Matemática Contemporânea, Vol. 5, 77–91
http://doi.org/10.21711/231766361993/rmc56

GEOMETRY AND SINGULARITIES OF ORTHOGONAL PROJECTIONS OF 3-MANIFOLDS IN $I\!R^5$

Dirce Hayashida Mochida

Introduction

Our aim in this paper is to analyse the generic contacts of a compact 3-manifold with hyperplanes in $I\!R^5$ from the geometrical viewpoint. This is done by considering the generic singularities of height functions on the 3-manifolds in the same spirit that it was done for surfaces in $I\!R^4$ in [6]. In fact, this form part of a more general study on submanifolds of codimension higher than 1, developped in [7].

The main tool in our work consists in using the canal hypersurface CM (that can be thought as the unit normal bundle) on the submanifold M, together with the natural projection $CM \xrightarrow{\xi} M$ restricted to the parabolic set of the hypersurface CM, whose generic singularities we study.

The singularities of the family of height functions on generic hypersurfaces of $I\!R^n$ were considered by J.W. Bruce [2] and M.C. Romero Fuster [11]. Their results relied on the following Looijenga's genericity theorem:

"There is a residual subset of embeddings $f : M^3 \hookrightarrow I\!R^5$, for which the family of height functions:

$$\begin{aligned} \lambda(f) : M \times S^4 &\longrightarrow I\!R \\ (m, v) &\longmapsto < f(m), v >= f_v(m) \end{aligned}$$

is locally stable as a family of functions on M with parameters on S^4" ([4]).

Now, the singularities of the orthogonal projections on submanifolds of $I\!R^n$ are tightly related to those of their canal hypersurfaces. So, we shall use here the already known facts on hypersurfaces, in order to get conclusions about the geometry of a 3-manifold M generically embedded in $\mathbb{R}^5$.

We shall see that the points of M can be classified according to the number of directions at them for which the germ of the corresponding height function is no Morse. Indeed, such a direction will be called binormal, and the hyperplane that passes through the considered point being orthogonal to it, will be the osculating hyperplane of the 3-manifold at the given point (by analogy to what happens with curves in 3-space). We shall prove then that there are at least 1 and at most 3 binormals at each point of a 3-manifold generically embedded in $\mathbb{R}^5$.

We shall also distinguish among the different points of M attending to the kind of degeneracies of the height functions on the binormal directions at them. For instance, we shall see that one of these has a singularity of fold type if and only if the torsion of a "special" normal section of the 3-manifold does not vanish at the given point. Otherwise, the singularity is more degenerate. We also characterize geometrically the umbilics of the height functions.

By means of the projection $\xi : CM \to M$ we shall associate to each binormal direction at a point, a tangent direction that we call asymptotic direction. As D. Mond [8] did for surfaces in $\mathbb{R}^4$, we shall see that the contacts of these tangent directions with the 3-manifold M are of higher order by showing that the projections of M onto their orthogonal hyperplanes are more degenerate than the corresponding to the other tangent directions.

The content of this paper is part of the author's Ph.D. Thesis at the ICMSC-USP - São Carlos. The author would like to thanks her advisers M.C. Romero Fuster and M.A. Soares Ruas for their enthusiasm and patient guidance. Without them this paper would not have been written.

1. Degenerate Directions of M

Let $f : \mathbb{R}^3, 0 \longrightarrow \mathbb{R}^5, 0$ given by $(x, y, z) \longmapsto (x, y, z, f_1(x, y, z), f_2(x, y, z))$, be the local expression of the embedding of the 3-manifold M in Monge's form.

With respect to these coordinates, the second fondamental form of $f(M)$ is characterized by two quadratic forms:

$$q_1(x,y,z) = a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2$$
$$q_2(x,y,z) = b_{11}x^2 + 2b_{12}xy + 2b_{13}yz + b_{22}y^2 + 2b_{23}yz + b_{33}z^2,$$

whose matrices we denote by $M(q_i),\ i = 1,2$.

The height function f_v, in a direction $v \in \mathbb{R}^5$ has the local form:

$$\begin{array}{ll} \mathbb{R}^3, 0 & \longrightarrow \mathbb{R}, 0 \\ (x,y,z) & \longmapsto f_v(x,y,z) = v_1x + v_2y + v_3z + v_4f_1(x,y,z) + v_5f_2(x,y,z), \end{array}$$

and its Hessian matrix $\mathcal{H}(f_v)(0)$ at zero is given by

$$\mathcal{H}(f_v)(0) = \begin{bmatrix} a_{11}v_4 + b_{11}v_5 & a_{12}v_4 + b_{12}v_5 & a_{13}v_4 + b_{13}v_5 \\ a_{12}v_4 + b_{12}v_5 & a_{22}v_4 + b_{22}v_5 & a_{23}v_4 + b_{23}v_5 \\ a_{13}v_4 + b_{13}v_5 & a_{23}v_4 + b_{23}v_5 & a_{33}v_4 + b_{33}v_5 \end{bmatrix}$$

