil group, the Shioda-Tate exact sequence gives a way of viewing the torsion part of this group. To be more precise, since the Shioda-Tate exact sequence gives the Mordell-Weil group as being isomorphic to the quotient group $\mathbf{NS}(X)/A$, and since it is a general fact that for Jacobian elliptic surfaces the Neron-Severi group $\mathbf{NS}(X)$ has no torsion, the torsion in the Mordell-Weil group $\mathbf{MW}(X)$ comes entirely from the fact that the subgroup A may not be saturated in $\mathbf{NS}(X)$. Since we have a bilinear form on the Neron-Severi group, we may write this as follows.

Corollary 2.4.6

$$\mathbf{MW}_{tor}(X) \cong A^{\perp\perp}/A.$$

2.5 Some lattice theory and applications

Let L be any free finitely generated $\mathbf{Z}$-module with a nondegenerate bilinear form $< -,- >$ with values in $\mathbf{Z}$. The form extend to a $\mathbf{Q}$-valued form on $L_{\mathbf{Q}} = L \otimes \mathbf{Q}$. Denote by $L^\#$ the module

$$L^\# = \{x \in L_{\mathbf{Q}} \mid < x, \ell > \in \mathbf{Z} \text{ for all } \ell \in L\}.$$

$L^\#$ is a free $\mathbf{Z}$-module containing L as a submodule of finite index; the quotient group $G_L = L^\#/L$ is a finite abelian group whose order is the discriminant of L.

$L^\#$ may be naturally identified with the dual module $L^* = \mathbf{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$, by sending $x \in L^\#$ to the functional $< x, - >$.

The intersection form on $\mathbf{NS}(X)$ gives a map $\mathbf{NS}(X) \to R^* = \mathrm{Hom}(R, \mathbf{Z})$, which after identifying R^* with $R^\#$ and passing to the quotient G_R of $R^\#$ by R, gives a map $\gamma : \mathbf{NS}(X) \to G_R$.

Note that $\gamma(A) = 0$, since S_0 and F do not meet any components generating R, and R goes to 0 in G_R. Therefore γ factors through $\mathbf{MW}(X)$, and we have a map (which we also call γ):

$$\gamma : \mathbf{MW}(X) \rightarrow G_R.$$

Since the lattice R splits as the direct sum of the local lattices, one for each singular fiber with 2 or more components, so does the dual lattice $R^{\#}$ and the finite group G_R:

$$G_R \cong \bigoplus_{c\in\Delta} G_c,$$

where G_c is the finite group for the lattice generated by the components of the singular fiber X_c not meeting the zero-section S_0. These lattices depend only on the type of the fiber, and the finite group can be computed once and for all; the result is given in Table 2.5.1. Included in this table is the Euler number e_c of the fiber, and the rank r_c of the lattice R_c, which is one less than the number of components.

Table 2.5.1: Local invariants of singular fibers

type of X_c	e_c	r_c	G_c
I_N	N	$N-1$	$\mathbf{Z}/N\mathbf{Z}$
I_{2N}^*	$N+2$	$N+4$	$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$
I_{2N+1}^*	$N+2$	$N+4$	$\mathbf{Z}/4\mathbf{Z}$
II	2	0	0
III	3	1	$\mathbf{Z}/2\mathbf{Z}$
IV	4	2	$\mathbf{Z}/3\mathbf{Z}$
IV^*	8	6	$\mathbf{Z}/3\mathbf{Z}$
III^*	9	7	$\mathbf{Z}/2\mathbf{Z}$
II^*	10	8	0

Focusing on the semistable case of a singular fiber X_c of type I_N, which is a cycle of smooth rational curves, we see that the number of components N is equal to the order of the local group G_c. This is no accident in this case; indeed, the components are arranged in a cycle, starting with the component that meets the zero section, and the map γ in the component of the singular fiber above c records simply which component of X_c the section S hits.

The map γ is not onto; it is interesting to ask which subgroups of G_R are hit by γ. The kernel of γ is the group $\mathbf{MW}_0(X)$ of sections which always meet the same component as does the zero-section S_0, in every fiber.

Lemma 2.5.1 $\mathbf{MW}_0(X)$ *is torsion-free if* $\deg(\mathbf{L}) \geq 1$.

Proof: Suppose that S is a torsion section of order n in $\mathbf{MW}_0(X)$. Hence $nS - nS_0$ restricts to 0 on the generic fiber X_η; therefore $nS - nS_0 = V$ with V vertical. However since S is in $\mathbf{MW}_0(X), S - S_0$ does not meet any vertical components; therefore neither does V, and so V must have square 0, forcing V to be a sum of fibers. Working in $\mathbf{NS}(X)$, we then have $nS - nS_0 = aF$ for some integer a.

Let $k = S \cdot S_0$ and $\ell = -\deg(\mathbf{L}) = S^2 = S_0^2$. By intersecting the above equation with S we obtain $n\ell - nk = a$, and by intersecting it with S_0 we get $nk - n\ell = a$; therefore $a = 0$ and $k = \ell$. However k is non-negative and ℓ is negative!. □

Corollary 2.5.2 *Suppose* $\deg(\mathbf{L}) \geq 1$. *Then* γ *maps the torsion part of the Mordell-Weil group injectively into* G_R.

In other words, a torsion section is determined by which components of the fibers it passes through.

3 Examples: Rational Elliptic Surfaces

Let us in this lecture begin to use some of the theory developed up to this point and analyse the case of rational elliptic surfaces. These are smooth minimal elliptic surfaces which are, in addition, rational surfaces. If in addition we assume that the surface has a section, then every such surface comes from a pencil of cubic curves in the plane. Most of this material is taken from the papers of the author and Ulf Persson (both individual and joint work): see [P], [M1], and [MP1].

Every rational elliptic surface has $\mathbf{P}^1$ as the base curve. The Euler number of the surface is 12, and so the associated line bundle has degree $d = 1$.

3.1 Extremal Rational Elliptic Surfaces

Let $f : X \to \mathbf{P}^1$ be a rational elliptic surface with section S_0. In this case $e = e(X) = 12$, and $K_X = -F$.

Let $U = < S_0, F >$ be the rank two unimodular sublattice of $\mathbf{NS}(X)$ generated by S_0 and F; as remarked previously, U splits off $\mathbf{NS}(X) : \mathbf{NS}(X) = U \oplus U^{\perp}$. Since $\mathbf{NS}(X)$ is unimodular with signature $(1,9)$ (X is a blow-up of $\mathbf{P}^2$ at nine points), and U is unimodular with signature $(1,1)$, we must have $U^{\perp}$ unimodular with signature $(0,8)$: i.e., $U^{\perp}$ is a negative definite unimodular lattice of rank 8.

The intersection form on $U^{\perp}$, moreover, is even, since $K_X \in U$ and X is rational; hence $U^{\perp}$ is abstractly isomorphic to a lattice of type E_8.

Recall from the last lecture that we have $\mathbf{MW}(X) \cong U^{\perp}/R$, where R is the sublattice of $U^{\perp}$ generated by components of fibers not meeting S_0.

Definition 3.1.1 *A smooth minimal elliptic surface $f : X \to C$ with section will be called* extremal *if $\rho = h^{1,1} = 2 + rank(R)$.*

In other words, X is extremal if X has maximal Picard number $h^{1,1}$ *and* the classes of S_0 and components of fibers generate $\mathbf{NS}(X)$ over $\mathbf{Q}$. We have an immediate corollary: X is extremal if and only if $\rho = h^{1,1}$ and $\mathbf{MW}(X)$ is finite.

In fact, for rational surfaces, the concept of extremal can be viewed in many ways:

Proposition 3.1.2 *Let X be a rational elliptic surface with section. Then the following are equivalent:*

(a) X is extremal

(b) The relative automorphism group $\mathrm{Aut}_C(X)$ is finite.

(c) The number of representations of X as a blow-up of $\mathbf{P}^2$ is finite.

(d) The number of smooth rational curves C with $C^2 < 0$ is finite.

(e) The number of reduced irreducible curves C with $C^2 < 0$ is finite.

