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SELF-TRANSLATION SURFACES

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Abstract

We study the singularities of self-translation surfaces and their Gauss maps. If α is a generic closed space curve, the locus of midpoints of secants of α is a Möbius strip with boundary α, and with normal crossings and crosscap singularities. We use the topology of this surface to obtain some new global geometric properties of closed space curves.

Given two curves $\alpha(s)$ and $\beta(t)$ in $\mathbf{R}^3$, a surface swept out by translates of the curve $\alpha(s)$ as each of its points describes a translate of $\beta(t)$ is called a *translation surface* of α and β (*cf.* [5, vol. 1, p. 137], [6, §71], [12, p. 109]). Any such surface is a translate of the surface $\alpha(s)+\beta(t)$, $(s,t)\in\mathbf{R}^2$, which is homothetic to the locus $\frac{1}{2}(\alpha(s)+\beta(t))$ of midpoints of line segments joining points $\alpha(s)$ with points $\beta(t)$.

We consider here the particular case when $\alpha=\beta$, which we call the *self-translation surface* of α. Suppose that $\alpha : S^1 \to \mathbf{R}^3$ is a closed curve, where $S^1=\mathbf{R}/2\pi\mathbf{Z}$. Then the self-translation surface of α is given by

$$T_\alpha : S^1\times S^1 \to \mathbf{R}^3,$$

$$T_\alpha(s,t)=\alpha(s)+\alpha(t).$$

Since $T_\alpha(s,t)=T_\alpha(t,s)$, we have symmetry about the diagonal, as in the case of the secant map studied by Bruce [3]. Therefore we cannot expect T_α to behave generically near the diagonal; to obtain local models we work in the space of

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$\mathbf{Z}_2$-invariant maps. We also 20 consider the self-translation surface of α to be parametrized by the Möbius strip $S^1 \times S^1/((s,t) \sim (t,s))$.

In this note we study the topology of self-translation surfaces of generic closed space curves. In the first section we obtain local normal forms, and we show that the self-translation surface is topologically stable. In the second section we apply this result to the geometry of closed space curves. In particular we show that a generic convex curve admits a family of inscribed parallelograms, and that a generic curve with less than four torsion zero points also has this property. In the third section we apply a general formula for the Euler number of a topologically stable surface with boundary to get a relation between the number of pairs of parallel tangents of a generic closed space curve, and the numbers of certain trisecant lines and inscribed octahedra. In the last section, we analyze the singularities of the Gauss map on the nonsingular part of the self-translation surface of a generic curve.

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1. Singularities of self-translation surfaces

Let α be a smooth closed space curve, and let T_α be the self-translation surface of α. A generic smooth curve is a regular embedding with nonvanishing curvature.

Lemma 1. *If $\alpha'(s_0) \neq 0$ and the curvature of α does not vanish at s_0, then the germ of T_α at (s_0, s_0) is equivalent, as a $\mathbf{Z}_2$-invariant map, to the germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$,*

$$(x, y) \mapsto (x, y^2, 0),$$

with $\mathbf{Z}_2$ action $(x, y) \mapsto (x, -y)$.

Proof. We choose local coordinates so that $s_0 = 0$ and the germ of T_α at 0 is given by

$$T_\alpha(s,t) = (s + t, s^2 + t^2 + p(s) + p(t), q(s) + q(t)),$$

where p and q vanish to second order at 0 (i.e. $p(s), q(s) \in (s)^3$). Now let $X = s + t$ and $Y = s - t$; we have

$$T_\alpha(X,Y) = (X, \frac{1}{2}(X^2 + Y^2) + p(X+Y) + p(X-Y), q(X+Y) + q(X-Y)).$$

Since $T_\alpha(s,t) = T_\alpha(t,s)$, we have $T_\alpha(X,Y) = T_\alpha(X,-Y)$. By the Malgrange preparation theorem we can write

$$T_\alpha(X,Y) = (X, \frac{1}{2}(X^2 + Y^2) + \tilde{p}(X,Y^2), \tilde{q}(X,Y^2)),$$

with $\tilde{p}(X,Y^2), \tilde{q}(X,Y^2) \in (X,Y)^3$.

