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COMPLETENESS PROPERTIES OF CERTAIN FORMATIONS

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All groups we consider are finite. It is well known that the product of supersolvable normal subgroups is not supersolvable in general (see Huppert [3]).

In [1], Asaad and Shaalan proved the following result:

Let $G = G_1G_2$ be a group such that G_1 and G_2 are supersolvable subgroups. If every subgroup of G_1 is permutable with every subgroup of G_2, then G is supersolvable.

If G_1 and G_2 are subgroups of a group G such that every subgroup of G_1 is permutable with every subgroup of G_2, we say that G_1 and G_2 are *totally permutable.*

In [6], Maier proved that Asaad and Shaalan's result is a special case of a general completeness property of all saturated formations which contain the class of supersolvable groups. In [6], the following theorem is proved:

Let $G = G_1G_2$ be a group such that G_1 and G_2 are totally permutable subgroups. Let $\mathcal{F}$ be a saturated formation which contains the class of supersolvable groups. If G_1 and G_2 lie in $\mathcal{F}$, then so does G.

In [2] we give a generalization for an arbitrary number of factors of Maier's result. In [2] is proved:

Theorem 1. *Let* $G = G_1G_2....G_r$ *be a group such that* $G_1, G_2, ..., G_r$ *are* *pairwise totally permutable subgroups of* G. *Let* $\mathcal{F}$ *be a saturated formation which contains the class of supersolvable groups. If for all* $i \in \{1, 2, ..., r\}$ *the subgroups* G_i *are in* $\mathcal{F}$, *then* $G \in \mathcal{F}$.

If G_1 and G_2 are totally permutable subgroups of a group G, then $< x, y >=< x >< y >=< y >< x >$ is a supersolvable subgroup, for each $x \in G_1$ and $y \in G_2$, by ([4], p. 722, Th. 10.1).

If G_1 and G_2 are subgroups of a group G and $\mathcal{L}$ is a non-empty class of groups, we say that G_1 and G_2 are *$\mathcal{L}$-connected*, whenever for each $x \in G_1$ and $y \in G_2$ we have $< x, y > \in \mathcal{L}$.

Assuming this definition, we prove the following:

Theorem 2. *Let* $G = G_1 G_2 .... G_r$ *be a group such that* $G_1, G_2, ..., G_r$ *are pairwise permutable subgroups of* G. *Let* $\mathcal{L} = \mathcal{N}$ *be the class of nilpotent groups and let* $\mathcal{F}$ *be a saturated formation such that* $\mathcal{N} \subseteq \mathcal{F}$. *If for every pair* $i, j \in \{1, 2, ..., r\}$, $i \neq j$, *the subgroups* G_i *and* G_j *are* $\mathcal{N}$*-connected* $\mathcal{F}$*-groups, then* $G \in \mathcal{F}$.

Proof. Suppose the theorem is false and let G be a counterexample of smallest order.
Since the hypothesis is inherited by quotients, we conlude that G has a unique minimal normal subgroup N. Since $\mathcal{F}$ is saturated, we have $\Phi(G) = 1$.

Let p be a prime number and $i, j \in \{ 1, 2, ..., r \}$, such that $i \neq j$. Let $x \in G_i$ be a p-element and $y \in G_j$ a p'-element. Since $< x, y >$ is nilpotent, we have that y centralizes x.

Let $P_i \in Syl_p(G_i)$. Since $\mathbf{O}^p(G_j)$ is generated by all p'-elements of G_j, we have $\mathbf{O}^p(G_j) \leq \mathbf{C}_G(P_i)$. For the definition of $\mathbf{O}^p(G_j)$ see ([7] p. 142).

Set $G_j^* = \bigcap_p \mathbf{O}^p(G_j)$. The above consideration implies that $G_i \leq \mathbf{C}_G(G_j^*)$. Since our argument is true for all $i \in \{1, 2, ..., r\}$, such that $i \neq j$, we have that $G_j^* \trianglelefteq G$.

(I) Suppose $G_j^* \neq 1$, for some $j \in \{ 1, 2, ..., r \}$.

Because of the uniqueness of N we have $N \leq G_j^*$.

(a) If N is solvable, then $N = \mathbf{C}_G(N)$ and $G_i \leq N \leq G_j^*$, for all $i \in \{ 1, 2, ..., r \}$, with $i \neq j$. It follows that $G = G_j \in \mathcal{F}$.

(b) If N is not solvable, then $\mathbf{C}_G(N) = G_i = 1$ for all $i \in \{ 1, 2, ..., r \}$ with $i \neq j$. Again we have $G = G_j \in \mathcal{F}$.

(II) Suppose $G_j^* = 1$ for all $j \in \{ 1, 2, ..., r \}$.

Now G_j is nilpotent for all $j \in \{ 1, 2, ..., r \}$. Hence, $G_j = P_j \times \mathbf{O}^p(G_j)$, for every prime number p.

Let $i, j \in \{ 1, 2, ..., r \}$ such that $i \neq j$. By ([4], p.676, Th. 4.7) we have that $P_i P_j \in Syl_p(G_i G_j)$. Hence $P_1 P_2 ... P_r \in Syl_p(G)$.

