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TOPOLOGY OF SPECIAL GENERIC MAPS INTO $\mathbf{R}^3$

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Abstract

A smooth map between manifolds with only definite fold singular points is called a special generic map. We classify up to diffeomorphism those simply connected 5-manifolds which admit special generic maps into $\mathbf{R}^3$. We also classify special generic maps of simply connected 4- and 5-manifolds into $\mathbf{R}^3$ up to a certain equivalence. Furthermore, we construct closed n-manifolds ($^\forall n \geq 4$) which admit smooth maps into $\mathbf{R}^3$ with only fold singular points but which do not admit special generic maps into $\mathbf{R}^3$.

1. Introduction

Let $f : M^n \to N^p$ $(n \geq p)$ be a smooth map of a closed n-dimensional manifold into a p-dimensional manifold. We say that a singular point $q \in M^n$ of f is a *fold singular point of index* λ if f has the normal form as follows for some local coordinate systems around q and $f(q)$:

$$\begin{aligned} y_i \circ f &= x_i \qquad (1 \leq i \leq p-1) \\ y_p \circ f &= -x_p^2 - \cdots - x_{p+\lambda-1}^2 + x_{p+\lambda}^2 + \cdots + x_n^2 \,, \end{aligned}$$

where $0 \leq \lambda \leq n-p+1$ is an integer. If $\lambda = 0$ or $n-p+1$, we say that q is a *definite fold singular point*. A smooth map $f : M^n \to N^p$ with only definite fold singular points is called a *special generic map*. Note that a definite fold singular point is the simplest singularity among all possible singularities of smooth maps $M^n \to N^p$. Furthermore, if $f : M^n \to N^p$ is stable and N^p is open, then f has necessarily a definite fold singular point. Thus, it is fundamental to study special generic maps in studying the global topology of stable maps.

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Special generic maps were first defined by Burlet-de Rham [**BdR**], who showed that a closed 3-manifold admits a special generic map into $\mathbf{R}^2$ if and only if it is diffeomorphic to S^3 or the connected sum of some S^2-bundles over S^1. They also classified such maps up to a certain equivalence. After that, special generic maps have been studied by several authors; Porto-Furuya [**PF**], Sakuma [**Sak1, Sak2**], Saeki [**Sae1**], Hiratuka [**Hi**], Carrara [**C**], Motta-Porto-Sakuma [**MPS**], Kikuchi-Saeki [**KS**] etc.. Note also that the case where $n = p$ has been studied by Èliašberg [**E1**].

In this paper we mainly discuss three topics about special generic maps. In §2, we classify the simply connected 5-manifolds which admit special generic maps into $\mathbf{R}^3$ up to diffeomorphism. We achieve the classification using results obtained in [**Sae1**] together with a theorem of Hatcher [**Ha**]. In §3, we construct closed n-manifolds ($n \geq 4$) which admit smooth maps into $\mathbf{R}^3$ with only fold singular points but which do not admit special generic maps into $\mathbf{R}^3$. For $n \geq 5$ the construction is explicit and elementary, while for $n = 4$, instead of constructing explicitly, we show that the Moishezon-Teicher surface [**MT1**] (complex 2-manifold) with zero signature gives such an example. This result suggests that there are certain obstructions to eliminating indefinite fold singular points globally. In §4, we recall some equivalence relations between special generic maps and classify the special generic maps of simply connected 4- and 5-manifolds into $\mathbf{R}^3$ up to these equivalences. Finally in §5, we pose some important problems concerning special generic maps, where we define a new diffeomorphism invariant of closed manifolds using special generic maps. We also discuss a cobordism theory of special generic maps.

Throughout the paper, all homology and cohomology groups are with integral coefficients unless otherwise indicated. All manifolds and maps are assumed to be C^∞.

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2. Simply connected 5-manifolds with special generic maps

Since $\pi_1(SO(k))$ ($k \geq 3$) is isomorphic to $\mathbf{Z}_2$, there are exactly two orthogonal S^{k-1}-bundles over S^2. We denote by $S^2 \tilde{\times} S^{k-1}$ the total space of the unique non-trivial orthogonal S^{k-1}-bundle over S^2.

The purpose of this section is to prove the following.

Theorem 2.1. *Let M^5 be a simply connected closed 5-manifold. Then M^5 admits a special generic map into $\mathbf{R}^3$ if and only if it is diffeomorphic to one of the following manifolds:*

$$\sharp^r S^2 \times S^3 \quad (r \geq 0) \quad \text{or}$$
$$(\sharp^{r-1} S^2 \times S^3) \sharp (S^2 \tilde{\times} S^3) \quad (r \geq 1),$$

where $\sharp$ denotes connected sum and the connected sum over an empty set is assumed to be S^5. Furthermore, no two manifolds in this list are diffeomorphic.

Proof. Suppose that a simply connected closed 5-manifold M admits a special generic map into $\mathbf{R}^3$. Then by [**Sae1**, Theorem 6.16], M is diffeomorphic to the connected sum of a homotopy 5-sphere and some smooth S^3-bundles over S^2. Note that every homotopy 5-sphere is diffeomorphic to the standard 5-sphere S^5. Furthermore, since the natural inclusion $O(4) \hookrightarrow \mathrm{Diff}(S^3)$ is a weak homotopy equivalence by Hatcher [**Ha**], every *smooth* S^3-bundle is *orthogonal*. Hence, M is diffeomorphic to the connected sum of S^5 and some copies of $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$.

Lemma 2.2. *$S^2 \tilde{\times} S^3 \sharp S^2 \tilde{\times} S^3$ is diffeomorphic to $S^2 \times S^3 \sharp S^2 \tilde{\times} S^3$.*

Proof. Let E be the 6-manifold with boundary obtained by attaching two 2-handles h_1 and h_2 to a 0-handle D^6 simultaneously along unlinked circles C_1 and C_2 in ∂D^6. Recall that the normal bundles of C_i in ∂D^6 are the trivial D^4-bundle and that we have canonical trivializations corresponding to embedded 2-disks with boundary C_i. Note also that $\pi_1(SO(4))$ is isomorphic to $\mathbf{Z}_2$. We attach h_1

and h_2 with the non-trivial framing corresponding to the non-zero element of $\pi_1(SO(4))$. Then it is easy to see that ∂E is diffeomorphic to $S^2\tilde{\times}S^3\sharp S^2\tilde{\times}S^3$. If we slide the 2-handle h_1 over the other 2-handle h_2 once, then the framing of h_1 becomes trivial. Thus the boundary of the resulting manifold is diffeomorphic to $S^2\times S^3\sharp S^2\tilde{\times}S^3$. Since handle sliding does not change the diffeomorphism type of the 6-manifold, we have the conclusion. □

Using the above lemma, we see that M is diffeomorphic to one of the 5-manifolds in the list of Theorem 2.1.

Conversely, the manifolds listed in Theorem 2.1 admit special generic maps into $\mathbf{R}^3$. See Remark 6.15 and Lemma 5.4 of [**Sae1**].

Next we show that no two manifolds in the list are diffeomorphic. First note that $b_2(S^2\times S^3)=b_2(S^2\tilde{\times}S^3)=1$ and hence that $b_2(\sharp^r S^2\times S^3)=b_2((\sharp^{r-1}S^2\times S^3)\sharp(S^2\tilde{\times}S^3))=r$, where b_2 denotes the second Betti number. Thus we have only to show that $\sharp^r S^2\times S^3$ is not diffeomorphic to $(\sharp^{r-1}S^2\times S^3)\sharp(S^2\tilde{\times}S^3)$ $(r\geq 1)$.