Now we can use the following result of Bruce on $\mathbf{Z}_2$-invariant germs of maps $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ (see the appendix of [3]): Let F be a germ of the form $F(x,y) = (x, g(x,y^2))$, where $g(u,v)$ satisfies $\partial g/\partial u(0) = 0$ and $\partial g/\partial v(0) \neq 0$. Then F is stable as a $\mathbf{Z}_2$-invariant germ, and it is equivalent to the germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$,

$$(x,y) \mapsto (x, y^2).$$

In our case $g(u,v) = \frac{1}{2}(u^2 + v) + \tilde{p}(u,v)$; we conclude that T_α is equivalent to the germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$,

$$(X,Y) \mapsto (X, Y^2, r(X,Y^2)).$$

Therefore, by making the change of variables $\tilde{Z} = Z - r(X,Y^2)$, we obtain the desired result.

□

Therefore, if the curve α is a regular embedding, the parametrization

$$\varphi_\alpha : S^1 \times S^1/((s,t) \sim (t,s)) \to \mathbf{R}^3$$

$$\varphi_\alpha(s,t) = \alpha(s) + \alpha(t)$$

of the self-translation surface of α is a regular embedding in a neighborhood of the boundary of the Möbius strip $S^1 \times S^1/((s,t) \sim (t,s))$.

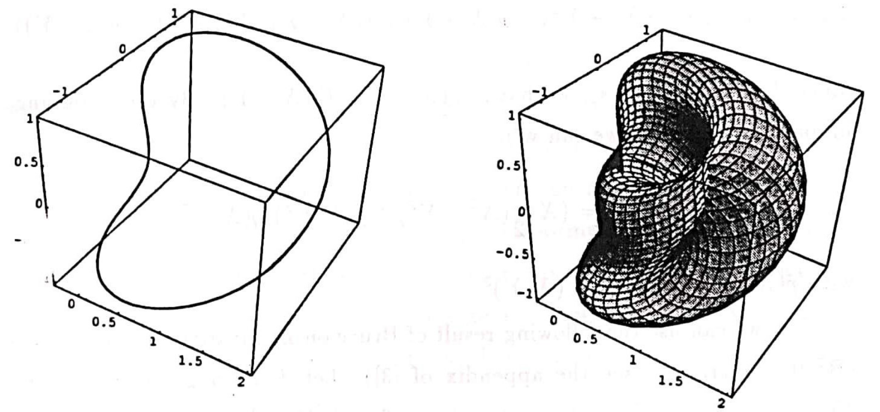

Figure 1 The space cardioid and its self-translation surface.

Theorem 1. *For an open dense subset of the space of closed smooth curves α in $\mathbf{R}^3$, the self-translation surface φ_α has one of the following local normal forms at each point:*

a) simple points:

i) regular interior point, $\varphi_\alpha(s,t) = (s,t,0)$.

ii) regular boundary point, $\varphi_\alpha(s,t) = (s+t,(s-t)^2,0)$.

iii) crosscap singularity, $\varphi_\alpha(s,t) = (s^2,t,st)$.

b) double points:

i) two regular interior points at which the surface has normal crossings.

ii) an interior point and a boundary point, both regular, at which the surface has normal crossings.

c) triple points: three regular points at which the surface has normal crossings.

Thus, except for crosscaps, the self-translation surface is immersed with normal crossings. From this we immediately get the following:

Corollary 1. *For a generic closed curve α in $\mathbf{R}^3$, φ_α is locally stable as a map of a surface with boundary, and T_α is locally stable as a $\mathbf{Z}_2$-invariant map.*

Figure 1 is a *Mathematica* plot of the self-translation surface of the *space cardioid*

$$\alpha(s) = ((1 + \cos s)\cos s, (1 + \cos s)\sin s, \sin s).$$

The surface has 3 crosscaps (corresponding to the values of the parameters $(s,t) = (0, \frac{2\pi}{3})$, $(0, \frac{4\pi}{3})$, $(\frac{2\pi}{3}, \frac{4\pi}{3})$), one boundary double point, and no triple points.

The theorem is a consequence of the following lemmas.