Since for all $i \in \{1, 2, ..., r\}$ we have $\mathbf{O}^p(G_i) \leq \mathbf{C}_G(P_1 P_2 ... P_r)$, we conclude that $G_i \leq \mathbf{N}_G(P_1 P_2 ... P_r)$ and therefore $P_1 P_2 .... P_r \trianglelefteq G$. It follows that $G \in \mathcal{N} \subseteq \mathcal{F}$.

□

The following example shows that the theorem 2 is not true when $\mathcal{N} \subsetneqq \mathcal{L} \subseteq \mathcal{F}$, without additional hypothesis (see also the Example given in [6]):

Example. Let $G = \mathbf{S}_4$ be the symmetric group of degree 4. Let G_1 be the normal subgroup of order 4 of G and let G_2 be a subgroup of order 6 of G. Clearly, $G = G_1 G_2$. Let $\mathcal{L} = \mathcal{F} = \mathcal{NA}$ be the class of all groups whose commutator subgroups are nilpotent. Clearly, G_1 and G_2 are $\mathcal{L}$-connected $\mathcal{F}$-groups, but $G \notin \mathcal{F}$.

In view of the fact that the finite simple groups are 2-generated, the following seems to be reasonable:

Conjecture. *Let* $\mathcal{S}$ *be the class of solvable groups. If the group* $G = G_1G_2....G_r$ *is the product of the pairwise permutable and pairwise* $\mathcal{S}$*-connected* $\mathcal{S}$*-subgroups* G_i, *then* G *is solvable.*

To mention the solution of a particular case of this conjecture, we introduce the following notation:

Let $\mathcal{T}$ be the class of groups having Sylow-tower for the prime numbers arranged in decreasing order.

Proposition. *Let* $G = G_1G_2....G_r$ *be a group such that* $G_1, G_2, ..., G_r$ *are pairwise permutable and pairwise* $\mathcal{T}$*-connected supersolvable subgroups of* G. *Then* G *is a* $\mathcal{T}$*-group. In particular,* G *is solvable.*

Proof. Suppose the proposition is false. Let G be a counterexample of smallest order with r least possible. Every quotient group of G satisfies the hypothesis of the proposition. Because of the minimality of $|G|$, every proper quotient group is a $\mathcal{T}$-group.

Let p denotes the largest prime number divisor of $|G|$. We may assume that p divides $|G_1|$.

We have to produce a nonidentity normal p-subgroup N of G.

Because of the supersolvability of G_1, we can choose $< x >$ a normal subgroup of G_1, with $| < x > | = p$. We show $< x >$ is subnormal in G. Then $N =< x >^G$ is a normal p-subgroup of G.
First we show that $r \leq 2$. If $r \geq 3$, then $H = G_1G_2....G_{r-1}$ and $K = G_1G_2...G_{r-2}G_r$ are $\mathcal{T}$-groups, since r is least possible. Hence $< x >$ is subnormal in H and K. By ([5], p. 239, Th. 7.7.1) we have that $< x >$ is

subnormal in $HK = G$. So $G = G_1G_2$.

Let $g \in G$. Write $g = g_1g_2$ with $g_1 \in G_1$ and $g_2 \in G_2$.
Since $< x > \trianglelefteq G_1$, we have that $x^{g_1} = x^i$ with $1 \leq i \leq p$.
By hypothesis $< x, g_2 >$ is a $\mathcal{T}$-group, thus $< x, g_2 >_p \trianglelefteq < x, g_2 >$, where $< x, g_2 >_p$ denotes the Sylow-p-subgroup of $< x, g_2 >$.
Therefore $x, x^{g_2} \in < x, g_2 >_p$ and $x^g = x^{g_1g_2} = (x^i)^{g_2} = (x^{g_2})^i \in < x, g_2 >_p$.
It follows that $< x, x^g >$ is p-group, for all $g \in G$. By ([7], p. 195, Th. 4.8) we have that $< x >$ is subnormal in G. □

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References

[1] Asaad, M. and Shaalan, A. *On the Supersolvability of Finite Groups* Arch. Math. Vol. **53**, (1989), 318 - 326..

[2] Carocca, A. *Produtos de subgrupos mutuamente f-permutáveis em grupos finitos*, Doctoral Theisis, Universidade de Brasília, (1992).

[3] Huppert, B. *Monomiale Darstellungen Endlicher Gruppen,* Nagoya Math. J. **6**,(1953) 93-94.

[4] ____________ *Endliche Gruppen I,* Springer-Verlag, Berlin/Heidelberg/New York, 1967

[5] Lennox, J. and Stonehewer,St. E. *Subnormal Subgroups of Groups,* Clarendon Press, Oxford (1987).

[6] Maier, R. *A completeness property of certain formations.* Bull. London Math. Soc. **24**,(1992), 540-544.

[7] Suzuki, M. *Group Theory I*, Springer-Verlag, Berlin/Heidelberg/New York, 1982.

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