Let $\det \mathcal{H}(f_v)(0) = 2^3.h_m(v_4,v_5)$, where

$$h_m(v_4,v_5) = A_1v_4^3 + 3A_2v_4^2v_5 + 3A_3v_4v_5^2 + A_4v_5^3, \tag{1}$$

with $A_1 = \det M(q_1)$

$A_2 = \frac{1}{3}\left[\sum_{i,j=1}^{3} b_{ij}\det(\alpha_{ij})\right]$, where α_{ij} is the 2×2-minor cofactor of the matrix $M(q_1)$ obtained by eliminating the $i^{\underline{th}}$ line and $j^{\underline{th}}$ column.

$A_3 = \frac{1}{3}\left[\sum_{i,j=1}^{3} a_{ij}\det(\beta_{ij})\right]$, where β_{ij} is defined as above, replacing $M(q_1)$ by $M(q_2)$.

$A_4 = \det M(q_2)$

We denote by A the matrix:

$$\begin{bmatrix} A_1A_3 - A_2^2 & \frac{1}{2}(A_1A_4 - A_2A_3) \\ \frac{1}{2}(A_1A_4 - A_2A_3) & A_2A_4 - A_3^2 \end{bmatrix} \tag{2}$$

Lemma 1: *m is a degenerate critical point of $f_v \Leftrightarrow v \in N_{f(m)}f(M)$ and the pair (v_4,v_5) of the coordinates of v in $N_{f(m)}f(M)$ is a solution of $h_m = 0$.*

If m is a degenerate critical point of f_v, v is called a degenerate direction at m.

Let $\Delta : M \longrightarrow I\!R$ *defined by* $\Delta(m) = \det A(m)$*, where* A *is given in (2):*

Lemma 2: *Let* m *be a critical point of* f_v*. Then,*

1. *If* $\Delta(m) > 0$*, there exists a unique degenerate direction* b *at* m*.*

2. *If* $\Delta(m) < 0$*, there are exactly three degenerate directions* b_i *(i= 1, 2, 3) at* m*.*

3. *If* $\Delta(m) = 0$*, and rank* $A(m) = 1$*, there exist two degenerate directions* b_i *(i= 1, 2) at* m*.*

4. *If rank* $A(m) = 0$*, there exists one degenerate direction* b *at* m*.*

Proof: The result follows from the real classification of the cubic form in two variables $h_m(v_4, v_5) = 0$.

Remark: When m is a degenerate critical point of f_b, the hyperplane H_b, orthogonal to b has a higher order contact with $f(M)$ at $f(m)$. Therefore, by analogy with curves in $I\!R^3$, we shall say that b is a binormal vector of $f(M)$ at $f(m)$ and H_b an osculating hyperplane.

Conditions (a), (b), (c) and (d) above depend only on the 3-jet of f_v. Furthermore, conditions (a) and (b) are open in $J^3(3;1)$, condition (c) holds on a algebraic subset of $J^3(3,1)$ of codimension one and condition (d) holds on an algebraic subset of $J^3(3,1)$ of codimension two. Then, these conditions hold for a residual set of embeddings f. For such a generic f, M can be decomposed in:

(i) two open regions:

$M_1(1) = \{m \in M$ /there exists a unique binormal direction at m, that is, $\Delta(m) > 0\}$

and

$M_3(1,1,1) = \{m \in M$ /there exist three binormal directions at m, that is $\Delta(m) < 0\}$.

(ii) a subset, locally defined as an algebraic subset of codimension 1:
$M_2(1,2) = \{m \in M$ /there are two binormal directions at m, that is, $\Delta(m) = 0$ and rank $A(m) = 1\}$
and

(iii) a subset, locally defined as an algebraic subset of codimension 2:
$M_1(3) = \{m \in M$ /there is a binormal direction at m, that is, rank $A(m) = 0\}$.