Lemma 2. *The critical points of the map T_α are the pairs (s,t) such that $\alpha'(s) \times \alpha'(t) = 0$. Moreover, for a generic curve α, all the critical points (s,t) with $s \neq t$ are crosscaps, and correspond to pairs of parallel tangents $\alpha'(s)$, $\alpha'(t)$.*

Proof. We have that $\partial T_\alpha/\partial s = \alpha'(s)$ and $\partial T_\alpha/\partial t = a'(t)$. But since the generic curve α is regular, we have $\alpha'(s), \alpha'(t) \neq 0$, for all s, t, and hence $\operatorname{rank} T_\alpha \geq 1$. Moreover, $\operatorname{rank} T_\alpha = 1$ if and only if $\alpha'(s)$ is parallel to $\alpha'(t)$.

If M and N are smooth manifolds, the rth *multijet space* ${}_rJ^k(M,N)$ is the set of r-tuples of k-jets of smooth maps from M to N. We have that (s,t) is a crosscap if and only if the 1-jet j^1T is transverse to S_1, where $S_1 \subset J^1(S^1 \times S^1, \mathbf{R}^3)$ denotes the submanifold of 1-jets of corank 1.

Consider the submersion

$$\begin{aligned} \pi : {}_2J^1(S^1, \mathbf{R}^3) &\rightarrow J^1(S^1 \times S^1, \mathbf{R}^3) \\ \pi(s_1, s_2, r_1^0, r_2^0, r_1^1, r_2^1) &= (s_1, s_2, r_1^0 + r_2^0, r_1^1, r_2^1) \end{aligned}$$

We have $\pi({}_2j^1\alpha) = j^1T_\alpha$; hence if ${}_2j^1\alpha$ is transverse to $\pi^{-1}(S_1)$, then j^1T_α is transverse to S_1, since π is a submersion. Now from the Thom transversality theorem we know that the set of curves α such that ${}_2j^1\alpha$ is transverse to $\pi^{-1}(S_1)$ is open and dense in the set of all smooth closed curves in $\mathbf{R}^3$. □

Lemma 3. *The self-translation surface of a generic curve has normal crossings at its self-intersection points. Moreover, the boundary of the surface does not contain triple points.*

Proof. First we prove that for generic α, the surface φ_α does not have self-intersections of multiplicity $n \geq 4$. If we denote the coordinates of the jet space ${}_8J^0(S^1, \mathbf{R}^3)$ by $r_1^0, r_2^0, \dots, r_8^0$, then the desired property is obtained by requiring transversality of the map ${}_8j^0\alpha : (S^1) \to {}_8J^0(S^1, \mathbf{R}^3)$ to the submanifold defined by the equations

$$r_1^0 + r_2^0 = \dots = r_7^0 + r_8^0.$$

Now this locus has codimension 9 in ${}_8J^0(S^1, \mathbf{R}^3)$, so the result follows from the Thom transversality theorem.

So we just need to analyze the structure of φ_α at double and triple points. We shall treat separately the three possibilities:

(i) two interior points with the same image,

(ii) an interior point with the same image as a boundary point,

(iii) three points with the same image.

To prove the assertion for case (i) suppose that we have $\alpha(s_1) + \alpha(t_1) = \alpha(s_2) + \alpha(t_2)$. Then it is enough to see that φ_α is regular at each of the two points $(s_i, t_i), s_i \neq t_i, i = 1, 2$, and that the tangent planes to φ_α at these points do not coincide. But this amounts to the following conditions:

(1) $\alpha'(s_i) \times \alpha'(t_i) \neq 0,\ i = 1, 2,$

(2) $(\alpha'(s_1) \times \alpha'(t_1)) \times (\alpha'(s_2) \times \alpha'(t_2)) \neq 0.$

The first condition holds provided that the image of the multijet map ${}_4j^1\alpha$: $(S^1)^{(4)} \to {}_4J^1(S^1, \mathbf{R}^3)$ does not intersect the subset of the multijet space ${}_4J^1(S^1, \mathbf{R}^3)$ defined by the equations

$$r_1^0 + r_2^0 = r_3^0 + r_4^0, \quad r_1^1 \times r_2^1 = 0,$$

where $r_1^0, r_2^0, r_3^0, r_4^0, r_1^1, r_2^1, r_3^1, r_4^1$ are the standard coordinates in ${}_4J^1(S^1, \mathbf{R}^3)$. But this subset has codimension 5, so it is avoided by a generic curve as a consequence of the Thom transversality theorem. To ensure the second condition, ${}_4j^1\alpha$ must avoid a codimension 5 subset of ${}_4J^1(S^1, \mathbf{R}^3)$.