Proof: The relative automorphism group $\mathrm{Aut}_C(X)$ is the group of automorphisms of X (all of which must preserve the elliptic fibration, since the elliptic fibration is given by $| -K_X |$) which induce the identity automorphism of the base curve C. Let τ be such an automorphism, and consider the image $\tau(S_0)$ of the zero section; it is again a section, and so the automorphism $\tau_{\tau(S_0)}^{-1} \circ \tau$ fixes S_0, where τ_S is the automorphism given by translation by S, for a section S. The automorphisms of X fixing S_0 form a finite group: this is the group of automorphisms of the generic fiber. Hence $\mathrm{Aut}_C(X)$ is finite if and only if the group of sections $\mathbf{MW}(X)$ is finite; this proves that (a) and (b) are equivalent. The implications $(e) \Leftrightarrow (d) \Leftrightarrow (c)$ are obvious. Since $K_X = -F$, a smooth rational curve E on X is exceptional if and only if it is a section, proving $(a) \Rightarrow (d)$, since in any case a smooth rational curve C must satisfy $-2 = C^2 + CK$, or $C^2 = CF - 2 \geq -2$, and the (-2)-curves are always finite in number: they are the components of reducible fibers. If C is reduced and irreducible with $C^2 < 0$, then we must have $-2 \leq C^2 + CK \leq -1$, forcing C to be smooth rational and either a (-1)- or (-2)-curve; this shows that (d) and (e) are equivalent. Finally, since any rational elliptic surface is a blow-up of $\mathbf{P}^2$, we have $(c) \Leftrightarrow (d)$. □

As consequences of extremality for rational elliptic surfaces, we have the following. For a fiber F, denote by $d(F)$ the discriminant of the lattice R_F generated by the components of the fiber F not meeting S_0; set $d(F) = 1$ if F is irreducible. This number $d(F)$ is the order of the local finite group G mentioned in the previous lecture; in particular, for an I_m fiber, we have $d = m$. Denote by $e(F)$ the Euler number of F, and by $r(F)$ the rank of the above lattice (which is one less than the number of components of F).

Proposition 3.1.3 *Let $f : X \to \mathbf{P}^1$ be an extremal rational elliptic surface with section. Then:*

(a) $disc(R) = \prod_F d(F) = | \mathbf{MW}(X) |^2$, and in particular is a perfect square.

(b) $\sum_F(e(F) - r(F)) = 4.$

(c) X *has, for singular fibers, either:*

- 4 *semistable fibers*
- 3 *singular fibers, exactly* 2 *of them semistable, or*
- 2 *non-semistable singular fibers.*

Proof: The lattice R is the orthogonal direct sum of the lattices R_F for each fiber F, and the discriminant of R_F is $d(F)$ by definition. Hence $\mathrm{disc}(R) = \prod_F d(F)$ is obvious. By general lattice theory, $\mathrm{disc}(R) = \mathrm{disc}(U^\perp) \cdot [U^\perp : R]^2$, and since $MW(X) \cong U^\perp/R$, and $U^\perp$ is unimodular, we have that $\mathrm{disc}(R) = |\, MW(X) \,|^2$. Since $e = \sum_F e_F = 12$ and $\mathrm{rank}(R) = \sum r_F = 8$, we have (b). Finally, (c) follows by noticing that for a semistable fiber F, $e(F) - r(F) = 1$, and for a non-semistable fiber F, $e(F) - r(F) = 2$. □

This Proposition allows us to classify all configurations of singular fibers on extremal rational elliptic surfaces. We give the list in Table 3.1.

Table 3.1.4: Possible configurations of singular fibers on extremal rational elliptic surfaces

Singular fibers	degree(J)	$\mid$ MW(X) $\mid$	Notation
II, II^*	0	1	X_{22}
III, III^*	0	2	X_{33}
IV, IV^*	0	3	X_{44}
I_0^*, I_0^*	0	4	$X_{11}(J)$, $J \in \mathbf{C}$
II^*, I_1, I_1	2	1	X_{211}
III^*, I_2, I_1	3	2	X_{321}
IV^*, I_3, I_1	4	3	X_{431}
I_4^*, I_1, I_1	6	2	X_{411}
I_1^*, I_4, I_1	6	2	X_{141}
I_2^*, I_2, I_2	6	4	X_{222}
I_9, I_1, I_1, I_1	12	3	X_{9111}
I_8, I_2, I_1, I_1	12	4	X_{8211}
I_6, I_3, I_2, I_1	12	6	X_{6321}
I_5, I_5, I_1, I_1	12	5	X_{5511}
I_4, I_4, I_2, I_2	12	8	X_{4422}
I_3, I_3, I_3, I_3	12	9	X_{3333}

Proof of the Table: There are three cases, corresponding to the number of singular fibers. I will only discuss here the case of 4 singular fibers, all of which must then be semistable by the previous Proposition. The possibilities the previous Proposition. The possibilities are to have $\{I_{n_1}, I_{n_2}, I_{n_3}, I_{n_4}\}$ with $\sum n_i = 12$ and $\prod n_i$ equal to a perfect square; this gives only the six sets in the table. □

A discussion of the other cases can be found in [MP1], as can a proof of the following theorem.

Theorem 3.1.4 *For every configuration of possible singular fibers in Table 3.1, there is a unique extremal rational elliptic surface with section with that configuration of singular fibers,* except *for the configuration* $\{I_0^*, I_0^*\}$*; these are classified by their* J*-invariant, which must be some constant* $J \in \mathbf{C}$*, and can be any complex number.*

The 6 semistable surfaces above were studied by Beauville [B1], and some authors have called these surfaces Beauville surfaces. This work has been extended to characteristic p by W. Lang [La2].

It would be a shame to have all of this theory without having seen a single equation, so below I present 2 explicit examples. In each case I give the Weierstrass coefficients A and B (in the homogeneous Weierstrass equation $y^2z = x^3 + Axz^2 + Bz^3$), the discriminant Δ, and the formulas for the sections. We use homogeneous coordinates $[u : v]$ for the base curve.

Example 3.1.5

Surface: X_{8211}

$A = -3(u^4+4u^2v^2+v^4)$, $B = 2u^6+12u^4v^2+15u^2v^4-2v^6$, $\Delta = -3^6u^2v^8(u^2+4v^2)$

Sections:

x	y	z
0	1	0

$u^2 + 2v^2$	0	1
$u^2 - v^2$	$\pm 3\sqrt{3}uv^2$	1

Example 3.1.6

Surface: X_{3333}

$A = -3u^4 + 24uv^3$, $B = 2u^6 + 40u^3v^3 - 16v^6$, $\Delta = 2^8v^3(u^3 + v^3)^3$

Sections:

x	y	z
0	1	0
$-3u^2$	$\pm 4i(u^3 + v^3)$	1
$(u - 2v)^2$	$\pm 4\sqrt{3}v(u^2 - uv + v^2)$	1
$(u - 2\omega v)^2$	$\pm 4\sqrt{3}v(\omega u^2 - \omega^2 uv + v^2)$	1
$(u - 2\omega^2 v)^2$	$\pm 4\sqrt{3}v(\omega^2 u^2 - \omega uv + v^2)$	1

where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.

3.2 Semistable Rational Elliptic Surfaces

In this section I would like to give a classification in a slightly different direction, namely of all possible configurations of singular fibers on *semistable* rational elliptic surfaces.

Let $f : X \to \mathbf{P}^1$ be a semistable elliptic surface with section, i.e., all fibers of f are of type I_n. Assume that X has s singular fibers, which are of types $I_{n_1}, \ldots, I_{n_s}$. In this case we will say that X *realizes* the unordered s-tuple $[n_1, \ldots, n_s]$; note that $\sum n_i$ is a multiple of 12: it is $e(X)$.

Conversely, given a set $[n_1, \ldots, n_s]$ of s positive integers whose sum is divisible by 12 (repetitions are allowed), we will say that $[n_1, \ldots, n_s]$ *exists as a semistable elliptic surface over* $\mathbf{P}^1$, or that simply the set *exists*, if there is a semistable elliptic surface with section over $\mathbf{P}^1$ with exactly s singular fibers of types $I_{n_1}, \ldots, I_{n_s}$.

For example, the Beauville rational elliptic surfaces realize $[9,1,1,1]$, $[8,2,1,1]$, $[6,3,2,1]$, $[5,5,1,1]$, $[4,4,2,2]$, and $[3,3,3,3]$; these six 4-tuples exist.

A rather elementary argument involving the J-map shows the following; I will omit the proof.

Lemma 3.2.1 *Assume that* $[n_1,\dots,n_s]$ *exists. Then* $[n_1,\dots,n_{i-1},a,b,n_{i+1},\dots,n_s]$ *exists, for any* $a,b \geq 1$ *with* $a+b = n_i$.

As a corollary of Lemma 3.2.1 and the existence of the six 4-tuples obtained by the Beauville surfaces, we have the following.