Lemma 2.3. *The second Stiefel-Whitney class of $S^2\tilde{\times}S^3$ does not vanish.*

Proof. Let $\xi: Y\to S^2$ be the non-trivial orthogonal $\mathbf{R}^4$-bundle over S^2. Since it is non-trivial, we see that its second Stiefel-Whitney class $w_2(\xi)\in H^2(S^2;\mathbf{Z}_2)$ does not vanish by obstruction theory. Let $\pi: Y'\to S^2$ be the unit disk bundle associated with ξ. Then we see that $w_2(Y')=\pi^*w_2(\xi)$, since $TY'\cong\pi^*(TS^2)\oplus\pi^*(\xi)$, $w_1(TS^2)=0$ and $w_2(TS^2)=0$. Moreover, we have that $w_2(S^2\tilde{\times}S^3)=w_2(\partial Y')=i^*w_2(Y')$, where $i:\partial Y'\to Y'$ is the inclusion map, and that $\pi^*: H^2(S^2;\mathbf{Z}_2)\to H^2(Y';\mathbf{Z}_2)$ and $i^*: H^2(Y';\mathbf{Z}_2)\to H^2(\partial Y';\mathbf{Z}_2)$ are isomorphisms. Hence $w_2(S^2\tilde{\times}S^3)$ does not vanish. □

By virtue of Lemma 2.3, $(\sharp^{r-1}S^2\times S^3)\sharp(S^2\tilde{\times}S^3)$ has non-trivial second Stiefel-Whitney class, while $w_2(\sharp^r S^2\times S^3)$ vanishes. Thus they are not diffeomorphic to each other. This completes the proof of Theorem 2.1. □

Remark 2.4. Using a result of Hatcher [**Ha**], we can improve some results of [**Sae1**] as above. For example, Proposition 3.4 and Corollary 4.2 of [**Sae1**] holds also for $n-p=3$, and exotic 7-spheres do not admit a special generic map into $\mathbf{R}^4$ (see §4 of [**Sae1**]).

3. Manifolds which do not admit special generic maps into $\mathbf{R}^3$

Let $f : M^n \to N^p$ be a smooth map with only fold singular points. Then it is natural to ask if the indefinite fold singular points can be eliminated by a homotopy so that f becomes a special generic map. The answer is "no" in general as follows.

Theorem 3.1. *For every $n \geq 4$, there exists a closed n-dimensional manifold M^n which admits a smooth map into $\mathbf{R}^3$ with only fold singular points but which does not admit a special generic map into $\mathbf{R}^3$.*

Remark 3.2. Note that every closed orientable 3-manifold admits a special generic map into $\mathbf{R}^3$ ([**E1**]). Furthermore, a result similar to the above proposition holds also for $\mathbf{R}^2$. This is proved using Theorem 5.1 of [**Sae1**] and a result of Levine [**L**]. It is enough to find a closed n-manifold ($n \geq 3$) with even Euler number which is not in the list of Theorem 5.1 of [**Sae1**].

Proof of Theorem 3.1. First suppose that n is odd and $n \geq 5$. Set $n = 2k+1$ ($k \geq 2$). We construct a desired manifold using a method due to Smale [**Sm**] as follows. Consider two disjoint embedded k-spheres in S^{2k+1} ($= \partial D^{n+1}$) with linking number q ($\in \mathbf{N}$). Let E be the $(n+1)$-manifold with boundary obtained by attaching two $(k+1)$-handles to the 0-handle D^{n+1} simultaneously along the k-spheres above. Here we assume that the handles are embedded in $\mathbf{R}^{n+1}$ and we choose the framings so that the attaching is realized in $\mathbf{R}^{n+1}$; i.e., so that E is immersed into $\mathbf{R}^{n+1}$. This is possible, since each attaching k-sphere is unknotted in S^{2k+1}. Then we see that the n-manifold $M^n = \partial E$ is stably parallelizable, since so is E. Thus, by **[E2]**, M^n admits a smooth map into $\mathbf{R}^3$ with only fold singular points. Furthermore, it is easy to see that M^n is simply connected. Thus, if M^n admits a special generic map into $\mathbf{R}^3$, it is homeomorphic to S^n or the connected sum of some S^{n-2}-bundles over S^2 ([**Sae1**, Proposition 6.13]).

Lemma 3.3. *Let X be an S^{n-2}-bundle over S^2. Then $\tilde{H}_i(X) = 0$ for $i \neq 2, n-2$.*

Proof. Let $\tilde{X}$ be the (topological) D^{n-1}-bundle associated with the S^{n-2}-bundle over S^2. Then $\tilde{X}$ is a topological $(n+1)$-manifold with boundary and X is canonically homeomorphic to $\partial\tilde{X}$. Since $\tilde{X}$ is homotopy equivalent to S^2, we have

$$\tilde{H}_i(\tilde{X}) \cong \begin{cases} 0 & (i \neq 2) \\ \mathbf{Z} & (i = 2). \end{cases}$$

Consider the following exact sequence:

$$\tilde{H}_{i+1}(\tilde{X}, \partial\tilde{X}) \to \tilde{H}_i(\partial\tilde{X}) \to \tilde{H}_i(\tilde{X}).$$

Since $\tilde{H}_{i+1}(\tilde{X}, \partial\tilde{X}) \cong \tilde{H}^{n-i}(\tilde{X})$, we see that $\tilde{H}_{i+1}(\tilde{X}, \partial\tilde{X}) = 0$ and $\tilde{H}_i(\tilde{X}) = 0$ for $i \neq 2, n-2$. Hence we have $\tilde{H}_i(X) \cong \tilde{H}_i(\partial\tilde{X}) = 0$. □

It follows from this lemma that we have the same conclusion for S^n and the connected sum of S^{n-2}-bundles over S^2.

Lemma 3.4. $H_k(M) \cong \mathbf{Z}_q \oplus \mathbf{Z}_q$.

Proof. Consider the following exact sequence:

$$H_{k+1}(E) \xrightarrow{\alpha} H_{k+1}(E, M) \to H_k(M) \to H_k(E).$$

Note that $H_{k+1}(E) \cong H_{k+1}(E, M) \cong \mathbf{Z} \oplus \mathbf{Z}$ and that $H_k(E) = 0$. Furthermore, by the Poincaré duality, the map α above corresponds to the intersection form

$H_{k+1}(E) \times H_{k+1}(E, M) \to \mathbf{Z}$ of E. Thus α has a matrix representative of the form

$$\begin{pmatrix} r & q \\ q & s \end{pmatrix},$$

where r and s are the self-intersection numbers of the embedded $(k+1)$-spheres S_1 and S_2 in E which consist of the cores of the $(k+1)$-handles and $(k+1)$-disks embedded in the 0-handle D^{n+1} with boundary the attaching spheres. Since S_1 and S_2 are embedded in $\mathbf{R}^{n+1}$ by the immersion $E \to \mathbf{R}^{n+1}$, we see that $r = s = 0$. Thus we have the conclusion. □

By Lemmas 3.3 and 3.4, we see that M^n is not homeomorphic to the connected sum of some S^{n-2}-bundles over S^2 if $q \neq 1$. Thus M^n cannot admit a special generic map into $\mathbf{R}^3$.

Next suppose that n is even and $n \geq 6$. Set $n = 2k$ $(k \geq 3)$. Consider a $(k-1)$-sphere and a k-sphere disjointedly embedded in S^{2k} $(= \partial D^{n+1})$ with linking number q $(\in \mathbf{N})$. Let E' be the $(n+1)$-manifold with boundary obtained by attaching a k-handle and a $(k+1)$-handle to the 0-handle D^{n+1} along the spheres above. Here, we realize the attaching in $\mathbf{R}^{n+1}$ as in the previous case. Set $M^n = \partial E'$. Note that M^n is simply connected, since $k \geq 3$.

Lemma 3.5. $H_{k-1}(M) \cong H_k(M) \cong \mathbf{Z}_q$.

Proof. Consider the following exact sequence:

$$\begin{aligned} &H_{k+1}(E') \xrightarrow{\alpha} H_{k+1}(E', M) \to H_k(M) \to \\ &\quad H_k(E') \xrightarrow{\beta} H_k(E', M) \to H_{k-1}(M) \to H_{k-1}(E'). \end{aligned}$$

Note that $H_{k+1}(E') \cong H_{k+1}(E', M) \cong H_k(E') \cong H_k(E', M) \cong \mathbf{Z}$ and that $H_{k-1}(E') = 0$. Furthermore, by the Poincaré duality, we see that α and β are the maps corresponding to the multiplication by q. Hence, we have the desired conclusion. □

Thus we see that the n-manifold M^n for $q \neq 1$ is the desired one as in the previous case.