For case (ii) we take $s, t, r \in S^1$ with $s \neq t$ such that $\alpha(r) = \alpha(s) + \alpha(t)$. We must show that φ_α is regular at (s, t) and that the tangent line to α at r is transverse to the tangent plane of φ_α at (s, t). These conditions translate into the requirement that ${}_3j^1\alpha$ does not meet the subsets of ${}_3J^1(S^1, \mathbf{R}^3)$ defined by the equations

$$\begin{aligned} r_1^0 &= r_2^0 + r_3^0, \quad r_2^1 \times r_3^1 = 0; \\ r_1^0 &= \tfrac{1}{2}(r_2^0 + r_3^0), \quad \det(r_1^1, r_1^2, r_3^1) = 0. \end{aligned}$$

But these subsets have codimension 5 and 4 respectively in ${}_3J^1(S^1, \mathbf{R}^3)$, and again the Thom transversality theorem ensures that they will be avoided by a generic curve.

For the third case we consider pairs (s_i, t_i) such that $\alpha(s_i) + \alpha(t_i) = p \in \mathbf{R}^3$, $i = 1, 2, 3$. We observe first that generically $s_i \neq t_i$, $i = 1, 2, 3$, and thus the triple point will not lie on the boundary of the self-translation surface. For otherwise the image of the map ${}_5j^0\alpha : (S^1)^{(5)} \to {}_5J^0(S^1, \mathbf{R}^3)$ would intersect the subset of codimension 6 defined by the conditions

$$r_1^0 = r_2^0 + r_3^0 = r_4^0 + r_5^0,$$

which can be avoided for generic α. On the other hand, the facts that φ_α is regular at the three points and that the three tangent planes meet in general position follow from the transversality of the multijet map ${}_6j^1\alpha : (S^1)^{(6)} \to {}_6J^1(S^1, \mathbf{R}^3)$ to the codimension 7 subset of ${}_6J^0(S^1, \mathbf{R}^3)$ given by the equations

$$r_1^0 + r_2^0 = r_3^0 + r_4^0 = r_5^0 + r_6^0, \quad \det(r_1^1 \times r_2^1, r_3^1 \times r_4^1, r_5^1 \times r_6^1) = 0.$$

□

2. Geometric consequences

To study the geometry of the curve α, we consider the locus of midpoints of secants of α, a singular Möbius strip with boundary α, parametrized by

$$\frac{1}{2}\varphi_\alpha(s,t) = \frac{1}{2}(\alpha(s) + \alpha(t)).$$

For a self-intersection point of φ_α of type b(i) in theorem 1, we have four distinct points s_1, s_2, s_3, s_4 such that

$$\frac{1}{2}(\alpha(s_1) + \alpha(s_2)) = \frac{1}{2}(\alpha(s_3) + \alpha(s_4)).$$

These four points define a parallelogram which is inscribed in the curve α. Moreover, if α is a generic curve such that its associated translation surface φ_α has a self-intersection, then there must be a 1-parameter family of such inscribed parallelograms, corresponding to the curve of double points of φ_α. This curve of double points may end either at a crosscap (type a(iii)) or a boundary double point (type b(ii)). At a crosscap the inscribed parallelogram degenerates to a line segment joining two points at which α has parallel tangents. At a boundary double point the inscribed parallelogram degenerates to a trisecant line of α with the property that the middle intersection point is equidistant from the other two intersection points. We'll call such a trisecant line a *symmetric trisecant.* Isolated points of the curve of double points may correspond to quadrisecants (degenerate parallelograms with distinct collinear vertices).

From these considerations we immediately get the following:

Corollary 2. *For a generic closed curve α, the number of pairs of parallel tangents has the same parity as the number of symmetric trisecants.*

For the space cardioid (Figure 1), there are two arcs of inscribed parallelograms. One arc joins two pairs of parallel tangents, and the other arc joins a symmetric trisecant to a pair of parallel tangents.