Corollary 3.2.2 *The following* s*-tuples exist.*

$s=4$:	$91^3, 821^2, 6321, 5^21^2, 4^22^2, 3^4$
$s=5$:	$81^4, 721^3, 631^3, 541^3, 62^21^2, 5321^2, 4^221^2, 432^21, 42^4, 3^321$
$s=6$:	$71^5, 621^4, 531^4, 52^21^3, 4^21^4, 4321^3, 42^31^2, 3^31^3, 3^22^21^2, 32^41, 2^6$
$s=7$:	$61^6, 521^5, 431^5, 42^21^4, 3^221^4, 32^31^3, 2^51^2$
$s=8$:	$51^7, 421^6, 3^21^6, 32^21^5, 2^41^4$
$s=9$:	$41^8, 321^7, 2^31^6$
$s=10$:	$31^9, 2^21^8$
$s=11$:	21^{10}
$s=12$:	1^{12}

Note that every s-tuple with $s \geq 6$ and $\sum n_i = 12$ is on the above list, i.e., can be obtained from the six 4-tuples:

Corollary 3.2.3 *Let* $[n_1,\dots,n_s]$ *be an* s*-tuple with* $\sum n_i = 12$. *Then if* $s \geq 6, [n_1,\dots,n_s]$ *exists.*

Our goal is to prove that the list of Corollary 3.2.2 is complete, i.e., that these are the only s-tuples with $\sum n_i = 12$ which exist. In view of the previous corollary, the necessity that s be at least 4, and the classification in the case $s = 4$, it suffices to show the following.

Proposition 3.2.4 *The three 5-tuples*

$$[5,2,2,2,1], [4,3,3,1,1], \text{ and } [3,3,2,2,2]$$

do not exist.

The proof of the above Proposition involves a bit more lattice theory, some of which we have already seen. I will only sketch the ideas, which are fairly straightforward; the proof involves only a mild computation after the outline is understood.

3.3 The impossibility of the three 5-tuples

Let L be a finitely generated free $\mathbf{Z}$-module, and let $< -,- >$ be an even non-degenerate $\mathbf{Z}$-valued symmetric bilinear form on L. $L_{\mathbf{Q}} = L \otimes_{\mathbf{Z}} \mathbf{Q}$ naturally inherits the bilinear form, which will still be non-degenerate and symmetric; moreover $L \subset L_{\mathbf{Q}}$ naturally. Define $L^{\#} = \{x \in L_{\mathbf{Q}} \mid < x, \ell > \in \mathbf{Z} \text{ for all } \ell \in L\}$. We of course have $L \subseteq L^{\#}$, since the form is $\mathbf{Z}$-valued on L. The natural map $\phi : L^{\#} \rightarrow L^{*} (= \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z}))$ defined by $\phi(x) = < x, - >$ is an isomorphism, and so we see that $L^{\#}$ is a free $\mathbf{Z}$-module with the same rank as L; in particular, L has finite index in $L^{\#}$.

Define $G_L = L^{\#}/L$, the so-called *discriminant-form group* of L. Its order is the absolute value of the discriminant of L,

$$\mid G_L \mid = \mid \mathrm{disc}(L) \mid,$$

since both sides are computed as the absolute value of the determinant of any matrix for $< -,- >$ on a $\mathbf{Z}$-basis of L.

If we define the *length* $\ell(G)$ of a finite abelian group to be the minimum number of generators of G, we also have that

$$\ell(G_L) \leq \mathrm{rank}(L),$$

since G_L is generated by the cosets of the rank(L) generators of $L^{\#}$.

One can define a $\mathbf{Q}/\mathbf{Z}$-valued quadratic form q_L, the *discriminant-form*, on G_L by setting, for x in $L^\#, q_L(x) = \frac{1}{2} < x, x >$ mod $\mathbf{Z}$. The reader should check that q_L is well-defined, and satisfies $q_L(nx) = n^2 q_L(x)$ for $n \in \mathbf{Z}$ and $x \in G_L$. Moreover, the function $q_L(x+y) - q_L(x) - q_L(y)$ is exactly the induced symmetric bilinear form $< -, - >$ on G_L, with values in $\mathbf{Q}/\mathbf{Z}$.

Example 3.3.1

Let L be the lattice of rank $N-1$ representing A_{N-1}. Then L is realized as the lattice R_c, where X_c is a fiber of type I_N on an elliptic surface with section. In particular, as we noted in Table 2.5.1, $G_L \cong \mathbf{Z}/N\mathbf{Z}$. A generator for G_L is afforded by the coset of the element $e_1^\# = \frac{-1}{N}\sum_{i=1}^{N-1} ie_i$, where $\{e_i\}$ is the natural basis of L, namely the classes of the components of the cycle I_N; these are numbered so that e_i meets $e_{i\pm1}$ around the cycle, and e_0 meets the zero section. The element $e_1^\#$ meets e_1 exactly once, and meets no other e_i; its image in L^* is the dual element to e_1.

A calculation gives that $q_L(e_1^\# \bmod L) = (1-N)/2N$, so that if we identify G_L with $\mathbf{Z}/N$ by sending $e_1^\#$ mod L to 1, we have that

$$q_{A_{N-1}}(a) = a^2(1-N)/2N.$$

It will be useful later to remark the following:

$$\begin{array}{l} \text{For the following lattices } L, \\ G_L \text{ has no nonzero } q_L\text{-isotropic elements} \\ \text{(i.e., elements } g \text{ with } q_L(g) = 0\text{):} \\ A_4, A_3, A_2 \oplus A_2, A_1 \oplus A_1 \oplus A_1. \end{array} \tag{3.3.2}$$

One of the main applications of this discriminant-form construction is to the analysis of embeddings of lattices. The following is a typical example.

Lemma 3.3.3 *There is a 1-1 correspondence between*

$$\left\{ \begin{array}{c} \textit{Intermediate lattices } M \\ L \subseteq M \subseteq L^\# \\ \textit{such that} \\ < -, - >|_M \textit{ is } \mathbf{Z}\textit{-valued and even} \end{array} \right\} \textit{ and } \left\{ \begin{array}{c} q_L\textit{-isotropic subgroups} \\ H \subseteq G_L \end{array} \right\}$$

Moreover if M corresponds to H, then $G_M \cong H^\perp/H$, and q_M is induced from q_L.

Of course by q_L-isotropic I mean that $q_L(h) = 0$ for every h in H. The lemma is easily proved; the correspondence is the usual one, sending an intermediate lattice M to M/L, and a subgroup H to $\pi^{-1}(H)$, where $\pi : L^\# \to G_L$ is the natural quotient map. I leave the details to the reader.

The last bit of lattice theory is the following.

Lemma 3.3.4 *Suppose that U is a unimodular lattice, and L_1 and L_2 are two nondegenerate sublattices of U, such that $L_1 = L_2^\perp$ and $L_2 = L_1^\perp$. Then there exists an isomorphism between G_{L_1} and G_{L_2} such that $q_{L_1} = -q_{L_2}$ under the isomorphism.*

Proof: The nondegeneracy of the L_i implies that $L_1 \oplus L_2$ is a sublattice of U; hence we may view U as an intermediate lattice between $L_1 \oplus L_2$ and $(L_1 \oplus L_2)^\#$, so that there exists a q-isotropic subgroup H of $G_{L_1 \oplus L_2}$ corresponding to U. Since U is unimodular, G_U is trivial, so that $H^\perp = H$ by the previous lemma. Note that since the L_i are orthogonal, $G_{L_1 \oplus L_2} \cong G_{L_1} \oplus G_{L_2}$. Let π_i be the projection of $G_{L_1 \oplus L_2}$ onto G_{L_i}.
claim: $\pi_i \,|_H \colon H \to G_{L_i}$ is an isomorphism for both i.
Why: First let us show injectivity: suppose that $\pi_1(h) = 0$, for $h \in$ H. Then $h = (0, g_2)$ for some g_2 in G_{L_2}. Hence there exists an element u in U, mapping to h, of the form $(0, x_2)$, where $x_2 \in L_2^\#$ and $g_2 = x_2 \bmod L_2$. Since $u \in L_1^\perp$, we must have $u \in L_2$, so that $x_2 \in L_2$ and $g_2 = 0$, whence $h = 0$. Therefore π_1 is injective on H; the argument for π_2 is the same.

The injectivity shows that $| H | \leq | G_{L_i} |$ for both i; since $H^\perp = H$, and the order of $H^\perp$ is the index of H (this is a general fact), we have that $| H |^2 = | G_{L_1 \oplus L_2} | = | G_{L_1} | \cdot | G_{L_2} |$. Hence $| H | = | G_{L_1} | = | G_{L_2} |$ and the injectivity also implies surjectivity. This proves the claim.