2. The Canal 4-Manifold of M in $\mathbb{R}^5$

The canal 4-manifold of M in $\mathbb{R}^5$ is defined as $CM = \{(m,v) \in M \times S^4 / v$ is orthogonal to $T_{f(m)}f(M)\}$. We denote by $\tilde{f}$ the natural embedding of CM into $\mathbb{R}^5$,

$$\begin{array}{ccc} CM & \xrightarrow{\tilde{f}} & \mathbb{R}^5 \\ (m,v) & \longmapsto & \tilde{f}(m,v) = f(m) + \varepsilon v, \quad \text{where} \end{array}$$

ε is a sufficiently small positive real number. We thus have two families of height functions $\lambda(f)$ and $\lambda(\tilde{f})$ respectively defined on M and CM, whose singularities are tightly related [10]. In fact, the singularities of $\tilde{f}_v$ at (m,v) and f_v at m are stably equivalent (Arnold, [1]), which implies that the family $\lambda(f)$ is locally versal if and only if $\lambda(\tilde{f})$ is also locally versal. The singularities that may appear for a generic f, are of one of the following types: Morse (A_1), fold (A_2), cusp (A_3), swallowtail (A_4), butterfly (A_5), elliptic or hyperbolic umbilic ($D_4^{\pm}$) and parabolic umbilic (D_5). Moreover, the singularities of the normal Gauss map, $\Gamma : CM \to S^4$ (also called generalized Gauss map on M) can be described in terms of those as follows:

Lemma 3: *Given a critical point $(m,v) \in CM$ of the height function $\tilde{f}_v$ (or equivalently, given a critical point $m \in M$ of f_v),*
m is a degenerate critical point of $f_v \Leftrightarrow (m,v)$ is a singular point of Γ.

Proof.: [see 10]

Let $\mathcal{K}_c : CM \to I\!R$ be the Gaussian curvature function on CM. The parabolic set, $\mathcal{K}_c^{-1}(0)$, of CM is the singular set of Γ. It can be shown that, for a generic f, $\mathcal{K}_c^{-1}(0)$ is a 3-submanifold except along a curve consisting of singular points of type Σ^2 or equivalently, umbilic points $(D_4^{\pm})$ or (D_5) of $\tilde{f}_v$. We denote this curve by $\Sigma^2(\Gamma)$.

Let $\xi : CM \to M$ be the natural projection of CM onto M, i.e, $\xi(m,v) = m$, and $\bar{\xi}$ its restrition to the submanifold $\mathcal{K}_c^{-1}(0) - \Sigma^2(\Gamma)$. At each point of $\mathcal{K}_c^{-1}(0) - \Sigma^2(\Gamma)$ there is a unique principal direction of zero curvature for CM. This field of directions is tangent to $\mathcal{K}_c^{-1}(0)$ along a surface made of points of type $\Sigma^{1,1}$, which is in turn tangent to this surface along a curve of points of type $\Sigma^{1,1,1}$. Moreover, at isolated points of type $\Sigma^{1,1,1,1}$, this principal direction is in fact tangent to this curve [see 3].

Let us consider

$B_i = \{(m,v) \in \mathcal{K}_c^{-1}(0) - \Sigma^2(\Gamma)/p \in M_i(\overbrace{1,\dots,1}^{i})\}$ $(i = 1,3)$
$B_2 = \{(m,v) \in \mathcal{K}_c^{-1}(0) - \Sigma^2(\Gamma)/p \in M_2(1,2)\}$ and
$B_1(3) = \{(m,v) \in \mathcal{K}_c^{-1}(0) - \Sigma^2(\Gamma)/p \in M_1(3)\}$ *in* $C(M)$.

Theorem 1:

(i) $\Sigma^2(\Gamma) \cap B_i = \emptyset$, $i = 1,3$

(ii) $\bar{\xi}_{|B_i} : B_i \to M_i$ $(i = 1$ *or* $3)$ *is a local diffeomorphism (more precisely it is a diffeomorphism when* $i = 1$*, and a triple covering when* $i = 3$*).*

(iii) $\Delta(m) = 0$*, rank* $A(m) = 1$ *and* m *is not an umbilic point* $\Leftrightarrow$ *there exists* $v \in S^4$ *such that* (m,v) *is a fold point of* $\bar{\xi}$*.*

(iv) *rank*$A(m) = 0$ *and* m *is not an umbilic point of* $M \Leftrightarrow$ *there exists* $v \in S^4$ *such that* (m,v) *is a cusp point of* $\bar{\xi}$*.*

Proof:

(i) Let $(p,v) \in \Sigma^2(\Gamma)$, that is, f_v has an umbilic singularity at p. Choosing coordinates such that $p = (0,0,0)$, $v = (0,0,0,0,1)$ and $f_v = z^2 + h.o.t.$, it is easy to compute $\Delta(p)$ and show that $\Delta(p) = 0$.

Now, we choose coordinates for CM such that $m = (0,0,0)$ and $v = (0,0,0,0,1)$. So, it is sufficient to notice that in (1), if the vector $v = (0,0,0,0,1)$ is a degenerate direction then $h_m(v_4,1) = A_1v_4^3 + 3A_2v_4^2 + 3A_3v_4 = \mathcal{K}_c(m,v_4) = 0$.