B. Segre [11] proved that if a closed curve α with nonvanishing curvature has no parallel tangents, then it has at least 4 zero torsion points. So we can assert:

Corollary 3. *A generic closed curve with less than 4 zero torsion points admits a family of inscribed parallelograms.*

On the other hand, we know that a generic convex curve has at least 4 zero torsion points (see [9] for a proof in the generic case, or [10] for C^3-embedded closed curves with nonvanishing curvature). Nevertheless, we have:

Corollary 4. *A generic convex curve admits a family of inscribed parallelograms.*

Proof. Otherwise the self-translation surface would have have no self-intersections. So the map $\frac{1}{2}\varphi_\alpha : S^1 \times S^1/((s,t) \sim (t,s)) \to \mathbf{R}^3$ would be an embedding of the Möbius strip with the convex curve α as boundary. Let S be the image of this embedding. Now α lies on the boundary of its convex hull H, which coincides with the convex hull of S. The convex hull H is a closed 3-ball, since α is nonplanar. The boundary of this 3-ball may also contain some interior points of S, but we can slightly perturb H to get another 3-ball H_ϵ whose intersection with S consists only of the boundary points of S. For let $g : \partial H \to [0,1]$ be a continuous function such that $g^{-1}(0) = \alpha(S^1)$. Now $\partial H - \alpha(S^1)$ is a C^1-surface, so it has a continuous outward unit normal vector field N. We then have, for sufficiently small $\epsilon > 0$, an embedding $f_\epsilon : \partial H \to \mathbf{R}^3$,

$$f_\epsilon(x) = x + \epsilon g(x) N(x),$$

that leaves $\alpha(S^1)$ fixed. Let H_ϵ be the closed 3-ball bounded by $f_\epsilon(\partial H)$. We have $S \subset H \subset H_\epsilon$, and $\partial S = S \cap \partial H_\epsilon$. If we collapse the boundary ∂H_ϵ to a point, we obtain a sphere which contains the embedded projective plane $S/\partial S$. But this is impossible. □

3. The Euler number of a self-translation surface

A triple point of the self-translation surface φ_α corresponds to an octahedron $\mathcal{O}$ inscribed in the curve, with the property that each pair of opposite faces of

$\mathcal{O}$ is parallel and congruent. (In other words, $\mathcal{O}$ is the dual of a parallelepiped.) We'll call such an octahedron a *parallel octahedron*. In this section, we use a variation of a result on topologically stable surfaces with boundary in $\mathbf{R}^3$, due to Izumiya and Marar [7] (*cf.* also [8]), to obtain a formula for the Euler number of the self-translation surface of α in terms of the numbers of parallel octahedra, pairs of parallel tangents, and symmetric trisecants of α.

Theorem 2. *For any compact surface M with boundary and topologically stable map $f : M \to \mathbf{R}^3$,*

$$\chi(f(M)) = \chi(M) + T + \frac{1}{2}(C - B),$$

where $\chi(M)$ is the Euler number of M, and B, T and C are the numbers of boundary double points, triple points and crosscaps of f, respectively.

Proof. The argument is similar to that used by Izumiya and Marar. Choose a stratification of f. If U_i and V_i are the strata of codimension $i = 0, 1, 2$, in M and $f(M)$, respectively, we have

$$\begin{aligned} \chi(M) &= \chi(U_0) - \chi(U_1) + \chi(U_2), \\ \chi(f(M)) &= \chi(V_0) - \chi(V_1) + \chi(V_2). \end{aligned}$$

We define the strata U_i and V_i as follows:

$U_0 = \{x \in \mathrm{int}(M) \,|\, f \text{ is nonsingular at } x, \#f^{-1}(f(x)) = 1\};$

$U_1 = U_{11} \cup U_{12},$
$U_{11} = \{x \in \partial M \,|\, \#f^{-1}(f(x)) = 1\},$
$U_{12} = \{x \in M \,|\, f^{-1}(f(x)) \cap \partial M = \emptyset,\ \#f^{-1}(f(x)) = 2\};$