Finally we consider the case $n = 4$. Let M^4 be the simply connected Moishezon-Teicher surface (complex 2-manifold) with zero signature [**MT1**]. By [**K2**] (see also [**K1**]), M^4 is spin, and hence it is stably parallelizable. Thus by [**E2**], M^4 admits a smooth map into $\mathbf{R}^3$ with only fold singular points. Suppose that M^4 admits a special generic map into $\mathbf{R}^3$. Then by [**Sae1**], M^4 is diffeomorphic to $X_1 \sharp X_2$, where neither of X_i has negative definite intersection form. (Note that the second Betti number of the Moishezon-Teicher surface is very big. See [**MT2**].) This contradicts a result of Donaldson [**D**]. Thus M^4 cannot admit a special generic map into $\mathbf{R}^3$. This completes the proof of Theorem 3.1. □

Note that the construction for the case $n = 2k$ $(k \geq 3)$ does not work for $n = 4$. This is because the constructed manifold M^n is not simply connected for $k = 2$.

Next we consider a more general class than the special generic maps. Let $f : M^n \to N^p$ $(n \geq p)$ be a smooth map with only fold singular points. We say that f is *simple* if every component of $f^{-1}(a)$ contains at most one singular point for all $a \in N^p$ ([**Sae2, Sak2**]). Note that a special generic map is always simple. Thus the class of simple maps is an intermediate class of the special generic maps and the maps with only fold singular points.

Proposition 3.6. *Let M^4 be a simply connected closed 4-manifold. If M^4 admits a simple map into N^3 with only fold singular points for a 3-manifold N^3, then it is homeomorphic to one of the following manifolds:*

$$\sharp^r S^2 \times S^2 \quad (r \geq 0) \quad \text{or}$$
$$(\sharp^{r-1} S^2 \times S^2) \sharp (S^2 \tilde{\times} S^2) \quad (r \geq 1),$$

where the connected sum over an empty set is assumed to be S^4.

Proof. By [**Sae2**], M^4 is null-cobordant in the oriented category; in particular, its signature vanishes. Then, by Freedman [**F**], M^4 must be homeomorphic to one of the above 4-manifolds. □

Since the 4-manifolds in the above list admit special generic maps into $\mathbf{R}^3$, it is natural to conjecture that if a simply connected closed 4-manifold admits a simple map into $\mathbf{R}^3$ with only fold singular points, then it admits also a special generic map into $\mathbf{R}^3$. This problem seems difficult, since a component of the singular set of a simple map is not necessarily homeomorphic to S^2. If all the components of the singular set of the given simple map are homeomorphic to S^2, then it would be possible to prove the conjecture using graph theoretical techniques as in [**Sae4**]. We do not know if there exists a simply connected 4-manifold which admits a smooth map into $\mathbf{R}^3$ with only fold singular points but which does not admit simple ones. In the case of the maps of 3-manifolds into $\mathbf{R}^2$, such 3-manifolds are known to exist [**Sae3**].

4. Classification of special generic maps

In this section, we discuss classifications of special generic maps up to some equivalence relations. First, we recall the definition of the Stein factorization. Let $f : M^n \to N^p$ be a special generic map of a closed n-manifold. For $x, y \in M^n$, define $x \sim y$ if $f(x) = f(y)(= a)$ and x, y are in the same connected component of $f^{-1}(a)$. Denote by W_f the quotient space of M^n by this equivalence relation and by $q_f : M^n \to W_f$ the quotient map. We have a unique map $\bar{f} : W_f \to N^p$ such that $f = \bar{f} \circ q_f$. The space W_f or the commutative diagram

$$\begin{array}{ccc} M^n & \xrightarrow{f} & N^p \\ {\scriptstyle q_f} \searrow & & \nearrow {\scriptstyle \bar{f}} \\ & W_f & \end{array}$$

is called the *Stein factorization* of f. Note that W_f can be given a structure of a smooth p-dimensional manifold so that $\bar{f}$ is an immersion and that $q_f \mid S(f) : S(f) \to \partial W_f$ is a diffeomorphism, where $S(f) = \{q \in M^n; \mathrm{rank} df_q < p\}$ is the *singular set* of f.

Definition 4.1. Let $f : M_1^n \to N_1^p$ and $g : M_2^n \to N_2^p$ be special generic maps of closed n-manifolds into p-manifolds. We say that f and g are *quasi-equivalent* if there exist diffeomorphisms $\Phi : M_1^n \to M_2^n$ and $\varphi : W_f \to W_g$ which make the following diagram commutative:

$$\begin{array}{ccc} M_1^n & \xrightarrow{\Phi} & M_2^n \\ q_f \downarrow & & \downarrow q_g \\ W_f & \xrightarrow{\varphi} & W_g. \end{array}$$

Note that this equivalence was first introduced by Burlet-de Rham [**BdR**] and was called "equivalence" (see also [**PF**]). They showed that two special generic maps $f : M_1^3 \to \mathbf{R}^2$ and $g : M_2^3 \to \mathbf{R}^2$ of closed orientable 3-manifolds are quasi-equivalent if and only if $b_1(M_1^3) = b_1(M_2^3)$ and $\sharp S(f) = \sharp S(g)$, where b_1 denotes the first Betti number and $\sharp S(f)$ ($\sharp S(g)$) denotes the number of connected components.

Definition 4.2. Let $f : M_1^n \to N_1^p$ and $g : M_2^n \to N_2^p$ be special generic maps. We say that f and g are *regularly equivalent* if there exist diffeomorphisms $\Phi : M_1^n \to M_2^n, \varphi : W_f \to W_g$ and $\psi : N_1^p \to N_2^p$ such that the diagram

$$\begin{array}{ccc} M_1^n & \xrightarrow{\Phi} & M_2^n \\ q_f \downarrow & & \downarrow q_g \\ W_f & \xrightarrow{\varphi} & W_g \end{array}$$

is commutative and that the immersions $\bar{f}$ and $\psi^{-1} \circ \bar{g} \circ \varphi : W_f \to N_1^p$ are regularly homotopic.

Note that this equivalence was first defined by Porto-Furuya [**PF**], although they did not consider the diffeomorphism $\psi : N_1^p \to N_2^p$. Here we have added this diffeomorphism so that the following proposition appears to be natural when compared with the most natural equivalence – the right-left equivalence.

Proposition 4.3. *Let $f : M_1^n \to N_1^p$ and $g : M_2^n \to N_2^p$ be special generic maps of closed n-manifolds. Then f and g are regularly equivalent if and only if there exist a smooth family of special generic maps $f_t : M_1^n \to N_1^p$ $(t \in [0,1])$ and diffeomorphisms $\Phi : M_1^n \to M_2^n$ and $\psi : N_1^p \to N_2^p$ such that $f_0 = f$ and $g = \psi \circ f_1 \circ \Phi^{-1}$.*

Note that the above proposition is proved by the same argument as in [**PF**] (see also [**Sae1**, §6]).

Remark 4.4. We see easily that the right-left equivalence implies the regular equivalence, which in turn implies the quasi-equivalence.

In the following, we discuss classifications of special generic maps of simply connected 4-manifolds into $\mathbf{R}^3$ up to the above equivalences.

First, we recall some of the results of [**Sae1**]. Let $f : M^4 \to \mathbf{R}^3$ be a special generic map of a simply connected closed 4-manifold into $\mathbf{R}^3$. Then each component of the singular set $S(f)$ is diffeomorphic to S^2 and the number of connected components is equal to $b_2(M^4)/2+1$, where b_2 denotes the second Betti number. Let $\Sigma(f)$ be the closed 3-manifold obtained by attaching 3-balls to the boundaries of W_f. Note that $\Sigma(f)$ is a homotopy 3-sphere, since $\pi_1(\Sigma(f)) \cong \pi_1(W_f) = \{1\}$ (see [**Sae1**, Proposition 3.9]). Denote by Θ the set of all diffeomorphism classes of homotopy 3-spheres. Furthermore, let $S(f) = S_1 \cup \cdots \cup S_s$ $(s = b_2(M^4)/2+1)$ be the components of $S(f)$. We fix an orientation of M^4 and set $n_i(f) = S_i \cdot S_i$, where $S_i \cdot S_i$ is the self-intersection number of S_i in M^4. We renumber the indices if necessary so that $n_1(f) \leq n_2(f) \leq \cdots \leq n_s(f)$. Note that, if we change the orientation of M^4, this sequence transforms to $-n_s(f) \leq \cdots \leq -n_1(f)$.