(Observe that $A_4 = \det M(q_2) = 0$, since $f_2 = f_v$ is degenerate).

Then,

(ii) $v_4 = 0$ is a simple root of $\mathcal{K}_c(m,v_4) = 0 \Leftrightarrow \frac{\partial \mathcal{K}_c}{\partial v_4}(0,0) \neq 0 \Leftrightarrow \bar{\xi}$ is a local diffeomorphism.

(iii) $v_4 = 0$ is a double root $\Leftrightarrow \frac{\partial \mathcal{K}_c}{\partial v_4}(0,0) = 0$ and $\frac{\partial^2 \mathcal{K}_c}{\partial v_4^2}(0,0) \neq 0 \Leftrightarrow (m,v)$ is a fold point of $\bar{\xi}$.

(iv) $\Delta(m) = 0$, rank$(A(m) = 1$ and m is not an umbilic $\Leftrightarrow v_4 = 0$ is a triple root of $\mathcal{K}_c(0,v_4) = 0$ and $(m,v) \in \mathcal{K}_c^{-1}(0) - \Sigma^2(\Gamma) \Leftrightarrow \frac{\partial \mathcal{K}_c}{\partial v_4}(0,0) = \frac{\partial^2 \mathcal{K}_c}{\partial v_4^2}(0,0) = 0$, $\frac{\partial^3 \mathcal{K}_c}{\partial v_4^3}(0,0) \neq 0$ and $(m,v) \in \mathcal{K}_c^{-1}(0) - \Sigma^2(\Gamma)$. This implies that v_4-direction is not tangent to the curve $B_1(3)$ and then (m,v) is cusp point of $\bar{\xi}$. Now, the converse follows easily.

Definition: *We call* asymptotic direction *of M at m to the direction $\theta \subseteq T_{f(m)}M$ image by $T\bar{\xi}$ of the unique principal direction of zero curvature of CM at (m,v).*

In the following theorem we show that, as expected, an asymptotic direction is a tangent direction at m with contact of higher order with the manifold. We measure the contact of the direction θ with the manifold by looking at the singularities of the projection:

$$\begin{aligned} p_\theta : M &\to \mathbb{R}^4 \\ m &\mapsto p_\theta(m) = f(m) - < \theta, f(m) > \theta \ , \quad \theta \in S^4. \end{aligned}$$

Then, we have:

Theorem 2: *The direction θ is not asymptotic at $m \in M$ if and only if the germ of p_θ at m is a cross-cap.*

Proof:

$$\begin{aligned} \mathbb{R}^3, 0 &\longrightarrow \mathbb{R}^5, (0,0) \\ (x,y,z) &\longmapsto (x,y,z,f_1(x,y,z),f_2(x,y,z)), \text{ where} \end{aligned}$$

$f_1(x,y,z) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \varepsilon_1(x,y,z)$

$f_2(x,y,z) = b_{11}x^2 + 2b_{12}xy + b_{22}y^2 + 2b_{12}xz + 2b_{23}yz + b_{33}z^2 + \varepsilon_2(x,y,z)$,

$\varepsilon_i \in m^3 \quad (i = 1,2)$.

Let us assume $\theta = (1,0,0,0,0)$ and we can suppose $q_2(x,y,z) = b_{11}x^2 + b_{22}y^2 + b_{33}z^2$, non degenerate. Then, $p_\theta(x,y,z) = (y, z, a_{11}x^2 + 2a_{12}xy + a_{22}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \varepsilon_1(x,y,z),\ b_{11}x^2 + b_{22}y^2 + b_{33}z^2 + \varepsilon_2(x,y,z))$ and in these coordinates systems, the condition so that f_θ has a cross-cap at 0 is $a_{12}b_{22} \neq 0$ and $a_{13}b_{33} \neq 0$. So, if 0 is not a cross-cap, then $a_{12} = a_{13} = 0$, implying that θ is an asymptotic direction.

Reciprocally, let us assume that θ is an asymptotic direction associated to the binormal vector v, chosen in such a way that $f_2 = f_v$.

Then, $(m,v) \in \Sigma^1(\Gamma)$ and, if necessary, by a change of coordinates in the source, we can take $f_2(x,y,z)$ in the form:

(i) $x^3 + \bar{f}_2(y,z)$ if $(m,v) \in \Sigma^{1,0}(\Gamma)$ or,

(ii) $x^k \pm xy^2 + \bar{f}_2(y,z)$ if $(m,v) \in \Sigma^{\overbrace{1,1,\ldots,1}^{k-2},0}(\Gamma)$ $(k = 4, 5 \text{ or } 6)$.