$U_2 = U_{21} \cup U_{22} \cup U_{23},$
$U_{21} = \{x \in M \,|\, f \text{ is singular at } x\},$
$U_{22} = \{x \in M \,|\, f^{-1}(f(x)) \cap \partial M \neq \emptyset,\ 20\#f^{-1}(f(x)) = 2\},$
$U_{23} = \{x \in M \,|\, \#f^{-1}(f(x)) = 3\}.$

On the other hand, let $V_i = f(U_i)$, and20 $V_{ij} = f(U_{ij})$, $i, j = 0, 1, 2$. Then the following relations hold:

$$\begin{aligned}\chi(U_{21}) &= \chi(V_{21}) = C,\\ \tfrac{1}{2}\chi(U_{22}) &= \chi(V_{22}) = B,\\ \tfrac{1}{3}\chi(U_{23}) &= \chi(V_{23}) = T.\end{aligned}$$

Therefore,

$$\chi(V_2) = C + B + T,$$

$$\chi(U_2) = C + 2B + 3T.$$

Furthermore,

$$\chi(U_{11}) = \chi(U_{22}) = \chi(V_{11}),$$

$$\chi(U_{12}) = 2\chi(V_{12}) = C + B + 6T,$$

$$\chi(U_0) = \chi(V_0).$$

Therefore we can write

$$\chi(M) = \chi(U_0) - (B + (C + B + 6T)) + (C + 2B + 3T),$$

and thus

$$\chi(U_0) = \chi(M) + 3T.$$

Finally we have

$$\begin{aligned}\chi(f(M)) &= (\chi(M) + 3T) - (\chi(U_{11}) + \tfrac{1}{2}\chi(U_{12})) + (C + B + T)\\ &= \chi(M) + 3T - \tfrac{1}{2}(C + 3B + 6T) + C + B + T\\ &= \chi(M) + T + \tfrac{1}{2}(C - B)\end{aligned}$$

as required. □

Remark. The difference between our result and that of Izumiya and Marar is that each of our boundary double points has just one of its preimages on the boundary of M, whereas in [7] both preimages lie on the boundary.

If α is a closed space curve with self-translation surface parametrized by $\varphi_\alpha : S^1 \times S^1/((s,t) \sim (t,s)) \to \mathbf{R}^3$, let S_α be the image of φ_α.

Corollary 5. *Let α be a generic closed curve with self-translation surface S_α. If α has c pairs of parallel tangents, b symmetric trisecants and t parallel inscribed octahedra, then*

$$\chi(S_\alpha) = t + \frac{1}{2}(c - b).$$

For example, if α is the space cardioid (Figure 1), then $c = 3$, $b = 1$, $t = 0$ and $\chi(S_\alpha) = 1$.

4. Singularities of the Gauss map

Let α be a generic smooth closed curve in $\mathbf{R}^3$, and let $T_\alpha : S^1 \times S^1 \to \mathbf{R}^3$ be the self-translation surface of α. Denote by C the subset of $S^1 \times S^1$ consisting of all crosscaps of T_α. Let $\pi : S^2 \to \mathbf{RP}^2$ be the natural projection. We define the Gauss map of the self-translation surface, $N_\alpha : S^1 \times S^1 - C \to \mathbf{RP}^2$, by

$$\begin{aligned} N_\alpha(s,t) &= \pi(\alpha'(s) \times \alpha'(t)), \ s \neq t, \\ N_\alpha(s,t) &= \pi(\alpha'(s) \times \alpha''(s)), \ s = t. \end{aligned}$$

20 Recall the following normal forms for map germs $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$. A *fold* is a germ equivalent to $(x,y) \mapsto (x, y^2)$, a *cusp* is a germ equivalent to $(x,y) \mapsto (x, y^3 + xy)$, and a *handkerchief* is a germ equivalent to $(x,y) \mapsto (x^2, y^2)$.