Lemma 4.5. $\sum_{i=1}^{s} n_i(f) = 0$.

Proof. This follows from Proposition 3.16 of [**Sae1**]. □

Set $\Lambda_s = \{(n_1, \cdots, n_s) \in \mathbf{Z}^s; n_1 \leq \cdots \leq n_s, \sum_{i=1}^s n_i = 0\}$. Define $(n_1, \cdots, n_s) \sim (n'_1, \cdots, n'_s)$ if $(n_1, \cdots, n_s) = (n'_1, \cdots, n'_s)$ or $(n_1, \cdots, n_s) = -(n'_s, \cdots, n'_1)$. Finally set $\tilde{\Lambda}_s = \Lambda_s / \sim$. Now the element $(\Sigma(f); (n_1(f), \cdots, n_s(f))) \in \Theta \times \tilde{\Lambda}_s$ is well-defined for a special generic map f.

Theorem 4.6. *The map which associates $(\Sigma(f); (n_1(f), \cdots, n_s(f)))$ with a special generic map $f : M^4 \to \mathbf{R}^3$ is a bijection to the set $\Theta \times \tilde{\Lambda}_s$ from the set of all regular equivalence classes of special generic maps of simply connected closed 4-manifolds of second Betti number $2(s-1)$ into $\mathbf{R}^3$.*

Proof. First, the above map is clearly well-defined, since regularly equivalent special generic maps have diffeomorphic Stein factorizations and diffeomorphic singular sets as submanifolds in the 4-manifolds. The injectivity follows from the same argument as in the proof of Proposition 6.11 of [**Sae1**]. Thus it suffices to show the surjectivity. Let Σ be a homotopy 3-sphere and take $(n_1, \cdots, n_s) \in \Lambda_s$. Let W be the compact 3-manifold obtained by removing s disjoint open 3-balls from Σ. We construct an orthogonal D^2-bundle over W as follows. Let $\partial W = S_1 \cup \cdots \cup S_s$ be the components of ∂W and $i_j : S_j \to W$ $(j = 1, \cdots, s)$ the inclusion maps. Note that S_j are diffeomorphic to S^2. We fix an orientation of W and orient S_j as the boundary of W. Let $\gamma_j \in H^2(S_j)$ be the generator corresponding to the orientation. Let $\pi_j : E_j \to S_j$ be the oriented orthogonal D^2-bundle over S_j with Euler class $n_j \gamma_j \in H^2(S_j)$. Consider the following exact sequence of cohomology:

$$H^2(W) \to H^2(\partial W) \xrightarrow{\alpha} H^3(W, \partial W).$$

Note that $H^3(W, \partial W) \cong \mathbf{Z}$, $H^2(\partial W) \cong \oplus_{j=1}^s H^2(S_j)$ and that $\alpha(n_j \gamma_j) = n_j \in \mathbf{Z}$. Since $\sum_{j=1}^s n_j = 0$ by our hypothesis, we see that $\alpha(\oplus_{j=1}^s n_j \gamma_j) = 0$. Hence, there exists an element $e \in H^2(W)$ such that $(i_j)^* e = n_j \gamma_j$ $(j = 1, \cdots, s)$ by the exactness of the above sequence. Then let $\pi : E \to W$ be the oriented orthogonal D^2-bundle over W with Euler class $e \in H^2(W)$. Note that $\pi | \pi^{-1}(S_j) : \pi^{-1}(S_j) \to S_j$ is isomorphic to π_j. Let $f : \partial E \to \mathbf{R}^3$ be a special generic map constructed as in [**Sae1**] from $\pi : E \to W$. Note that $M^4 = \partial E$ is a closed simply connected 4-manifold with second Betti number $2(s-1)$. Furthermore, we see easily form the construction that $\Sigma(f)$ is diffeomorphic to Σ and that $(n_1, \cdots, n_s) \sim (n_1(f), \cdots, n_s(f))$. This completes the proof of Theorem 4.6. □

Remark 4.7. Let $f : M^4 \to \mathbf{R}^3$ be the special generic map constructed as above from $(\Sigma; (n_1, \cdots, n_s)) \in \Theta \times \tilde{\Lambda}_s$. Then M^4 is spin if and only if all the integers n_i are even. This is proved as follows. For $s = 1$, this is obvious, since $n_1 = 0$ and M^4 is a homotopy 4-sphere. Thus we assume $s \geq 2$. Let $S(f) = S'_1 \cup \cdots \cup S'_s$ be the connected components of $S(f)$ with $S'_i \cdot S'_i = n_i$. Furthermore, set $S''_j = q_f^{-1}(C_j)$ $(j = 1, \cdots, s-1)$, where $q_f : M^4 \to W$ is the quotient map in the Stein factorization of f and C_j is a properly embedded arc in W whose end points are in S_j and S_s and which is normal to ∂W. We may assume that C_j are pairwise disjoint. Then S''_j is a topologically embedded 2-sphere in M^4 and we see that the intersection matrix of $S'_1, \cdots, S'_{s-1}, S''_1, \cdots, S''_{s-1}$ is equal to

$$\left(\begin{array}{ccc|ccc} n_1 & & 0 & \pm 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & n_{s-1} & 0 & & \pm 1 \\ \hline \pm 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & \pm 1 & & & \end{array}\right).$$

(See the proof of Proposition 3.15 of [**Sae1**].) Since this $2(s-1)$ by $2(s-1)$ matrix is unimodular and $\operatorname{rank} H_2(M) = 2(s-1)$, we see that $S'_1, \cdots, S'_{s-1}, S''_1, \cdots, S''_{s-1}$ are a base of $H_2(M)$ and that the intersection form of M^4 is represented by the above matrix. Thus the intersection form is even if and only if n_i are all even. (Note that $\sum_{i=1}^{s} n_i = 0$ by Lemma 4.5.) Recall that a simply connected closed 4-manifold is spin if and only if its intersection form is even. Hence, if n_i are all even, then M^4 is homeomorphic to $\sharp^{s-1} S^2 \times S^2$; otherwise M^4 is homeomorphic to $(\sharp^{s-2} S^2 \times S^2) \sharp (S^2 \tilde{\times} S^2)$.

Remark 4.8. By the same argument, we see that the same map as in Theorem 4.6 induces a bijection to the set $\Theta \times \tilde{\Lambda}_s$ from the set of all *quasi-equivalence* classes of special generic maps of simply connected closed 4-manifolds of second Betti number $2(s-1)$ into $\mathbf{R}^3$. Thus, in this case, the classification up to quasi-equivalence and that up to regular equivalence coincide.

Remark 4.9. Burlet-de Rham [**BdR**] have shown that there are only finitely many quasi-equivalence classes of special generic maps of a fixed closed 3-manifold into $\mathbf{R}^2$. Theorem 4.6 (Remark 4.8) shows that this is not true in general for special generic maps of closed 4-manifolds into $\mathbf{R}^3$, since the set $\tilde{\Lambda}_s$ contains infinitely many elements if $s \geq 2$.

For special generic maps of closed simply connected 5-manifolds into $\mathbf{R}^3$, we use the second Stiefel-Whitney class instead of the Euler class and can obtain a similar result as follows. Set $\Lambda'_s = \{(n'_1, \cdots, n'_s) \in (\mathbf{Z}_2)^s; \sum_{i=1}^s n'_i = 0\}$ and $\tilde{\Lambda}'_s = \Lambda'_s / \mathcal{G}_s$, where the symmetric group $\mathcal{G}_s$ of order s acts on Λ'_s naturally. For a special generic map $f : M^5 \to \mathbf{R}^3$ of a closed simply connected 5-manifold into $\mathbf{R}^3$, let $S(f) = S_1 \cup \cdots \cup S_s$ be the connected components of $S(f)$. It is not difficult to show that $s = \text{rank} H_2(M^5) + 1$. Define $n'_i(f) \in \mathbf{Z}_2$ to be the second Stiefel-Whitney class $w_2(\nu_i) \in H^2(S_i) \cong \mathbf{Z}_2$ of the normal bundle ν_i of S_i in M^5. Furthermore, we define $\Sigma(f) \in \Theta$ as before. Then *the map which associates* $(\Sigma(f); (n'_1(f), \cdots, n'_s(f)))$ *with* f *is a bijection to the set* $\Theta \times \tilde{\Lambda}'_s$ *from the set of all regular equivalence (quasi-equivalence) classes of special generic maps of closed simply connected* 5*-manifolds of second Betti number* $s-1$ *into* $\mathbf{R}^3$. (For the proof, we use a result of Hatcher [**Ha**] as in §2. Details of the proof are left to the reader.) In particular, if the set Θ of all diffeomorphism classes of homotopy 3-spheres is finite, the number of regular equivalence (quasi-equivalence) classes of special generic maps of a fixed closed simply connected 5-manifold into $\mathbf{R}^3$ is finite, since the set $\tilde{\Lambda}'_s$ is finite.