This will imply that $\theta = (1,0,\ldots,0)$ and

$p_\theta(x,y,z) \sim (y, z, a_{11}x^2 + 2x(a_{12}y + a_{13}z), x^3)$ in case (i)

or

$p_\theta(x,y,z) \sim (y, z, a_{11}x^2 + 2x(a_{12}y + a_{13}z), x^k \pm xy^2)$ in case (ii).

The normal forms (i) and (ii) are more degenerate than a cross-cap ([8]).

3. Geometric Characterization of Singularities of Height Functions on M

For each unit vector $\theta \in T_{f(m)}f(M)$, let γ_θ be the curve obtained as the intersection of $f(M)$ and the 3-space containing $N_{f(m)}f(M)$ and θ. Such curve is called *normal section* of $f(M)$ in the direction θ.

We have seen that a height function $f_v : M \to I\!R$ has a degenerate singularity at m, if and only if, v is a binormal vector of $f(M)$ at $f(m)$. From now on, we shall consider a height function $f_v : M \to I\!R$, where v is a binormal vector of $f(M)$ at $f(m)$, and we denote by θ the asymptotic direction associated to v. Then, we characterize followingly the type of degenerate singularities that may generically appear, into:

Theorem 3: *For each $m \in M$ such that $\Delta(m) \neq 0 : m \in \bar{\xi}(\Sigma^{1,0}(\Gamma)) \Leftrightarrow \gamma_\theta$ has a nonvanishing normal torsion at m.*

Now, if γ_θ has vanishing normal torsion at m, then $m \in \bar{\xi}(\Sigma^{1,1}(\Gamma))$ and we have that:

(i) m is a cusp singularity of $f_v \Leftrightarrow \theta$ is transversal to $\bar{\xi}(\Sigma^{1,1,0}(\Gamma))$

(ii) m is a swallowtail singularity of $f_v \Leftrightarrow \theta$ is tangent to $\bar{\xi}(\Sigma^{1,1,0}(\Gamma))$ and transversal to $\bar{\xi}(\Sigma^{1,1,1,0}(\Gamma))$.

(iii) m is a butterfly singularity $\Leftrightarrow \theta$ is tangent to $\bar{\xi}(\Sigma^{1,1,1,0}(\Gamma))$, with first order contact.

Proof: We saw that m is a degenerate critical point of $f_v \Leftrightarrow (m, v) \in \mathcal{K}_c^{-1}(0)$. Furthermore, if $\Delta(m) \neq 0$ then $m \in M_1(1)$ when $\Delta(m) > 0$ and $m \in M_3(1,1,1)$ when $\Delta(m) < 0$.

We can choose orthogonal systems of coordinates to obtain:

(i) f locally given by:

$$\begin{array}{ccl} I\!R^3, 0 & \longrightarrow & I\!R^5, 0 \\ (x, y, z) & \longmapsto & (x, y, z, f_1(x, y, z), f_2(x, y, z)), \end{array}$$

$f_1(x, y, z) = q_1(x, y, z) + M_1 x^3 + \ldots$

$f_2(x, y, z) = \bar{f}_2(x, y) + z^2$ with $\bar{f}_2(x, y) = y^2 + P_1 x^3 + 3P_2 x^2 y + 3P_3 x y^2 + P_4 y^3 + Q_1 x^4 + \ldots + Q_5 y^4 + R_1 x^5 + \ldots + S_1 x^6 + \ldots$ and $f_2 = f_v$.

(ii) $\mathcal{K}_c$ locally given by:

$$\begin{array}{rcl} \mathbb{R}^3 \times \mathbb{R}, 0 & \longrightarrow & \mathbb{R}, 0 \\ (x,y,z,v_4) & \longmapsto & \mathcal{K}_c(x,y,z,v_4) = A_1(x,y,z)v_4^3 + 3A_2(x,y,z)v_4^2 + \\ & & 3A_3\,(x,y,z)v_4 + A_4(x,y,z), \end{array}$$

where:

$$\begin{array}{rcl} A_1(x,y,z) & = & \det M(q_1(x,y,z)) \\ A_2(x,y,z) & = & \frac{1}{3}[(f_{1_{xx}} f_{1_{yy}} f_{2_{zz}} + f_{1_{xx}} f_{1_{zz}} f_{2_{yy}} + f_{1_{yy}} f_{1_{zz}} f_{2_{xx}}) + \\ & + & 2(f_{1_{xz}} f_{1_{yz}} f_{2_{xy}} + f_{1_{xy}} f_{1_{xz}} f_{2_{yz}}) + (f_{1_{xy}} f_{1_{yz}} f_{2_{xz}}) - \\ & - & (f_{1_{xz}} f_{1_{yy}} f_{2_{xz}} + f_{1_{xz}}^2 f_{2_{yy}} + f_{1_{xz}} f_{1_{yy}} f_{2_{xz}}) - \\ & - & (2f_{1_{xx}} f_{1_{yz}} f_{2_{yz}} + f_{1_{yz}}^2 f_{2_{xx}}) - (2f_{1_{xy}} f_{1_{zz}} f_{2_{xy}} + f_{1_{xy}}^2 f_{2_{zz}})] \\ A_3(x,y,z) & = & \frac{1}{3}[(f_{1_{xx}} f_{2_{yy}} f_{2_{zz}} + f_{1_{yy}} f_{2_{xx}} f_{2_{zz}} + f_{1_{zz}} f_{2_{xx}} f_{2_{yy}}) + \\ & + & 2(f_{1_{xz}} f_{2_{xy}} f_{2_{yz}} + f_{2_{xz}} f_{1_{yz}} f_{2_{xy}} + f_{1_{xy}} f_{2_{xz}} f_{2_{yz}}) - \\ & - & (f_{1_{xz}} f_{2_{xz}} f_{2_{yy}} + f_{1_{yy}} f_{2_{xz}}^2 + f_{1_{xz}} f_{2_{yy}} f_{2_{xz}}) - \\ & - & (2f_{1_{yz}} f_{2_{xx}} f_{2_{yz}} + f_{1_{xx}} f_{2_{yz}}^2) - (f_{2_{xy}}^2 f_{1_{zz}} + 2f_{1_{xy}} f_{2_{xy}} f_{2_{zz}})] \\ A_4(x,y,z) & = & \det M(q_2(x,y,z)). \end{array}$$

Now, the only principal direction of zero curvature at (m,v) is $e_1 = (1,0,0,0,0)$ and, hence, $D\bar{\xi}(m,v)e_1 = e_1 = \theta$. Under these conditions, γ_θ has the following parametrization:

$$\begin{array}{rl} \gamma_\theta : (\mathbb{R},0) & \longrightarrow (\mathbb{R}^5, 0) \\ s & \longmapsto \gamma_\theta(s) = (s, 0, 0, a_{11}s^2 + ..., P_1 s^3 + ...) \end{array}$$

Then, γ_θ has non zero torsion $\Leftrightarrow P_1 \neq 0 \Leftrightarrow m$ is a fold point of f_v. So, if $P_1 = 0, (m,v) \in \Sigma^{1,1}(\Gamma)$ and the rest of the proof follows from the description of the singularities of the Gauss mapping Γ, from Lemma 3 and Theorem 1 (part (ii)).

The characterization of the singularities of the height functions on the points of $\Delta^{-1}(0)$ away from the umbilics is given by Theorem 4 when $m \in M_2(1,2)$ and by Theorem 5 when $m \in M_1(3)$. Using a transversality's argument we can see that, generically, singularities of butterfly type do not occur in $M_2(1,2)$ and singularities more degenerate than cusps are away from $M_1(3)$.

Theorem 4: *For each $m \in M$ such that $\Delta(m) = 0$, rank $A(m) = 1$ and m is not an umbilic point for f_v:*

(i) *m is a fold singularity of $f_v \Leftrightarrow \theta$ is transversal to $M_2(1,2)$.*

(ii) *If m is an A_k singularity of $f_v(k = 3$ or $4)$ then θ is tangent to $M_2(1,2)$.*

Moreover,

m is a cusp singularity of $f_v \Leftrightarrow \bar{\xi}(\Sigma^{1,1,0}(\Gamma))$ is transversal to $M_2(1,2)$ at m.

m is a swallowtail singularity of $f_v \Leftrightarrow \bar{\xi}(\Sigma^{1,1,0}(\Gamma))$ is tangent to $M_2(1,2)$ at m.

Proof: We recall that $(m,v) \in B_2 \Leftrightarrow m \in M_2(1,2)$ and m is not an umbilic point of $f_v \Leftrightarrow (m,v)$ is a fold point of $\bar{\xi}$. Then, there exists a neighbourhood U of (m,v) in B_2 such that $(x,y,z,v_4) \in U \Leftrightarrow \mathcal{K}_c(x,y,z,v_4) = \frac{\partial \mathcal{K}_c}{\partial v_4}(x,y,z,v_4) = 0$ and $\frac{\partial^2 \mathcal{K}_c}{\partial v_4^2}(x,y,z,v_4) \neq 0$, that is