To finish the proof, define $f : G_{L_1} \to G_{L_2}$ by $f = (\pi_2 \,|_H) \circ (\pi_1 \,|_H)^{-1}$; f is an isomorphism by the claim. If $h \in H$, then, since H is isotropic, we have

$$0 = q_{L_1 \oplus L_2}(h) = q_{L_1}(\pi_1(h)) + q_{L_2}(\pi_2(h)),$$

proving that the quadratic forms for the L_i are opposite in sign. □

Corollary 3.3.5 *Suppose L is a nondegenerate sublattice of a unimodular lattice U. Then $G_{L^{\perp\perp}} \cong G_{L^\perp}$ and $q_{L^{\perp\perp}} = -q_{L^\perp}$.*

Proof: Just apply the previous lemma to $L^\perp$ and $L^{\perp\perp}$. □

We are now in a position to prove Proposition 3.2.4, i.e., to prove that the three 5-tuples $[5,2,2,2,1]$, $[4,3,3,1,1]$, and $[3,3,2,2,2]$ do not exist.

Suppose that $\pi : X \rightarrow C$ is a semistable rational elliptic surface with section, realizing the s-tuple $[n_1, \dots, n_s]$; i.e., there are exactly s singular fibers of types $I_{n_1}, \dots, I_{n_s}$, and $\sum n_i = 12$. Let R be the sublattice of $NS(X)$ generated by the components of fibers not meeting the zero section S_0; R is a lattice of rank $\sum(n_i - 1) = 12 - s$, isomorphic to $\oplus_i A_{n_i-1}$. Let U be the lattice generated by S_0 and the fiber F; since $\mathbf{NS}(X)$ is unimodular (X is rational), so is $U^\perp$, and since $K_X \in U, U^\perp$ is even; in fact $U^\perp$ is isomorphic to the E_8 lattice, but we do not need to know that. In any case $U^\perp$ has rank 8, since U has rank 2 and $\mathbf{NS}(X)$ has rank 10. Therefore:

If $[n_1, \dots, n_s]$ exists, then $\oplus_i A_{n_i-1}$ embeds into a unimodular lattice of rank 8. (3.3.6)

Now suppose further that $s = 5$; then $\text{rank}(R) = 7$, so that $K = R^\perp$ in $U^\perp$ has rank 1. Therefore G_K is cyclic, and by Corollary 3.3.5, so is $G_{R^{\perp\perp}}$. The inclusion $R \subseteq R^{\perp\perp}$ is between lattices of the same rank, so that we can view $R^{\perp\perp}$ as an intermediate lattice between R and $R^\#$; therefore $R^{\perp\perp}$ corresponds to an isotropic subgroup H of G_R with $G_{R^{\perp\perp}} \cong H^\perp/\text{H}$. Thus:

If $[n_1, \dots, n_5]$ exists,
there is an isotropic subgroup H of G_R (3.3.7)
with $H^\perp/H$ cyclic.

In particular:

If $G_{\oplus A_{n_i-1}}$ is not cyclic,
and has no nonzero isotropic elements, (3.3.8)
then $[n_1, \dots, n_5]$ does not exist.

The final ingredient is provided by the next lemma.

Lemma 3.3.9 *For the lattices*

$$A_1 \oplus A_1 \oplus A_1 \oplus A_2 \oplus A_2, \qquad A_1 \oplus A_1 \oplus A_1 \oplus A_4, \text{ and } A_2 \oplus A_2 \oplus A_3,$$

the discriminant-form groups have no nonzero isotropic elements.

Proof: Since (2,3), (2,5), and (3,4) are relatively prime, any isotropic element of these lattices must decompose into isotropic elements of the summands $A_1 \oplus A_1 \oplus A_1, A_2 \oplus A_2, A_3$, and A_4: no cancellation is possible. This forces any isotropic element to be zero by (3.3.2). □

Since the discriminant-form groups for the three lattices above are not cyclic, applying the above lemma to ([3.3.8]) proves Proposition 3.2.4.

The reader interested in the lattice theory used here should consult Nikulin's paper [Ni]. The methods used here were also successful in computing the possible configurations of fibers on semistable $K3$ elliptic surfaces; this appears in [MP2].

4 Torsion in the Mordell-Weil Group

4.1 Review of Basic Facts

In this lecture I want to develop some interesting results concerning the torsion in the Mordell-Weil group. Specifically, I want to return to the question raised previously: which components of singular fibers does a torsion section hit? This material is taken mostly from [M6]. Let us introduce some specific notation to address this.

Recall that if e is the topological Euler number of X, then $e = 12\chi$.

We suppose that the map f has s singular fibers $F_1, \ldots, F_s$, each semistable, i.e., each a cycle of rational curves (of type "I_m" in Kodaira's notation). Indeed, let us say that the fiber F_j is of type I_{m_j}. Therefore

$$e = 12\chi = \sum_{j=1}^{s} m_j.$$

Choose an "orientation" of each fiber F_j and write the m_j components of F_j as

$$C_0^{(j)}, C_1^{(j)}, \ldots, C_{m_j-1}^{(j)},$$

where the zero section S_0 meets only $C_0^{(j)}$ and for each k, $C_k^{(j)}$ meets only $C_{k\pm1}^{(j)} \bmod m_j$. If $m_j = 1$, then $F_j = C_0^{(j)}$ is a nodal rational curve, of self-intersection 0. If $m_j \geq 2$, then each $C_k^{(j)}$ is a smooth rational curve with self-intersection -2.

If we denote by $\mathbf{NS}(X)$ the Neron-Severi group of the elliptic surface X, we can consider the sublattice $A \subseteq \mathbf{NS}(X)$ generated by the zero-section S_0 and the components $C_k^{(j)}$ of the singular fibers. The class of the fiber F is of course in this sublattice A; indeed, the only relation among these classes is that

$$F \equiv \sum_{k=0}^{m_j-1} C_k^{(j)} \text{ for each } j = 1, \ldots, s. \tag{4.1.1}$$

If we let U denote the sublattice of A spanned by F and S_0, then U is a unimodular lattice of rank 2, and so splits off A. Its orthogonal complement R is freely generated by the components $C_k^{(j)}$ for $j = 1 \ldots s$ and $k \neq 0$. R is therefore a direct sum of s root lattices, with the j^{th} summand isomorphic to A_{m_j-1}, generated by $C_k^{(j)}$ for $k \neq 0$.

The Shioda-Tate formula for the Mordell-Weil group $\mathbf{MW}(X)$ of sections of X is derived from the exact sequence

$$0 \to A \to \mathbf{NS}(X) \to \mathbf{MW}(X) \to 0 \tag{4.1.2}$$

where the first map is the inclusion of the sublattice A into $\mathbf{NS}(X)$, and the second map is the fiber-by-fiber summation map, sending a divisor class $D \in \mathbf{NS}(X)$ to the closure of the sum of the points of D on the generic fiber of X. We therefore obtain information about both the rank and the torsion of the Mordell-Weil group $\mathbf{MW}(X)$. If we denote by ρ the Picard number of X, which is the rank of the Neron-Severi group, we see that since A has rank equal to $2 + \sum_{j=1}^{s}(m_j - 1)$,

$$\mathbf{rank}\, \mathbf{MW}(X) = \rho - 2 - \sum_{j=1}^{s}(m_j - 1) = s + \rho - 2 - e. \tag{4.1.3}$$

The torsion $\mathbf{MW}_{tor}(X)$ in the Mordell-Weil group corresponds to those classes in $\mathbf{NS}(X)$ for which some multiple lies in the sublattice A. We see that

$$\mathbf{MW}_{tor}(X) \cong A^{\perp\perp}/A.$$

For any lattice N, denote by $N^{\#}$ the dual lattice $\mathbf{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. The inclusion of A into $A^{\perp\perp}$ gives the sequence of inclusions

$$A \subseteq A^{\perp\perp} \subseteq (A^{\perp\perp})^{\#} \subseteq A^{\#}$$

which shows that the quotient $A^{\perp\perp}/A$ is isomorphic to a subgroup of the discriminant-form group $G_A = A^{\#}/A$. Note that $A^{\#} = U^{\#} \oplus R^{\#}$ since $A = U \oplus R$ and since $U^{\#} = U$ (U is unimodular), we have $G_A \cong R^{\#}/R = \bigoplus_{j=1}^{s} A^{\#}_{m_j-1}/A_{m_j-1}$. A computation shows that for the root lattices of type A, $A^{\#}_{m_j-1}/A_{m_j-1} \cong \mathbf{Z}/m_j\mathbf{Z}$; therefore we have that the torsion part $\mathbf{MW}_{tor}(X)$ of the Mordell-Weil group of sections of X is isomorphic to a subgroup of $\bigoplus_{j=1}^{s} \mathbf{Z}/m_j\mathbf{Z}$.