Theorem 3. *Let α be a generic smooth closed space curve with torsion τ. Given a point $(s,t) \in S^1 \times S^1 - C$ with $s \neq t$, the germ of N_α at (s,t) is*

1) *a fold if $\alpha'(t)$ is parallel to the osculating plane of α at s, but $\alpha''(t)$ is not, and $\tau(s) \neq 0$,*

2) *a cusp if $\alpha'(t)$ is parallel to the osculating plane of α at s, but $\alpha''(t)$ is not, $\tau(s) = 0$ and $\tau'(s) \neq 0$,*

3) *a handkerchief if the osculating planes of α at s and t are parallel, $\tau(s) \neq 0$ and $\tau(t) \neq 0$.*

Proof. To describe the singularities of the Gauss map $N = N_\alpha$ we consider the family of height functions $H : S^1 \times S^1 \times S^2 \to \mathbf{R}$,

$$H(s,t,v) = (\alpha(s) + \alpha(t)) \cdot v$$

and analyze the singularties of H_v for all $v \in S^2$. Now (s,t) is a singularity of H_v if and only if $\alpha'(s) \times \alpha'(t)$ is parallel to v. So, unless $\alpha'(s) \times \alpha'(t) = 0$ (which occurs only if $(s,t) \in C$ or $s = t$), we have that v must be a normal vector to the self-translation surface. This singularity is nondegenerate, and hence a regular point of N, if and only if the Hessian of H_v has maximum rank at the given point (s,t). But this means that $\alpha'(s)$ is not parallel to the osculating plane of α at t and $\alpha'(t)$ is not parallel to the osculating plane of α at s.

The singularities of N thus occur at points for which rank HessH_v is less than 2, $[v] = N(s,t)$. Suppose rank Hess $H_v(s,t) = 1$, which occurs when either $\alpha'(t)$ is parallel to the osculating plane of α at s or $\alpha'(s)$ is parallel to the osculating plane of α at t. In this case, N has a singularity of type S_1, which is a fold or a cusp according to whether the torsion of α does not vanish or vanishes at s (or t respectively). On the other hand, suppose rank Hess $H_v(s,t) = 0$, which corresponds to the case when the osculating planes at s and t are parallel. Here the map N has a handkerchief singularity provided the torsion of α does not vanish at either s or t. (This is true for generic curves.) □

Corollary 6. *If α is a generic closed curve with no parallel tangents, then the germ of the Gauss map N_α at (s,t) is stable for $s \neq t$.*

Remark. A component of the set of critical values of the Gauss map N is dual (in the sense of classical projective geometry) to a component of the set of critical values of the unit secant map $\tilde{S}$ studied by Bruce in [?]. More precisely, N takes the diagonal to the binormal curve of α in $\mathbf{R}\mathbf{P}^2$, whereas $\tilde{S}$ takes the diagonal to the tangent curve of α in $\mathbf{R}\mathbf{P}^2$, and these curves are dual (*cf.* [4]). Furthermore the other component of the set of critical values of $\tilde{S}$ consists of the unit secants corresponding to bitangent planes of α, which is dual to the Maxwell stratum of the family of height functions on the curve (*cf.* [9]). (The family of height functions on the curve is the restriction of the family of height functions on the self-translation surface.)

Remark. The Gauss map N of the self-translation surface of the curve α is a Lagrangian map with generating family H, the family of height functions on the surface (*cf.* [2]). The classification of singularities of N given in theorem 3 corresponds to the following classification of singularities of H.

Let $v \in S^2$ be normal to $T_\alpha(s,t)$. If the tangent vector $\alpha'(t)$ is not parallel to the osculating plane of α at s, and $\alpha'(s)$ is not parallel to the osculating plane of α at t, then the germ of H_v at (s,t) has type A_1. In case (1) of theorem 3, the germ of H_v has type A_2, in case (2) the germ of H_v has type A_3 and in case (3) the germ of H_v has type D_4. Except in case (3), the germ at (s,t,v) of the family H is versal, so the germ of N at (s,t) is a stable Lagrangian germ.

At points of the boundary $s=t$ of the self-translation surface T_α, the height function H has boundary singularities (*cf.* [1], [2, 17.4]). If $v \in S^2$ is normal to $\alpha(s)$, but v is not normal to $T_\alpha(s,s)$ (*i.e.* v is not parallel to the binormal vector of $\alpha(s)$), then the germ of H_v at (s,s) has a boundary singularity of type A_1. If v is normal to $T_\alpha(s,s)$ and $\tau(s) \neq 0$, then the germ of H_v at (s,s) has type C_3. If v is normal to $T_\alpha(s,s)$, $\tau(s)=0$ and $\tau'(s) \neq 0$, then the germ of H_v at (s,s) has type $K_{4,2}$.

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