(1) $A_1(x,y,z)v_4^3 + 3A_2(x,y,z)v_4^2 + 3A_3(x,y,z)v_4 + A_4(x,y,z) = 0$

(2) $A_1(x,y,z)v_4^2 + 2A_2(x,y,z)v_4 + A_3(x,y,z) = 0$

(3) $A_1(x,y,z)v_4 + A_2(x,y,z) \neq 0$

Since the discriminant of $\frac{\partial \mathcal{K}_c}{\partial v_4}(0,0,0,v_4)$ is $4A_2^2(0,0,0) > 0$, it follows that $A_2^2 - A_1A_3$ is positive in a neighbourhood of the origin and then equation (2) defines a function $v_4 = v_4(x,y,z)$, such that $\mathcal{K}_c(x,y,z,v_4(x,y,z)) = 0$ reduces to $\Delta(x,y,z) = 0$. Furthermore, in a neighbourhood of the origin, rank $A(x,y,z) = 1$, which implies that (x,y,z) belongs to a neighbourhood of m in $M_2(1,2)$. In other words, $\bar{\xi}(B_2) \subset M_2(1,2)$. Observe that if m is an A_k singularity of $f_v(k = 3$ or $4)$, $\Delta_x(0,0,0) = 0$.

Now, the result follows as in theorem 3, from Lema 3, from the description of the singularities of Γ and from theorem 1 (iii).

We must remark that if $\Delta_{xx}(0,0,0) = -P_2^2\{P_2(24Q_1 - 36P_2^2) - \frac{3}{2}[12(M_1 + a_{33}P_1)]^2\} \neq 0$, the contact of the asymptotic direction with $M_2(1,2)$ is of order one. Although this condition is verified for a residual set of embeddings, it does not follow from the conditions defining a cusp or swallowtail point.

Theorem 5: *Let $m \in M_1(3)$ and v the unique binormal direction at m. Then:*

(i) *m is a fold singularity of $f_v \Leftrightarrow \theta$ is transversal to $M_1(3)$.*

(ii) *m is a cusp singularity of $f_v \Leftrightarrow \theta$ is tangent to $M_1(3)$ with contact of first* *order.*

Proof: The points of $B_1(3)$ are the cusp singularities of $\bar{\xi}$. As in the Theorem 4, if U is a neighbourhood of (m,v) in $B_1(3)$, then

$$(x,y,z,v_4)\in U \Leftrightarrow \begin{cases} (1)=(2)=(3)=0 \\ \quad and \\ (4) A_1(x,y,z)\neq 0, \end{cases} \qquad (Theorem 4)$$

that is, locally, $\bar{\xi}(B_1(3)) = M_1(3)$. So, $M_1(3)$ is given by

$$\begin{cases} K_1 = A_2^2 - A_1A_3 = 0 \\ K_2 = A_1A_4 - A_2A_3 = 0. \end{cases}$$

Now, the result follows observing that θ is transversal to $K_1 \cap K_2$ if only if $M_1 + a_{33}P_1 \neq 0$. □

In the following theorem we discuss the umbilic points for a generic embedding f. Recall that if m is an umbilic point of f_v then $(m,v)\in \Sigma^2(\Gamma)$. In this case, CM has two principal directions with zero curvature and two principal directions with non zero-curvature, one of which projects onto a tangent direction of $f(M)$ at $f(m)$. Let us denote by M_s the section of M by the 4-space orthogonal to this unique direction with non zero principal curvature wich is tangente to $f(M)$ at $f(p)$. Locally, this section is a surface embedded in $\mathbb{R}^4$ and, hence, we have the concepts of inflection point as in [4]. That is, the curvature ellipse of $f(M)$ at $f(m)$ degenerate on a radial segment of straight line. We call that this inflection point of real type when $f(m)$ belongs to the curvature ellipse and of imaginary type when it doesn't. An inflection point is flat when $f(m)$ is an end point of the curvature ellipse, [6].

Theorem 6:

(a) *m is an elliptic or hyperbolic umbilic for f_v $\Leftrightarrow$ m is a non degenerate inflection point of M_s. Furthermore, in a neighbourhood of m, $\Delta^{-1}(0)$ is diffeomorphic to a cartesian product of $\Delta_s^{-1}(0)$ by an interval. The generic models for $\Delta_s^{-1}(0)$ were given in [6]. (Figure 1)*

(b) m is a parabolic umbilic $\Leftrightarrow$ m is a degenerate inflection point of M_s. (Figure 2)