A choice of orientation for each singular fiber F_j gives an identification of the cyclic group of components $C_k^{(j)}$ for $k = 0, \ldots, m_j - 1$ with $\mathbf{Z}/m_j\mathbf{Z}$. For any section S of f, and any singular fiber F_j, denote by $k_j(S)$ the index of the component of F_j which S meets. Thus

$$S \cdot C_l^{(j)} = \delta_{lk_j(S)} \tag{4.1.4}$$

where δ above is the Kronecker delta. Note that with our notation above, $k_j(S_0) = 0$ for every j. The numbers $\{k_j(S)\}$ will be called the *component numbers* of the section S. The upshot of the remarks above is that, for torsion sections, this assignment of integers $\{k_j(S)\}$ to S is 1-1. Thus a torsion section is determined by its component numbers (but not all sets of component numbers can occur).

In this lecture we will study the properties of these encodings of torsion sections for semistable elliptic surfaces. Although it is clear from the above discussion that the component numbers $k_j(S)$ are well-defined $\mathbf{mod} m_j$, it is more useful for our purposes to take them to be integers in the range $0, \ldots, m_j - 1$.

Two facts are necessary for what follows.

Lemma 4.1.5 *a) If S_1 and S_2 are two different sections in* $\mathbf{MW}(X)$ *with $S_1 - S_2$ torsion, then $S_1 \cdot S_2 = 0$, i.e., they are disjoint.*

b) If S is any section in $\mathbf{MW}(X)$*, then $S \cdot S = -\chi$.*

4.2 The Divisor Class of a Torsion Section

Let S be a torsion section of the semistable elliptic surface $f : X \to C$. In this section we wish to write down the class of S in $\mathbf{NS}(X)$.

Since the class of S lies in $L^{\perp\perp}$, we have that S is a $\mathbf{Q}$-linear combination of the zero section S_0, and the fiber components $C_k^{(j)}$. Because (4.1.1) is the only relation among these classes, the classes S_0, F, and $C_k^{(j)}$ for $j = 1, \ldots, s$ and $k \neq 0$ form a basis for the module spanned by all these classes.

For fixed $j = 1, \ldots, s$ and $k = 1, \ldots, m_j - 1$, set

$$\begin{aligned} D_k^{(j)} &= (m_j - k)\sum_{i=1}^{k} iC_i^{(j)} + k\sum_{i=k+1}^{m_j-1}(m_j - i)C_i^{(j)} \\ &= (m_j - k)[C_1^{(j)} + 2C_2^{(j)} + \ldots + kC_k^{(j)}] + \\ &\quad\ k[(m_j - k - 1)C_{k+1}^{(j)} + \ldots + 2C_{m_j-2}^{(j)} + C_{m_j-1}^{(j)}]. \end{aligned}$$

We set $D_0^{(j)} = 0$.

Note that

$$D_k^{(j)} \cdot S_0 = 0 \tag{4.2.1}$$

for every k.

The following is a straightforward check:

Lemma 4.2.2 *Fix $j = 1, \ldots, s$, and indices k and l with $0 \leq k, l \leq m_j - 1$. Then*

$$D_k^{(j)} \cdot C_l^{(j)} = \begin{cases} m_j & \text{if } l = 0 \text{ and } k \neq 0 \\ -m_j & \text{if } l = k \text{ and } k \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is useful to note that the above lemma can be re-expressed using the Kronecker delta function as

$$D_k^{(j_1)} \cdot C_l^{(j_2)} = m_j \delta_{j_1 j_2}(1 - \delta_{k0})(\delta_{l0} - \delta_{lk}). \tag{4.2.3}$$

Theorem 4.2.4 *With the above notation, if S is a torsion section of order n, then*

$$S \equiv S_0 + (S - S_0 \cdot S_0)F - \sum_{j=1}^{s} \frac{1}{m_j} D^{(j)}_{k_j(S)}.$$

Proof: Since the intersection form on A is nondegenerate, we need only check that both sides of the equation intersect the generators S_0, F, and the $C_l^{(j)}$ in the same number. Let us begin with S_0. Using (4.2.1) and the equations

$$S \cdot F = S_0 \cdot F = 1,$$

we have that the right hand side intersects S_0 to $S \cdot S_0$, agreeing with the left hand side. Similarly, F meets only the S_0 term on the right hand side, so the intersections with F are also equal.

Now fix indices $j = 1, \dots, s$ and $l = 0, \dots, m_j - 1$, and let us check the intersection with $C_l^{(j)}$; Lemma 4.2.2 is the critical part of the computation. Intersecting with the right hand side gives $S_0 \cdot C_l^{(j)} - \frac{1}{m_j} D^{(j)}_{k_j(S)} \cdot C_l^{(j)}$, which reduces to $\delta_{l0} - (1 - \delta_{k_j(S)0})(\delta_{l0} - \delta_{lk_j(S)})$ using (4.2.3). This is after some simplification equal to $\delta_{lk_j(S)}$, which is also $S \cdot C_l^{(j)}$ by (4.1.4). □

The formula for the linear equivalence class of S given above can be simplified somewhat in the case that S is not the zero section S_0. Then $(S \cdot S_0) = 0$ and $(S_0 \cdot S_0) = -\chi$ by Lemma 4.1.5, so we have

$$S \equiv S_0 + \chi F - \sum_{j=1}^{s} \frac{1}{m_j} D^{(j)}_{k_j(S)}. \tag{4.2.5}$$

4.3 The Quadratic Relation for the Component Numbers

Let us now take the above formula for the torsion section S and intersect it with S itself. If $S = S_0$, then of course all $k_j(S) = 0$, both sides of the equation are S_0, so we recover no information. However if $S \neq S_0$, the formula (4.2.5) is nontrivial. Dotting S with the right-hand side gives

$$\chi - \sum_{j=1}^{s} \frac{1}{m_j} D^{(j)}_{k_j(S)} \cdot S.$$

Since S meets only the curve $C^{(j)}_{k_j(S)}$ in the j^{th} singular fiber, and it meets it exactly once, we have

$$D^{(j)}_{k_j(S)} \cdot S = (m_j - k_j(S))k_j(S).$$

Therefore the above reduces to

$$\chi - \sum_{j=1}^{s} k_j(S)(1 - \frac{k_j(S)}{m_j}).$$

Finally, using Lemma 4.1.5.2, dotting the left-hand side with S gives $-\chi$. Hence we obtain the following formula, which we call the *quadratic relation* for the component numbers.

Proposition 4.3.1 *Let S be a torsion section of f, not equal to the zero section S_0. Then*

$$\sum_{j=1}^{s} k_j(S)(1 - \frac{k_j(S)}{m_j}) = 2\chi(\mathcal{O}_X).$$

Note that the quadratic relation is independent of the choice of orientation of each singular fiber, as it should be. (If one reverses the orientation of F_j, then k_j is replaced by $m_j - k_j$ if $k_j \neq 0$, so that the two terms of each summand are simply switched.)

4.4 The Component Number Sums

In this section I want to develop formulas for the component number sums $\sum_j k_j(S)$ for a torsion section S. Such formulas follow rather directly from the quadratic relation for the component numbers. Suppose first that S has order 2. Then for each j with $k_j(S) \neq 0$, we must have m_j even and $k_j(S) = m_j/2$. Therefore the quadratic relation reduces to the following.

Corollary 4.4.1 *Let S be an order 2 section of f. Then*

$$\sum_{j=1}^{s} k_j(S) = 4\chi(\mathcal{O}_X).$$

Note that in the order 2 case, each k_j is either 0 or $m_j/2$, and hence the component numbers for a torsion section of order 2 are independent of the orientation of the fibers. This is clearly not true for other values of the component numbers. In general, reversing the orientation of the fiber F_j changes the component number from k_j to $m_j - k_j$. Hence the analogue of Corollary (4.4.1) for torsion sections of order at least 3 must take this into account.

To this end define a function $d_m : \{0, \ldots, m-1\} \to \{0, \ldots, [m/2]\}$ by setting

$$d_m(k) = \min\{k, m-k\}.$$

Note that for any given torsion section S, it is possible to choose the orientations of the fibers so that the component numbers $k_j(S)$ are minimal, that is, $k_j(S) = d_{m_j}(k_j(S))$ for each j. If this condition holds, we say that the section S has *minimal* component numbers.

Next suppose that a torsion section S has order $n \geq 3$. Then both S and $2S$ are nonzero torsion sections of X. If for each index j, we choose the orientation of the components so that $0 \leq k_j(S) \leq m_j/2$, then we have the formulas

$$k_j(2S) = \begin{cases} 0 & \text{if } k_j(S) = m_j/2, \text{ and} \\ 2k_j(S) & \text{if } k_j(S) < m_j/2 \end{cases} \tag{4.4.2}$$

for every j. Therefore applying Proposition 4.3.1 to $2S$, after dividing by 2 we obtain

$$\sum_{j \text{ with } k_j(S) < m_j/2} k_j(S)\left(1 - 2\frac{k_j(S)}{m_j}\right) = \chi(\mathcal{O}_X).$$

Multiplying the formula of Proposition 4.3.1 by 2 gives

$$\sum_{j \text{ with } k_j(S) = m_j/2} k_j(S) + \sum_{j \text{ with } k_j(S) < m_j/2} k_j(S)\left(2 - 2\frac{k_j(S)}{m_j}\right) = 4\chi(\mathcal{O}_X),$$

and subtracting the above two equations yields the following.