Note also that if $n'_i(f) = 0$ for $i = 1, \cdots, s$, then M^5 is diffeomorphic to $\sharp^{s-1} S^2 \times S^3$; otherwise M^5 is diffeomorphic to $(\sharp^{s-2} S^2 \times S^3) \sharp (S^2 \tilde{\times} S^3)$.

For special generic maps $f : M^n \to \mathbf{R}^3$ of closed simply connected n-manifolds ($n \geq 6$) into $\mathbf{R}^3$, we do not know if such classifications as above are possible. The difficulty lies in the fact that a smooth D^{n-1}-bundle is not necessarily orthogonal if $n \geq 6$.

Finally we note that everything in the above argument works under the hypothesis that $H_1(M) = 0$ or $H_1(M; \mathbf{Z}_2) = 0$, where M is the source 4- or 5-manifold. Denote by $\Theta(\mathbf{Z})$ (or $\Theta(\mathbf{Z}_2)$) the set of all diffeomorphism classes of closed 3-manifolds Σ^3 with $H_*(\Sigma^3) \cong H_*(S^3)$ (resp. $H_*(\Sigma^3; \mathbf{Z}_2) \cong H_*(S^3; \mathbf{Z}_2)$). Such 3-manifolds are called $\mathbf{Z}$-homology (resp. $\mathbf{Z}_2$-homology) 3-spheres. Then we have the following.

Proposition 4.10. (1) *Let M^4 be a closed 4-manifold with $H_1(M^4) = 0$. Then M^4 admits a special generic map into $\mathbf{R}^3$ if and only if it is diffeomorphic to one of the following manifolds:*

$$(\sharp^r S^2 \times S^2)\sharp\Sigma^4 \quad (r \geq 0) \quad \text{or}$$
$$(\sharp^{r-1} S^2 \times S^2)\sharp(S^2 \tilde{\times} S^2)\sharp\Sigma^4 \quad (r \geq 1),$$

where Σ^4 is a $\mathbf{Z}$-homology 4-sphere which is the boundary of $\Delta^3 \times D^2$ for some $\mathbf{Z}$-homology 3-ball Δ^3.

(2) *Let M^5 be a closed 5-manifold with $H_1(M) = 0$ (or $H_1(M; \mathbf{Z}_2) = 0$). Then M^5 admits a special generic map into $\mathbf{R}^3$ if and only if it is diffeomorphic to one of the following manifolds:*

$$(\sharp^r S^2 \times S^3)\sharp\Sigma^5 \quad (r \geq 0) \quad \text{or}$$
$$(\sharp^{r-1} S^2 \times S^3)\sharp(S^2 \tilde{\times} S^3)\sharp\Sigma^5 \quad (r \geq 1),$$

where Σ^5 is a $\mathbf{Z}$-homology (resp. $\mathbf{Z}_2$-homology) 5-sphere which is the boundary of $\Delta^3 \times D^3$ for some $\mathbf{Z}$-homology (resp. $\mathbf{Z}_2$-homology) 3-ball Δ^3.

Proposition 4.11. (1) *The map which associates $(\Sigma(f); (n_1(f), \cdots, n_s(f)))$ with a special generic map $f : M^4 \to \mathbf{R}^3$ is a bijection to the set $\Theta(\mathbf{Z}) \times \tilde{\Lambda}_s$ from the set of all regular equivalence (quasi-equivalence) classes of special generic maps of closed 4-manifolds with vanishing $\mathbf{Z}$-coefficient first homology and with second Betti number $2(s-1)$ into $\mathbf{R}^3$.*

(2) *The map which associates* $(\Sigma(f); (n'_1(f), \cdots, n'_s(f)))$ *with a special generic map* $f : M^5 \to \mathbf{R}^3$ *is a bijection to the set* $\Theta(\mathbf{Z}) \times \tilde{\Lambda}'_s$ *(or* $\Theta(\mathbf{Z}_2) \times \tilde{\Lambda}'_s$*) from the set of all regular equivalence (quasi-equivalence) classes of special generic maps of closed* 5*-manifolds with vanishing* $\mathbf{Z}$*-coefficient (resp.* $\mathbf{Z}_2$*-coefficient) first homology and with second Betti number* $s-1$ *into* $\mathbf{R}^3$.

The proofs of these propositions are similar to the simply connected case and are left to the reader.

Remark 4.12. For the 4-dimensional case, we cannot replace $\mathbf{Z}$ with $\mathbf{Z}_2$. This is because a D^2-bundle over a $\mathbf{Z}_2$-homology 3-ball Δ^3 is not necessarily trivial, since $H^2(\Delta^3) \cong H_1(\Delta^3)$ does not necessarily vanish.

5. Problems

We end this paper by posing some problems.

In [**Sae1**], we have determined the diffeomorphism classes of closed n-manifolds which admit special generic maps into $\mathbf{R}^2$. The remaining problems are as follows.

Problem 5.1. Classify up to diffeomorphism the closed n-manifolds which admit special generic maps into $\mathbf{R}^2$.

Note that, in the list of Theorem 5.1 of [**Sae1**], there are some repetitions of diffeomorphism classes as is pointed out in [**Sae1**, Remark 5.7].

Problem 5.2. Classify up to quasi-equivalence (or regular equivalence) the special generic maps of closed n-manifolds into $\mathbf{R}^2$.

For a closed n-dimensional manifold M^n, define $S(M^n)$ to be the set of the integers p with $1 \le p \le n$ such that there exists a special generic map $f : M^n \to \mathbf{R}^p$. Of course this set is an invariant of the diffeomorphism class of M^n.

Problem 5.3. Study the set $S(M^n)$.

Here we list some of its properties and some examples of the calculation of this invariant.

(5.3.1) If M^n is not null-cobordant, then $S(M^n) = \emptyset$ ([**KS**]).

(5.3.2) $S(S^n) = \{1, 2, \cdots, n\}$.

Proof. This is obvious, since we can construct special generic maps $S^n \to \mathbf{R}^p$ $(1 \leq {}^\forall p \leq n)$ by restricting the standard projection $\mathbf{R}^{n+1} \to \mathbf{R}^p$ to the unit n-sphere. □

(5.3.3) If $S(M^n) \supset \{1, n-3\}, \{1, n-2\}$ or $\{1, n-1\}$, then M^n is diffeomorphic to S^n. In particular, M^n is diffeomorphic to S^n if and only if $S(M^n) = \{1, 2, \cdots, n\}$.

Proof. If $S(M^n) \ni 1$, then M^n is diffeomorphic to S^n for $n \leq 6$ and is a homotopy n-sphere if $n \geq 7$. Furthermore, for $n \geq 7$, if $S(M^n) \ni n-3, n-2$ or $n-1$ for a homotopy n-sphere M^n, then M^n is diffeomorphic to S^n by [**Sae1**, Corollary 4.2] (see also Remark 2.4 in §2 of the present paper). □

(5.3.4) If Σ^n is an exotic n-sphere $(n \geq 7)$, then $\{1, 2, n\} \subset S(\Sigma^n) \subset \{1, 2, \cdots, n-4, n\}$.

This follows from [**Sae1**]. In particular, (5.3.2) and (5.3.4) show that the invariant is *not* a *homeomorphism* type invariant in general. Note that this fact could be proved using the Moishezon-Teicher surface [**MT1**] of Theorem 3.1 together with Freedman's classification result in dimension 4 [**F**].