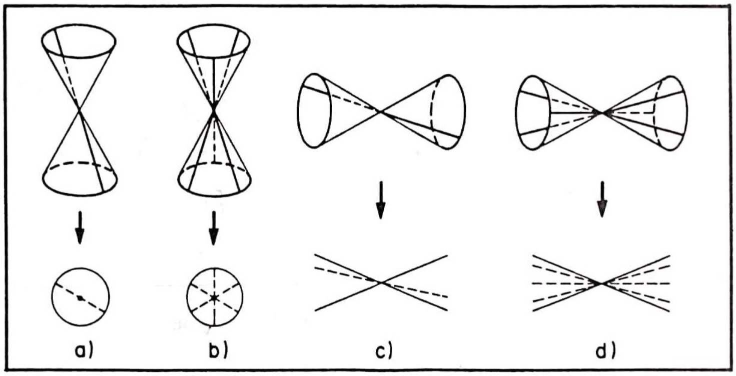

Figure 1. (a), (b), (c) and (d) show a non degenerate inflection point

Proof of Theorem 6: Let m be an umbilic point of f_v. As in Proprosition 2, we assume that f is in Monge's form, with $f_v = f_2$. Now, we use the Splitting-Lemma to write f_2 as:

(a) $x^3 \pm xy^2 + z^2$ if m is an elliptic or hyperbolic umbilic, or

(b) $x^4 + xy^2 + z^2$ if m is a parabolic umbilic.

Then, $z = 0$ produces a section M_s of M with the desired properties.

Notice that, in these coordinates, the 2-jet of the Gaussian curvature function, $j^2(\mathcal{K}_c)(0)$ is independent of z. Furthermore, in (b), m is a singular point of the curve $\Delta_s^{-1}(0)$.

Reciprocally we may assume, with no loss of generality, that $z = 0$ is the section M_s. In this case, f has the local form:

$$\begin{array}{rcl} (\mathbb{R}^3, 0) & \longrightarrow & \mathbb{R}^5, (0,0) \\ (x,y,z) & \longmapsto & (x,y,z,\bar{f}_1(x,y,z), z^2 + \bar{f}_2(x,y)) \end{array}$$

with $(0,0)$ singular point of $\bar{f}_2$ of type $D_4^{\pm}$, in case (a), and D_5, in case (b). Then, m is an umbilic point of $f_2(x,y,z)$.

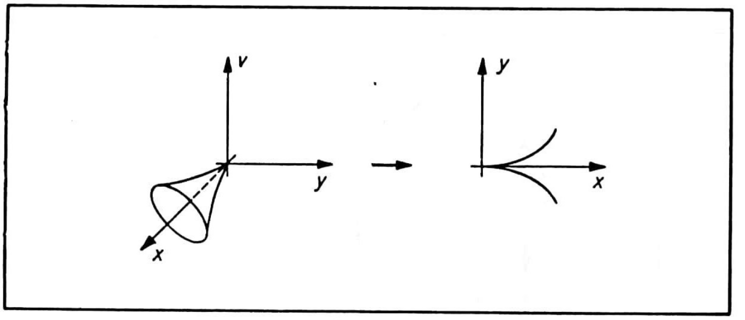

Figure 2. Inflection point of flat type.

References

[1] Arnold, V.I., *Singularities of differentiable maps.* Monographs in Mathematics, vol. 82.

[2] Bruce, J.W., *Wavefronts and parallels in Euclidean space.* Math. Proc. Cambridge Philos. Soc. 93, pags. 323-333 (1983).

[3] Golubitsky, M. and Guillemin, V., *Stable mappings and their singularities.* Springer Verlag, New York, 1973.

[4] Litlle, J.A., *On singularities of submanifolds of higher dimensional euclidean space.* Annali Mat. Pura et Appl., (ser. 4A) 83, (1969), 261-336.

[5] Looijenga, E. J. N., *Structural stability of smooth families of C^∞-functions*, Ph.D. Thesis, University of Amsterdam (1974).

[6] Mochida, D.K.; Romero Fuster, M.C.; Ruas, M.A.S., *The geometry of surfaces in 4-space from a contact view point*, (to appear in Geometriae Dedicata).

[7] Mochida, D.K., *Geometria genérica de subvariedades em codimensão maior que 1 em $\mathbb{R}^n$*, tese de Doutorado - ICMSC-USP (1993).

[8] Mond, D., Ph.D. Thesis, University of Liverpool (1982).

[9] Montaldi, J.A., *On generic composites of maps*, (to appear)

[10] Rodrígues, L., *Geometria das Subvariedades*, Monografias de Matemática n$^{\underline{o}}$26 - IMPA - Rio de Janeiro.

[11] Romero Fuster, M.C., *Sphere stratifications and the Gauss map*, Proceedings of the Royal Society of Edinburg, 95-A, (1983), 115-136.

Departamento de Matemática
Universidade Federal de São Carlos
56570-905 - São Carlos, SP
E-mail adress: ddhm@power.ufscar.br.