Corollary 4.4.3 *Let S be a section of order $n \geq 3$ with minimal component numbers $\{k_j(S)\}$. Then*

$$\sum_{j=1}^{s} k_j(S) = 3\chi(\mathcal{O}_X).$$

In case the orientations of the fibers are not chosen so as to give S minimal component numbers, a similar formula holds, expressed with the d_m function.

Corollary 4.4.4 *Let S be a section of order $n \geq 3$. Then*

$$\sum_{j=1}^{s} d_{m_j}(k_j(S)) = 3\chi(\mathcal{O}_X).$$

4.5 The Distribution Numbers for a Torsion Section

I would like to apply the formulas for the component number sums of a non-zero torsion section S to its multiples αS. To this end extend the function d_m to $d_m : \mathbf{N} \to \{0, \ldots, [m/2]\}$ by defining

$$d_m(q) = \min\{q - m[q/m], m(1 + [q/m]) - q\};$$

this function is simply the distance from q to the nearest multiple of m. Note that

$$d_{ab}(qb) = d_a(q)b \tag{4.5.1}$$

for any integers a, b, and q. We remark that in fact $d_m(q)$ can be defined for rationals m and q, in the same way, as the distance from q to the nearest integral multiple of m; it takes values in $\mathbf{Q} \cap [0, m/2]$. Formula (4.5.1) still holds.

The utility of this notation for our application comes from noticing that if S has a component numbers $\{k_j(S)\}$, then the minimal component numbers for αS are $\{d_{m_j}(\alpha k_j(S))\}$. Using (4.5.1) we may write this as

$$d_{m_j}(k_j(\alpha S)) = d_n(\alpha k_j(S)n/m_j)\frac{m_j}{n} \tag{4.5.2}$$

for any $n > 0$.

If S' is a section whose order divides n, with a minimal encoding $\{k_j(S')\}$, define rational numbers $M_{i,n}(S')$ for $i = 0, \ldots, [n/2]$ by

$$M_{i,n}(S') = \Big(\sum_{j \text{ with } k_j(S')=im_j/n} m_j\Big)/12\chi.$$

Roughly speaking, $M_{i,n}(S')$ is the fraction of the total sum $\sum_j m_j = 12\chi$ contributed by fibers where S' meets one of the two components which are exactly "distance i" from the component meeting the zero-section S_0. (This distance is measured in units of m_j/n.) We will call these fractions $M_{i,n}(S')$ the *distribution numbers* for the section S'.

With this notation we may write the results on component number sums as

$$\sum_i iM_{i,n}(S') = \begin{cases} n/3 & \text{if } S' \text{ has order 2, and} \\ n/4 & \text{otherwise,} \end{cases} \tag{4.5.3}$$

where S' is a nonzero torsion section whose order divides n.

Now fix a section S of order exactly n. As long as αS is not the zero-section S_0, the above notation may be applied to αS, whose order will divide n for every α.

Suppose that $k_j(S) = im_j/n$, so that the fiber F_j contributes to the fraction $M_{i,n}(S)$. Then $k_j(\alpha S) = d_n(\alpha i)m_j/n$ by (4.5.2), so that F_j contributes to the fraction $M_{d_n(\alpha i),n}(\alpha S)$. Therefore in the component number sum equation (4.5.3) for αS, the fiber F_j occurs with the weight $d_n(\alpha i)$. Hence the component number sum equation for αS reduces to

$$\sum_i d_n(\alpha i)M_{i,n}(S) = \begin{cases} n/3 & \text{if } \alpha = n/2, \text{ and} \\ n/4 & \text{otherwise.} \end{cases} \tag{4.5.4}$$

This equation is valid for $1 \leq \alpha, i \leq [n/2]$.

Let $\mathbf{A}_n$ be the square matrix of size $[n/2]$ whose αi^{th} entry is $d_n(\alpha i)$. Let $\mathbf{v}_n$ be a vector of length $[n/2]$, which if n is odd has every coordinate equal to $n/4$, and if n is even has every coordinate equal to $n/4$ except the last, which is $n/3$. Finally let $\mathbf{M}_n(S)$ be the column vector of $M_{i,n}(S)$'s. The equations of 4.5.4 can then be expressed in matrix form as follows.

Lemma 4.5.5 *If S is a nonzero torsion section of order exactly n, then*

$$\mathbf{A}_n\mathbf{M}_n(S) = \mathbf{v}_n.$$

Lemma 4.5.6 *The matrix $\mathbf{A}_n$ is invertible.*

The proof of the above Lemma is rather involved and not very enlightening. It involves generalized Bernoulli numbers associated to even Dirichlet characters for the cyclic group of order n. For a proof in the prime case, the reader may consult [M6].

Using this Lemma, we can prove that for a torsion section S of odd prime order p, the component numbers (in the minimal case) are equi-distributed in the sense that the fractions $M_{i,p}(S)$ for $i \neq 0$ are all the same.

Corollary 4.5.7 *Let p be an odd prime and let S be a section of f of order exactly p, with minimal component numbers $\{k_j(S)\}$. Then for every integer i between 1 and $(p-1)/2$,*

$$M_{i,p}(S) = \Big(\sum_{\{j | k_j(S) = im_j/p\}} m_j \Big)/12\chi = \frac{2p}{(p^2-1)}.$$

Proof: For p prime, each row of the matrix $\mathbf{A}_p$ consists of the integers $1, 2, \ldots, (p-1)/2$, in some order. Therefore the row sums of $\mathbf{A}_p$ are constant, equal to $(p^2-1)/8$. Thus the vector $\mathbf{v}_p$, all of whose coordinates are $p/4$, is an eigenvector of $\mathbf{A}_p$ with eigenvalue $(p^2-1)/8$. Hence the vector $\mathbf{M}_p$, all of whose coordinates are $2p/(p^2-1)$, is a solution to the matrix equation $\mathbf{A}_p\mathbf{M}_p = \mathbf{v}_p$. Since $\mathbf{A}_p$ is invertible, this solution is unique, and so by Lemma 4.5.5 the vector $\mathbf{M}_p(S)$ must be this constant vector $\mathbf{M}_p$. □

Corollary 4.5.8 *Let p be an odd prime and let S be a section of f of order exactly p. Then*

$$M_{0,p}(S) = \frac{1}{p+1}.$$

Proof: Since all of the fractions $M_{i,p}(S)$ for $i = 1, \ldots, (p-1)/2$ are equal to $2p/(p^2-1)$, their sum is $p/(p+1)$. Since the sum of all $M_{i,p}(S)$, including the fraction $M_{0,p}(S)$, is 1, we must have $M_{0,p}(S) = 1/(p+1)$. □

The above Corollary 4.5.8 also appears in [MP3].

Since each of the numbers $12\chi M_{i,p}(S)$ is an integer, and since p and p^2-1 are always relatively prime, we obtain the following divisibility result, which improves a divisibility result in [MP3].

Corollary 4.5.9 *Let p be an odd prime and suppose that f admits a section of order exactly p. Then p^2-1 divides 24χ.*

Using these results one can "divine" the genus of the modular curve and the Euler number of the modular surface, without knowing that such objects exist! (If they did exist, then they should have minimum allowable genus and minimum allowable Euler number; the minimum is gotten from the above, and this is indeed the correct genus and Euler number.)

For a complete treatment of elliptic modular surfaces, the reader should consult [Shm].

5 How torsion affects the genus, the rank, and the Euler number

5.1 Preliminaries and a lower bound for the number s of singular fibers

The purpose of this lecture is to study restrictions on the torsion groups of elliptic surfaces over an arbitrary base curve. In fact the approach is dual to this: we'll study the restrictions on the base curve (its genus), the rank of the Mordell-Weil group of sections, and the Euler number of the surface, given information about the group of torsion sections.

We will retrict ourselves to minimal semistable Jacobian elliptic surfaces, since if $f : X \to C$ has a non-semistable fiber, then the order of the torsion group is at most 4, and these small groups present no essential problems in the theory, only annoying complications. Most of this material is taken from [MP3].