(5.3.5) $S(S^p \times S^q) = \{m+1, m+2, \cdots, p+q\}$, where $m = \min\{p, q\}$.

Proof. Assume $p = m \leq q$. For $1 \leq s \leq q$, we have a special generic map $g : S^q \to \mathbf{R}^s$. Consider the map

$$f : S^p \times S^q \xrightarrow{\mathrm{id} \times g} S^p \times \mathbf{R}^s \xrightarrow{\eta} \mathbf{R}^{p+s},$$

where η is an embedding. Then we see that f is a special generic map and that $S(S^p \times S^q) \supset \{p+1, p+2, \cdots, p+q\}$. If there exists a special generic map

$S^p \times S^q \to \mathbf{R}^s$ with $1 \le s \le p$, then, by [**Sae1**, Corollary 3.12], $H_p(S^p \times S^q)$ must vanish. This is a contradiction. Thus we have proved (5.3.5). □

(5.3.6) If the i-th Stiefel-Whitney class $w_i(M^n) \in H^i(M^n; \mathbf{Z}_2)$ $(i \ge 1)$ of M^n does not vanish, then $S(M^n) \subset \{m+1, m+2, \cdots, n-i+1\}$, where $m = \min\{i, n-i\}$.

Proof. If $S(M^n)$ contains an element $p > n-i+1$, then $w_j(M^n)$ must vanish for ${}^\forall j > n-p+1$ by [**Sae1**, Corollaries 3.18 and 7.5]. This contradicts to our assumption. Furthermore, if $S(M^n)$ contains an element $p \le i, n-i$, then $H^j(M^n; \mathbf{Z}_2)$ must vanish for $p \le {}^\forall j \le n-p$ by [**Sae1**, Remark 3.21]. This is a contradiction. □

(5.3.7) Let M^{2n} $(n \equiv 0, 1, 2, 4 \pmod 8), n \ge 1)$ be the total space of an orthogonal S^n-bundle over S^n with non-trivial Stiefel-Whitney class $w_n(M^{2n}) \in H^n(S^n; \mathbf{Z}_2)$. (Note that $\pi_{n-1}(O(n+1)) \cong \mathbf{Z}$ for $n-1 \equiv 3, 7 \pmod 8$ and $\mathbf{Z}_2$ for $n-1 \equiv 0, 1 \pmod 8$.) Then $S(M^{2n}) = \{n+1\}$.

Proof. $S(M^{2n}) \subset \{n+1\}$ follows from (5.3.6). Furthermore, since the corresponding element in $\pi_{n-1}(O(n+1))$ comes from $\pi_{n-1}(O(n))$ by the homomorphism induced by the inclusion $O(n) \to O(n+1)$, we have a fiber-wise Morse function $g : M^{2n} \to \mathbf{R}$ which has exactly two critical points on each fiber. Then the map

$$f : M^{2n} \xrightarrow{\pi \times g} S^n \times \mathbf{R} \xrightarrow{\eta} \mathbf{R}^{n+1}$$

is a special generic map, where $\pi : M^{2n} \to S^n$ is the bundle projection and η is an embedding. Hence we have proved (5.3.7). □

We can define similar invariants using smooth maps with only fold singular points or simple maps instead of special generic maps. Denote by $F(M^n)$ (or $SF(M^n)$) the set of the integers p with $1 \le p \le n$ such that there exists a (simple) smooth map $f : M^n \to \mathbf{R}^p$ with only fold singular points.

Problem 5.4. Study the set $F(M^n)$ and $SF(M^n)$.

We see easily that $S(M^n) \subset SF(M^n) \subset F(M^n)$. Some results concerning these sets can be found in [**Sae2, Sae3, SK, E2**].

Next we discuss a cobordism theory of special generic maps.

Definition 5.5. Let $f : M_1^n \to N_1^p$ and $g : M_2^n \to N_2^p$ $(n \geq p)$ be special generic maps of closed n-manifolds into p-manifolds.

(1) We say that f and g are *right-left cobordant* if there exists a smooth map $F : \tilde{M}^{n+1} \to \tilde{N}^{p+1}$ of a compact $(n+1)$-manifold with boundary to a $(p+1)$-manifold with boundary with the following properties.

(a) $\partial \tilde{M}^{n+1} = M_1^n \amalg M_2^n$ (disjoint union).

(b) $\partial \tilde{N}^{p+1} = N_1^p \amalg N_2^p$ (disjoint union).

(c) $F^{-1}(N_i^p) = M_i^n$ $(i = 1, 2)$.

(d) $F|M_1^n : M_1^n \to N_1^p$ and $F|M_2^n : M_2^n \to N_2^p$ are right-left equivalent to f and g respectively.

(e) $F|\mathrm{Int}\tilde{M}^{n+1} : \mathrm{Int}\tilde{M}^{n+1} \to \mathrm{Int}\tilde{N}^{p+1}$ is a special generic map.

(f) For every $q \in \partial \tilde{M}^{n+1}$, there exist local coordinates $(x_0, \cdots, x_n)$ around q in $\tilde{M}^{n+1}$ and $(y_0, \cdots, y_p)$ around $F(q)$ in $\tilde{N}^{p+1}$ such that $\partial \tilde{M}^{n+1}$ and $\partial \tilde{N}^{p+1}$ correspond to $x_0 = 0$ and $y_0 = 0$ respectively and that F has one of the following forms:

$$y_i \circ F = x_i \qquad (i = 0, \cdots, p) \qquad \text{or}$$

$$\begin{cases} y_i \circ F = x_i & (i = 0, \cdots, p-1) \\ y_p \circ F = x_p^2 + \cdots + x_n^2. \end{cases}$$

(2) We say that f and g are *right cobordant* if in (1) $\tilde{N}^{p+1} = N_1^p \times I$, where $I = [0, 1]$.

(3) We say that f and g are *left cobordant* if in (1) $\tilde{M}^{n+1} = M_1^n \times I$.

(4) We say that f and g are *concordant* if in (1) $\tilde{M}^{n+1} = M_1^n \times I$ and $\tilde{N}^{p+1} = N_1^p \times I$.

Furthermore, when M_i^n are oriented $(i = 1, 2)$, we say that f and g are *oriented* right-left cobordant (right cobordant, left cobordant or concordant) if $\tilde{M}^{n+1}$ is oriented and $\partial \tilde{M}^{n+1} = M_1^n \amalg (-M_2^n)$ in (a), where $-M_2^n$ is M_2^n with the reversed orientation, and if the diffeomorphisms $M_1^n \to M_1^n$ and $M_2^n \to M_2^n$ giving the right-left equivalence of (d) are orientation preserving.

We say that the map F above is a right-left (right or left) *cobordism* (or a *concordance*) between f and g.

First we discuss the classification up to concordance.

(5.5.1) If two special generic maps $f : M_1^n \to N_1^p$ and $g : M_2^n \to N_2^p$ are regularly equivalent, then they are concordant. Furthermore, if M_i^n are oriented and the diffeomorphism $\Phi : M_1^n \to M_2^n$ in Definition 4.2 is orientation preserving, then f and g are oriented concordant.

Proof. Let $f_t : M_1^n \to N_1^p$ $(t \in [0,1])$ be a smooth family of special generic maps as in Proposition 4.3. Then we see that the map $F : M_1^n \times I \to N_1^p \times I$ defined by $F(x,t) = (f_t(x), t)$ $(x \in M_1^n, t \in I)$ gives the desired concordance. □

(5.5.2) If special generic maps $f : M^n \to N_1^p$ and $g : M^n \to N_2^p$ are left cobordant (or concordant), then $S(f)$ and $S(g)$ are $\mathbf{Z}_2$-homologous; i.e., there exists a diffeomorphism $\Phi : M^n \to M^n$ such that $\Phi_*[S(f)]_2 = [S(g)]_2$ in $H_{p-1}(M^n; \mathbf{Z}_2)$, where $[S(f)]_2 \in H_{p-1}(M; \mathbf{Z}_2)$ (or $[S(g)]_2$) is the homology class represented by $S(f)$ (resp. $S(g)$). Furthermore, if $\tilde{N}^{p+1}$ is oriented, then $S(f)$ and $S(g)$ are $\mathbf{Z}$-homologous, where $F : M^n \times I \to \tilde{N}^{p+1}$ is the map giving the left cobordism between f and g, $S(f)$ and $S(g)$ are oriented as the boundaries of W_f and W_g respectively, which are oriented so that $\bar{f} : W_f \to N_1^p$ and $\bar{g} : W_g \to N_2^p$ are orientation preserving, and N_i^p $(i = 1, 2)$ are oriented as the boundary of $\tilde{N}^{p+1}$.