We will use the notation of the previous lecture:

e = the topological Euler number of X

χ = the holomorphic Euler characteristic of X

ρ = the Picard number of X, the rank of the Neron-Severi group $\mathbf{NS}(X)$

Rk = the rank of the Mordell-Weil group $\mathbf{MW}(X)$ of sections of X

s = the number of singular fibers $F_1, \ldots, F_s$ of X

m_j = the number of components in F_j, which is of type I_{m_j}

g = the genus of the base curve C

$q = h^1(\mathcal{O}_X)$ and $p_g = h^2(\mathcal{O}_X)$

$h^{p,q}$ = the (p,q)-Hodge number of X.

We have already seen that

$$e = 12\chi = \sum_j m_j.$$

The Shioda-Tate formula is

$$\rho = 2 + \mathbf{Rk} + \sum_j (m_j - 1) = 2 + \mathbf{Rk} + \chi - s. \tag{5.1.1}$$

Assuming that X is not a product (which we will always do), then $q = g$, so that the Hodge diamond for X is

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ p_g & & h^{1,1} & & p_g \\ & g & & g & \\ & & 1 & & \end{array}$$

and therefore $12\chi = e = 2 + 2p_g + h^{1,1} - 4g = 2\chi + h^{1,1} - 2g$, so that $h^{1,1} = 10\chi + 2g$. This of course forces

$$\rho \leq 10\chi + 2g \tag{5.1.2}$$

since ρ is always at most $h^{1,1}$. Combining everything we get the following.

Proposition 5.1.3 *Let X be a smooth minimal semistable elliptic surface which is not a product. Then*

$$s \geq 2\chi + 2 - 2g + \mathbf{Rk}$$

with equality if and only if X has maximal Picard number $\rho = h^{1,1}$.

Proof: We have

$$\begin{aligned} s &= 12\chi - \sum_j (m_j - 1) \\ &= 12\chi - (\rho - 2 - \mathbf{Rk}) = 12\chi + 2 + \mathbf{Rk} - \rho \quad \text{(from 5.1.1)} \\ &\geq 12\chi + 2 + \mathbf{Rk} - (10\chi + 2g) \quad \text{(from 5.1.2)} \\ &= 2\chi + 2 - 2g + \mathbf{Rk} \end{aligned}$$

as claimed. Since the inequality comes from $\rho \leq h^{1,1}$, we have the last statement. □

5.2 Translation by torsion sections and an upper bound for s

The next ingredient is a combinatorial fixed point analysis of actions given by translations by torsion sections. Let T be a torsion subgroup of MW, let S_0 be the zero section of T, and let S be any other torsion section in T. Denote by τ_S the automorphism of X given by translation (in the group law of the fibers) by S. Via the τ_S's the torsion group T acts on X, preserving the fibration f.

We will say that a singular fiber F_j has *isotropy group* $H \subseteq T$ under the above action if H is the set of sections which meet the component $C_0^{(j)}$, that is, meet the same component of F_j as does the zero-section S_0.

The following lemma is proved in [MP2].

Lemma 5.2.1 *Let $f : X \to C$ be a smooth minimal semistable elliptic surface with torsion group $T \subset \mathbf{MW}_{tor}(X)$.*

a) The elliptic fibration f induces an elliptic fibration $\overline{f}_T : X/T \to C$.

b) $\overline{f}_T$ has exactly s singular fibers also, one under each singular fiber of f.

c) The image of an I_m fiber with isotropy H, under the quotient map, is a a cycle of $\frac{m}{|T/H|}$ smooth rational curves meeting at singular points which are rational double points of type $A_{|H|-1}$. These double points are the only singularities of X/T. The image of a smooth fiber is a smooth fiber.

d) Let Y_T be the smooth elliptic surface obtained by resolving the singularities of X/T, and let $f_T : Y_T \to C$ be the induced elliptic fibration; it is a smooth minimal semistable elliptic surface also. The singular fiber of f_T corresponding to an I_m fiber of f with isotropy group H is a fiber of type $I_{m|H|^2/|T|}$.

e) X and Y_T have the same Euler number e.

f) If the singular fiber F_j of type I_{m_j} has isotropy group H_j, then $e|\,T\,| = \sum_j m_j|\,H\,|^2$.

We will say that a cyclic subgroup $H \subseteq T$ is *cyclic and cocyclic* (abbreviated "*c&c*") if both H and T/H are cyclic. Only *c&c* subgroups of T can possibly be isotropy groups of singular fibers. For each *c&c* subgroup H of T, let $i_{T,H}$ be the number of nodes of singular fibers of X with isotropy group H under the action of T. Note that, for example, $i_{\{0\},\{0\}} = e$.

Proposition 5.2.2 *With the above notations:*

a) $\sum_{H \subseteq T,\, H\, c\&c} | H |^2 i_{T,H} = e \mid T \mid$.

b) For each c&c $H \subseteq T$*,* $\sum_{G\, c\&c,\, H \subseteq G \subseteq H} i_{T,G} = i_{H,H}$.

Proof: Statement (a) is simply Lemma 5.2.1(f), the sum begin organized over the subgroups instead of the singular fibers. The second statement is obtained by noticing that a node of a singular fiber of X is fixed by H under the induced H-action if and only if it is fixed by a subgroup of T containing H, under the full T-action. □

We are now in a position to illustrate the argument we will use. Suppose simply that f has a section of order p for some prime number p. That section generates a torsion subgroup T of order p, and the only relevant isotropy numbers are $i_{T,T}$ and $i_{T,\{0\}}$. By the previous proposition these numbers satisfy

$$i_{T,T} + i_{T,\{0\}} = i_{\{0\},\{0\}} = 12\chi$$

and

$$i_{T,\{0\}} + p^2 i_{T,T} = 12\chi p.$$

Solving this gives $i_{T,\{0\}} = 12\chi p/(p+1)$ and $i_{T,T} = 12\chi/(p+1)$. (We could also have computed these numbers using the results of the last lecture.)

Now, every singular fiber has isotropy either $\{0\}$ or T; those with trivial isotropy must have m_j divisible by p, since then T embeds into the group of components, which has order m_j. For fixed χ, one maximizes the number of singular fibers s by having $i_{T,T}$ fibers of type I_1 with isotropy T and $i_{T,\{0\}}/p$ fibers of type I_p with isotropy $\{0\}$. Hence the maximum number of singular fibers is $i_{T,T} + i_{T,\{0\}}/p = 24\chi/(p+1)$: $s \leq 24\chi/(p+1)$.

Combining this upper bound with the lower bound given by Proposition 5.1.3, we see that

$$2\chi + 2 - 2g + \mathbf{Rk} \leq 24\chi/(p+1) \tag{5.2.3}$$

if there is p-torsion in the Mordell-Weil group.

The above statement is the prototype of the conditions obtained by assuming a certain torsion subgroup T of $\mathbf{MW}(X)$.

Corollary 5.2.4 *Assume that there is p-torsion in the Mordell-Weil group of X. Then $p^2 - 1$ divides 24χ. Moreover:*

a) If $g = 0$ then $p \leq 7$. If $g = 0$ and χ is odd then $p \leq 5$.

b) If $g = 0$ and $p = 2$ then $\mathbf{Rk} \leq 6\chi - 2$; if $g = 0$ and $p = 3$ then $\mathbf{Rk} \leq 4\chi - 2$; if $g = 0$ and $p = 5$ then $\mathbf{Rk} \leq 2\chi - 2$; if $g = 0$ and $p = 7$ then $\mathbf{Rk} \leq \chi - 2$;

b) If $g = 1$ then $p \leq 11$.

c) If $g \geq 1$ and $\chi \geq 7g - 6$ then $p \leq 11$.

d) If χ is odd then $p \neq 7$. If χ is not divisible by 5 then $p \neq 11$.

5.3 The Size of an abelian group of length at most 2

I claim that the above analysis can be carried out for any possible torsion group T. The trick is to use the quotient construction to get an upper bound on the number s of singular fibers of X. This upper bound should depend linearly on the Euler number e, so let us in fact divide everything we see by e to try to obtain more invariant numbers.

Define then for each cyclic and cocyclic subgroup H of T, the number

$$\gamma_{T,H} = i_{T,H}/e$$

which is the fraction of the nodes which have isotropy H under the T action. A priori these fractions depend not only on T and H but on the representation of T as a group of torsion sections on X.

Now if the above situation occurs, then a fiber of type I_m with isotropy H must have m divisible by $\mid T \mid / \mid H \mid$. ($T/H$ embeds into the group of components.) Hence for fixed Eulèr number e, the maximum number of singular fibers s would be achieved when every singular fiber with isotropy H has $\mid T \mid / \mid H \mid$ components. This leads to a maximum of $\sum_H i_{T,H}/(\mid T \mid / \mid H \mid)$ singular fibers.