The proof of (5.5.2) is easy and is left to the reader as an exercise.

(5.5.3) Let f and $g : S^2 \times S^2 \to \mathbf{R}^3$ be special generic maps. If $(n_1(f), n_2(f)) \neq (n_1(g), n_2(g))$ in $\tilde{\Lambda}_2$, then they are not concordant.

Proof. Let $S(f) = S_1 \cup S_2$ and $S(g) = S_1' \cup S_2'$ be the connected components. Orient $\mathbf{R}^3$ arbitrarily. Then S_i and S_i' $(i = 1, 2)$ are canonically oriented as the boundaries of W_f and W_g respectively, where W_f and W_g are oriented so that $\bar{f}$ and $\bar{g}$ are orientation preserving. It is not difficult to see that there exists a base $\xi, \eta \in H_2(S^2 \times S^2)$ such that $\xi^2 = \eta^2 = 0, \xi \cdot \eta = 1, [S_1] = \xi + a\eta$ and $[S_2] = -\xi + a\eta$ for some integer a. Note that $2|a| = |[S_1]^2| = |[S_2]^2| = |n_1(f)| = |n_2(f)|$. Then $[S_1] + [S_2] = 2a\eta$ and hence the maximal positive integer which divides $[S_1] + [S_2]$ is equal to $|n_1(f)| = |n_2(f)|$. Similarly, we see that the maximal positive integer which divides $[S_1'] + [S_2']$ is equal to $|n_1(g)| = |n_2(g)|$. Since $|n_i(f)| \neq |n_j(g)|$ by our hypothesis, there is no diffeomorphism $\Phi : S^2 \times S^2 \to$

$S^2 \times S^2$ such that $\Phi_*([S_1]+[S_2]) = [S_1']+[S_2']$. This implies that f and g are not concordant by (5.5.2). □

Note that we have the same result also for $S^2\tilde{\times}S^2$.

The above fact (5.5.3) shows that there are infinitely many concordance classes of special generic maps of $S^2 \times S^2$ (or $S^2\tilde{\times}S^2$) into $\mathbf{R}^3$. We do not know if the concordance implies the regular equivalence in this case. This is true, if the classical Poincaré conjecture is true, for example (see Theorem 4.6).

Next we define cobordism groups of special generic maps. Let $\Gamma(n,p)$ (or $\tilde{\Gamma}(n,p)$) ($n \geq p$) denote the set of all (oriented) right cobordism classes of special generic maps of closed n-manifolds into $\mathbf{R}^p$. Note that $\Gamma(n,p)$ and $\tilde{\Gamma}(n,p)$ are abelian groups with respect to the following operation. For special generic maps $f : M_1^n \to \mathbf{R}^p$ and $g : M_2^n \to \mathbf{R}^p$ of closed n-manifolds, we have parallel translations $t_i : \mathbf{R}^p \to \mathbf{R}^p$ ($i = 1,2$) such that $t_1 \circ f(M_1^n) \subset \mathbf{R}^p_{++}$ and $t_2 \circ g(M_2^n) \subset \mathbf{R}^p_{--}$, where $\mathbf{R}^p_{++} = \{x_p > 0\}$ and $\mathbf{R}^p_{--} = \{x_p < 0\}$. Then the addition of the right cobordism classes of f and g is defined to be the class represented by the special generic map $(t_1 \circ f) \amalg (t_2 \circ g) : M_1^n \amalg M_2^n \to \mathbf{R}^p$. Note that this is well-defined up to (oriented) right cobordism. The identity element is represented by the obvious map $\emptyset \to \mathbf{R}^p$ and the inverse of the class of $f : M^n \to \mathbf{R}^p$ is represented by $\rho \circ f$ (or $\rho \circ f : -M^n \to \mathbf{R}^p$, if M^n is oriented), where the diffeomorphism $\rho : \mathbf{R}^p \to \mathbf{R}^p$ is defined by $\rho(x_1,\cdots,x_p) = (x_1,\cdots,x_{p-1},-x_p)$. We call $\Gamma(n,p)$ (or $\tilde{\Gamma}(n,p)$) the *(oriented) cobordism group* of special generic maps of n-manifolds into $\mathbf{R}^p$. Furthermore, we say that a special generic map $f : M^n \to \mathbf{R}^p$ is *(oriented) null-cobordant* if it is (oriented) right cobordant to the map $\emptyset \to \mathbf{R}^p$.

Note that we could define cobordism groups of special generic maps using arbitrary p-dimensional manifolds instead of $\mathbf{R}^p$ as target manifolds. However, if we allow arbitrary p-manifolds, we should consider all bundle maps $M^n \to N^p$, since such maps are special generic maps by the very definition. Thus, the above definition of the cobordism groups seems reasonable.

Here we list some properties and examples of cobordisms.

(5.5.4) Let $\alpha : \Gamma(n,p) \to \mathcal{N}(n)$ (or $\tilde{\alpha} : \tilde{\Gamma}(n,p) \to \Omega^{SO}(n)$) be the forgetful map defined by $\alpha[f : M^n \to \mathbf{R}^p] = [M^n]$, where $\mathcal{N}(n)$ (resp. $\Omega^{SO}(n)$) denotes the usual unoriented (resp. oriented) cobordism group of n-manifolds. Then α and $\tilde{\alpha}$ are the zero maps ([**KS**]).

(5.5.5) Let S^n be the unit sphere in $\mathbf{R}^{n+1}$ and $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^p$ $(n \geq p)$ the standard projection. Then the special generic map $f : S^n \to \mathbf{R}^p$ defined by $f = \pi | S^n$ is (oriented) null-cobordant.

Proof. Set $\tilde{M}^{n+1} = \{(x_1, \cdots, x_{n+2}) \in S^{n+1}; x_{n+2} \geq 0\}$ and define $F : \tilde{M}^{n+1} \to \mathbf{R}^{p+1}_+$ by $F = \tilde{\pi}|\tilde{M}^{n+1}$, where $\tilde{\pi} : \mathbf{R}^{n+2} \to \mathbf{R}^{p+1}$ is defined by $\tilde{\pi}(x_1, \cdots, x_{n+2}) = (\pi(x_1, \cdots, x_{n+1}), x_{n+2})$ and $\mathbf{R}^{p+1}_+ = \{(x_1, \cdots, x_{p+1}) \in \mathbf{R}^{p+1}; x_{p+1} \geq 0\}$. Note that $F(\tilde{M}^{n+1}) \subset \mathbf{R}^p \times [0,2]$. Then we see that $\varphi \circ F$ gives the desired null-cobordism for a diffeomorphism $\varphi : \mathbf{R}^p \times [0,2] \to \mathbf{R}^p \times [0,1]$. □

(5.5.6) Let $\pi : \tilde{E}^{n+1} \to W^p$ be an orthogonal D^{n-p+1}-bundle over a compact parallelizable p-manifold W^p with boundary $(n \geq p)$. Then the special generic map $f : \partial\tilde{E}^{n+1} \to \mathbf{R}^p$ constructed from π as in [**Sae1**] is null-cobordant. Furthermore, if $\tilde{E}^{n+1}$ is oriented, f is oriented null-cobordant.

Proof. We construct a map $F : \tilde{E}^{n+1} \to \mathbf{R}^p \times [0,\infty)$ as follows. Let $C \cong \partial W^p \times [0,1]$ be a closed collar neighborhood of ∂W^p in W^p such that ∂W^p corresponds to $\partial W^p \times \{0\}$. Set $B = \{(x_0, \cdots, x_{n-p+2}) \in \mathbf{R}^{n-p+3}; \sum_{i=0}^{n-p+2} x_i^2 = 1, x_0 \geq 0, x_{n-p+2} \geq 0\}$ and $B_0 = \{(x_0, \cdots, x_{n-p+2}) \in B; x_{n-p+2} = 0\}$. Note that there exists a diffeomorphism $\varphi : B \to D^{n-p+1} \times I$ such that $\varphi(B_0) = D^{n-p+1} \times \{1\}$ and that φ is equivariant with respect to the actions of $O(n-p+1)$, where $O(n-p+1)$ acts on $B \subset [0,\infty) \times \mathbf{R}^{n-p+1} \times [0,\infty)$ standardly on the $(x_1, \cdots, x_{n-p+1})$-coordinates and $O(n-p+1)$ acts on $D^{n-p+1} \times I$ standardly on the first factor. Define $g : B \to [0,\infty)$ by $g(x_0, \cdots, x_{n-p+2}) = x_0$ and $h : B \to [0,\infty)$ by $h(x_0, \cdots, x_{n-p+2}) = x_{n-p+2}$. Note that g and h are invariant under the $O(n-p+1)$-action on B. Note also that the images of g and h are contained in $[0,1]$. Define $\pi_1 : \pi^{-1}(C) \to \partial W^p$ by the composition

$$\pi_1 : \pi^{-1}(C) \xrightarrow{\pi} C(\cong \partial W^p \times I) \xrightarrow{p_1} \partial W^p,$$

where p_1 is the projection to the first factor. Note that π_1 is a $(D^{n-p+1} \times I)$-

bundle. Define $G : \tilde{E}^{n+1} \to [0,\infty)$ by $G(x) = g \circ \varphi^{-1}(x)$ for $x \in \tilde{E}^{n+1} - \pi^{-1}(C)$, where we identify $\pi^{-1}(\pi(x))(\ni x)$ with $D^{n-p+1} \times \{1\}(\cong B_0)$, and by $G(x) = g \circ \varphi^{-1}(x)$ for $x \in \pi^{-1}(C)$, where we identify $\pi_1^{-1}(\pi_1(x))(\ni x)$ with $D^{n-p+1} \times I(\cong B)$. Furthermore, define $H : \pi^{-1}(C) \to [0,1]$ by $H(x) = h \circ \varphi^{-1}(x)$, where we identify $\pi_1^{-1}(\pi_1(x))(\ni x)$ with $D^{n-p+1} \times I(\cong B)$. Note that G and H are well-defined smooth maps. Finally define $F : \tilde{E}^{n+1} \to \mathbf{R}^p \times [0,\infty)$ by $F(x) = (\eta \circ \pi(x), G(x))$ for $x \in \tilde{E}^{n+1} - \pi^{-1}(C)$ and by $F(x) = (\eta(\pi_1(x), 1 - H(x)), G(x))$ for $x \in \pi^{-1}(C)$, where $\eta : W^p \to \mathbf{R}^p$ is an immersion whose existence is guaranteed by [**Ph**] and we identify $C(\subset W^p)$ with $\partial W^p \times [0,1]$. Then it is not difficult to see that $\psi \circ F$ is a desired map giving the null-cobordism of f for some embedding $\psi : \mathbf{R}^p \times [0,\infty) \to \mathbf{R}^p \times [0,1]$. □

(5.5.7) The cobordism groups $\Gamma(n,p)$ and $\tilde{\Gamma}(n,p)$ are trivial for $n-3 \leq p \leq n-1$.

Proof. By [**Sae1**] and Remark 2.4, we see that every special generic map is obtained as in (5.5.6) for $n-3 \leq p \leq n-1$. Then (5.5.7) follows from (5.5.6). □

(5.5.8) Suppose that $f : M_1^n \to \mathbf{R}^n$ and $g : M_2^n \to \mathbf{R}^n$ are special generic maps of closed oriented n-manifolds. If f and g are oriented right cobordant, then $M_1^n \amalg (-M_2^n)$ bounds a compact parallelizable manifold.

Proof. Let $F : \tilde{M}^{n+1} \to \mathbf{R}^n \times I$ be an oriented right cobordism of f and g. Then by the same argument as in [**Sae1**], we see that $\tilde{M}^{n+1}$ is stably parallelizable. Since $\partial\tilde{M}^{n+1} \neq \emptyset$, this implies that $\tilde{M}^{n+1}$ is parallelizable. □

(5.5.9) Suppose that $f : M_1^n \to \mathbf{R}$ and $g : M_2^n \to \mathbf{R}$ are special generic maps of closed oriented n-manifolds. If f and g are oriented right cobordant, then $M_1^n \amalg (-M_2^n)$ bounds a compact parallelizable manifold.

Proof. Let $F : \tilde{M}^{n+1} \to \mathbf{R} \times I$ be an oriented right cobordism between f and g. We may assume that $f(M_1^n)$ and $g(M_2^n)$ are contained in $\mathbf{R}_{++}$ and that $F(\tilde{M}^{n+1})$ is contained in $\mathbf{R}_{++} \times I$. Set $Z = (M_1^n \times I) \cup \tilde{M}_1^{n+1} \cup \tilde{M}_2^{n+1} \cup (M_2^n \times I)$, where $\tilde{M}_1^{n+1} = \tilde{M}_2^{n+1} = \tilde{M}^{n+1}$ and $M_j^n \times \{i\}$ $(i = 0,1; j = 1,2)$ is identified with $M_j^n \subset \partial\tilde{M}_{i+1}^{n+1}$. Note that Z is a closed $(n+1)$-manifold and that it is orientable. Let $F_1 : M_1^n \times I \to \mathbf{R} \times [-1,0]$ be an oriented null-cobordism of $f \amalg (\rho \circ f)$ with $F_1(M_1^n \times \{0,1\}) \subset \mathbf{R} \times \{0\}$ and let $F_2 : M_2^n \times I \to \mathbf{R} \times [1,2]$ be an oriented null-cobordism of $g \amalg (\rho \circ g)$ with $F_2(M_2^n \times \{0,1\}) \subset \mathbf{R} \times \{1\}$, where $\rho : \mathbf{R} \to \mathbf{R}$ is the map defined by $\rho(x) = -x$. Then define $\tilde{F} : Z \to \mathbf{R} \times \mathbf{R}$ by $\tilde{F}|\tilde{M}_1^{n+1} = F, \tilde{F}|\tilde{M}_2^{n+1} = (\rho \times \mathrm{id}) \circ F$ and $\tilde{F}|M_i \times I = F_i$ $(i = 1, 2)$. We see easily that $\tilde{F}$ is a special generic map into $\mathbf{R}^2$. Since Z is a closed orientable $(n+1)$-manifold, by [**Sae1**], Z is diffeomorphic to $(\sharp_{i=1}^r S^1 \times \Sigma_i^n)\sharp\Sigma^{n+1}$ for some homotopy spheres Σ_i^n and Σ^{n+1}. In particular, Z is stably parallelizable, since so are homotopy spheres. Since $\tilde{M}^{n+1}$ is embedded in Z, we see that $\tilde{M}^{n+1}$ is also stably parallelizable and hence that it is parallelizable, since $\partial \tilde{M}^{n+1} \neq \emptyset$. □

By [**E1**], every homotopy n-sphere admits a special generic map into $\mathbf{R}^n$, since it is stably parallelizable. Furthermore, for $n \geq 5$, every homotopy n-sphere admits a special generic map into $\mathbf{R}$ (i.e., a Morse function with exactly two critical points). It is known that there exist homotopy n-spheres Σ^n ($n = 8, 9, 10, 13, \cdots$) which do not bound compact parallelizable $(n+1)$-manifolds. By (5.5.8) and (5.5.9), special generic maps of such homotopy n-spheres Σ^n into $\mathbf{R}^n$ (or into $\mathbf{R}$) are not oriented right cobordant to the standard special generic map $S^n \to \mathbf{R}^n$ (resp. $S^n \to \mathbf{R}$) (cf. (5.3.2) and (5.5.5)). In particular, *the cobordism groups* $\tilde{\Gamma}(n,n)$ *and* $\tilde{\Gamma}(n,1)$ *are not trivial in general.*

Problem 5.6. Study the cobordism groups $\Gamma(n,p)$ and $\tilde{\Gamma}(n,p)$ of special generic maps of closed n-manifolds into $\mathbf{R}^p$. For example, are they finitely generated?

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