Expressing this in terms of the γ's lead us to the following provisional definition:

Definition 5.3.1 *Let T be a finite abelian group of length at most 2. Define the* size *of T, denoted by* **Size**(T), *by*

$$\mathbf{Size}(T) = \frac{1}{\mid T \mid} \sum_{H \subseteq T,\, H \text{ c\&c}} \mid H \mid \gamma_{T,H}.$$

This then gives the following almost by definition:

Proposition 5.3.2 *Assume that X admits the finite group T as a group of sections. Then*

$$s \leq e \cdot \mathbf{Size}(T)$$

Now we must address two questions. Firstly, is **Size**(T) well-defined, independent of the way T lives as a group of sections? Secondly, what is **Size**(T)?

For the first question, in fact it is possible to show that the γ's are well-defined, which is of course enough to prove that **Size**(T) is.

We can re-write the equations of Proposition 5.2.2 as

$$\sum_{H \subseteq T,\, H \text{ c\&c}} \mid H \mid^2 \gamma_{T,H} = \mid T \mid \tag{5.3.3}$$

and

$$\sum_{G \text{ c\&c},\, H \subseteq G \subseteq H} \gamma_{T,G} = \gamma_{H,H} \tag{5.3.4}$$

for each *c&c* $H \subseteq T$.

Lemma 5.3.5 *For every finite abelian group T of length at most 2, the above linear equations for the fractions $\gamma_{T,H}$ have full rank, and determine these numbers, independently of any representation of T as a group of torsion sections of an elliptic surface.*

This is a rather long computation, which one can find in [MP3], Now that we have this in hand, we have no choice but to compute the γ's! This was also done in [MP3], and after plugging them into the **Size** formula, the result is the following.

Theorem 5.3.6 *Assume that M divides N. Then*

$$\mathbf{Size}(\mathbf{Z}/M \times \mathbf{Z}/N) = \frac{1}{N}\prod_p [1 + \nu_p(N/M)(\frac{p-1}{p+1})].$$

(Here $\nu_p(k)$ is the p-order of k, i.e., the largest exponent ν such that p^ν divides k.)

5.4 Applications to genus, rank, and Euler number

Comparing the lower bound of Proposition 5.1.3 with the upper bound of Proposition 5.3.2, we have the following, on which all of the applications are based:

Theorem 5.4.1 *Assume that X admits T as a group of torsion sections. Then*

$$2\chi + 2 - 2g + \mathbf{Rk} \leq 12\chi \cdot \mathbf{Size}(T).$$

There is also a divisibility condition on χ which will not be discussed further in these notes; it is the analogue of the result (from the last lecture) that $p^2 - 1$ divides 24χ if there is p-torsion in $\mathbf{MW}(X)$.

Let us draw the promised corollaries of the above theorem. Suppose first that $g = 0$, i.e., that X is an elliptic surface over $\mathbf{P}^1$. In this case Theorem 5.4.1 can be written as

$$\mathbf{Rk} \leq (12\mathbf{Size}(T) - 2)\chi - 2, \tag{5.4.2}$$

which implies that $\mathbf{Size}(T) > 1/6$, since $\mathbf{Rk} \geq 0$ and $\chi \geq 1$. There are a finite number of T's with $\mathbf{Size}(T) > 1/6$, and for each of these there is a corresponding

bound on the rank Rk of the Mordell-Weil group, and a divisibility statement for χ. We present the list in Table 5.4.2.

Table 5.4.2: Possible Mordell-Weil groups when $g = 0$

T	Size(T)	Rk
$\{0\}$	1	$\mathbf{R}k \leq 10\chi - 2$
$\mathbf{Z}/2$	2/3	$\mathbf{R}k \leq 6\chi - 2$
$\mathbf{Z}/3$	1/2	$\mathbf{R}k \leq 4\chi - 2$
$\mathbf{Z}/4$	5/12	$\mathbf{R}k \leq 3\chi - 2$
$\mathbf{Z}/5$	1/3	$\mathbf{R}k \leq 2\chi - 2$
$\mathbf{Z}/6$	1/3	$\mathbf{R}k \leq 2\chi - 2$
$\mathbf{Z}/7$	1/4	$\mathbf{R}k \leq \chi - 2$
$\mathbf{Z}/9$	2/9	$\mathbf{R}k \leq (2/3)\chi - 2$
$\mathbf{Z}/10$	2/9	$\mathbf{R}k \leq (2/3)\chi - 2$
$\mathbf{Z}/12$	5/24	$\mathbf{R}k \leq (1/2)\chi - 2$
$\mathbf{Z}/2 \times \mathbf{Z}/2$	1/2	$\mathbf{R}k \leq 4\chi - 2$
$\mathbf{Z}/2 \times \mathbf{Z}/4$	1/3	$\mathbf{R}k \leq 2\chi - 2$
$\mathbf{Z}/3 \times \mathbf{Z}/3$	1/3	$\mathbf{R}k \leq 2\chi - 2$
$\mathbf{Z}/2 \times \mathbf{Z}/6$	1/4	$\mathbf{R}k \leq \chi - 2$
$\mathbf{Z}/4 \times \mathbf{Z}/4$	1/4	$\mathbf{R}k \leq \chi - 2$
$\mathbf{Z}/2 \times \mathbf{Z}/8$	5/24	$\mathbf{R}k \leq (1/2)\chi - 2$
$\mathbf{Z}/3 \times \mathbf{Z}/6$	2/29	$\mathbf{R}k \leq (2/3)\chi - 2$
$\mathbf{Z}/5 \times \mathbf{Z}/5$	1/5	$\mathbf{R}k \leq (2/5)\chi - 2$

This list of 19 possible groups when $g = 0$ first appeared in Cox and Parry's article [CP1]. If one takes into account the divisibility conditions, and considers those possible when $\chi = 1$, (which is the rational elliptic surface case), one obtains the list of Cox [C3]. If one considers those possible when $\chi = 2$, (the $K3$ case), then we reproduce Cox's list again in this case [C3]. Cox's methods used the theory of elliptic modular surfaces.

Now suppose the base curve C has $g = 1$. In this case Theorem 5.4.1 can be written as

$$\mathbf{Rk} \leq (12 \cdot \mathbf{Size}(T) - 2)\chi$$

and so we must have $\mathbf{Size}(T) \geq 1/6$, since $\mathbf{Rk} \geq 0$ and $\chi \geq 1$. Thus any group occurring for $g = 0$ can occur for $g = 1$, and in fact the only additional groups T are those with $\mathbf{Size}(T) = 1/6$ exactly. These are listed below, with the following notation: (N) denotes $\mathbf{Z}/N$, and (M, N) denotes $\mathbf{Z}/M \times \mathbf{Z}/N$.

Groups with $\mathbf{Size}(T) = 1/6 :\ (11), (14), (15), (2, 10), (2, 12), (3, 9), (4, 8), (6, 6)$.

Note that in general, Theorem 5.4.1 can be written as

$$\mathbf{Rk} \leq (12 \cdot \mathbf{Size}(T) - 2)\chi - 2 + 2g$$

and so for any genus g, if $\mathbf{Size}(T) < 1/6$, then T cannot occur for large values of χ. Therefore the above set of groups (those with $\mathbf{Size}(T) \leq 1/6$) are, asymptotically for χ, the only groups that can occur.

A more precise asymptotic result is the following.

Proposition 5.4.3 *Suppose* $\chi \geq 2g - 2$. *Then* $\mathbf{Size}(T) \geq 1/12$.

Proof: We may suppose that $S = \mathbf{Size}(T)$ is less than $1/6$ and that $g \geq 2$. In this case $12S - 2 < 0$, so that $(12S - 2)\chi \leq (12S - 2)(2g - 2)$. Hence

$$\begin{aligned} 0 \leq \mathbf{Rk} &\leq (12S - 2)\chi - 2 + 2g \\ &\leq (12S - 2)(2g - 2) - 2 + 2g \\ &= (12S - 1)(2g - 2) \end{aligned}$$

which forces $12S - 1 \geq 0$ since $g \geq 2$. □

A corollary of this Proposition is that, by listing all the groups T with $\mathbf{Size}(T) \geq 1/12$, one sees by inspection that they all have order at most 144. (The group $(12, 12)$ is the largest on the list.) Therefore one has:

$$\text{If } \chi \geq 2g - 2, \text{ then } \mid T \mid \leq 144$$

which is a result of Hindry and Silverman [H-S].

Another easy consequence of the above formulation is the following, which gives a bound on the rank of the Mordell-Weil group.

Proposition 5.4.4 *Suppose* $\mathbf{Size}(T) \leq 1/6$. *Then* $\mathbf{Rk} \leq 2g - 2$.

The above first appeared in [C3]. .

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